Chapter 5

Variations Of Mean Labeling

5.1 Introduction

An edge labeling of a graph $G$ is an assignment $f$ of labels to the edges of $G$ that induces a label for each vertex $v$ depending on the labels of the edges incident on it. Edge labelings such as edge-magic labeling, $(a,d)$-anti magic labeling and vertex-graceful labeling are some of the interesting labelings found in the dynamic survey of graph labeling by Gallian [6]. In fact B. D. Acharya [1] has introduced vertex-graceful graphs, as an edge-analogue of graceful graphs.

Observe that, in a variety of practical problems, the arithmetic mean, $X$, of a finite set of real numbers $\{x_1, x_2, ..., x_n\}$ serves as a better estimate for it, in the sense that $\sum(x_i - X)$ is zero and $\sum(x_i - X)^2$ is the minimum. If it is required to use a single integer in the place of $X$ then $\text{Round}(X)$ does this best, in the sense that $\sum(x_i - \text{Round}(X))$ and $\sum(x_i - \text{Round}(X))^2$ are minimum, where
Round($Y$), nearest integer function of a real number, gives the integer closest to $Y$; to avoid ambiguity, it is defined to be the nearest even integer in the case of half integers. This motivates us to define an edge-analogue of the mean labeling introduced by R. Ponraj [28].

A mean labeling $f$ is an injection from $V$ to the set $\{0, 1, 2, ..., q\}$ such that the set of edge labels defined by the rule $Round(\frac{f(u) + f(v)}{2})$ for each edge $uv$ is $\{1, 2, ..., q\}$. In the first part of this chapter we introduce an edge labeling of graphs called Vertex-mean labeling and investigate this for some classes of standard graphs. In the Second part of this chapter we introduce another variation of mean labeling called triangular mean labeling and investigate this for some standard graphs.

### 5.2 Vertex Mean Labeling

![Figure 5.1: Some V-mean graphs](image)

Figure 5.1: Some V-mean graphs
Definition 5.2.1. A vertex-mean labeling of a \((p,q)\) graph \(G = (V,E)\) is defined as an injection \(f : E \rightarrow \{0,1,2,...,q_*\}\), \(q_* = \max(p,q)\) such that the function \(f^V : V \rightarrow \mathbb{N}\) defined by the rule \(f^V(v) = \text{Round} \left( \frac{\sum_{e \in E_v} f(e)}{d(v)} \right)\) satisfies the property that \(f^V(V) = \{f^V(u) : u \in V\} = \{1,2,...,p\}\), where \(E_v\) denotes the set of edges in \(G\) that are incident at \(v\) and \(\mathbb{N}\) denotes the set of all natural numbers. A graph that has a vertex-mean labeling is called vertex-mean graph or V-mean graph.

Henceforth we call vertex-mean as V-mean. We obtain necessary conditions for a graph to be a V-mean graph and we present some results on this new notion in this chapter. In Figure 5.1 we give some V-mean graphs and in Figure 5.2 some non-V-mean graphs are presented.
5.2.1 Necessary Conditions

Following observations are obvious from Definition 5.2.1.

Observation 5.2.2. If $G$ is a $V$-mean graph then no $V$-mean labeling assigns 0 to a pendant edge.

Observation 5.2.3. The graph $K_2$ and disjoint union of $K_2$ are not $V$-mean graphs, as any number assigned to an edge $uv$ leads to assignment of same number to each of $u$ and $v$. Thus every component of a $V$-mean graph has at least two edges.

Observation 5.2.4. The minimum degree of any $V$-mean graph is less than or equal to three i.e., $\delta \leq 3$ as $\text{Round}(\frac{0+1+2+\cdots+n}{n+1}) \geq 2$ for $n \geq 3$. Thus graphs that contain a $r$-regular graph, where $r \geq 4$ as spanning sub graph are not $V$-mean graphs and any 3-edge-connected $V$-mean graph has a vertex of degree three.

Observation 5.2.5. If $f$ is a $V$-mean labeling of a graph $G$ then either (1) or (2) of the following is satisfied according as the induced vertex label $f^V(v)$ is obtained by rounding up or rounding down.

\[
 f^V(v)d(v) \leq \sum_{e \in E_v} f(e) + \frac{1}{2}d(v) \quad (5.1)
\]

\[
 f^V(v)d(v) \geq \sum_{e \in E_v} f(e) - \frac{1}{2}d(v) \quad (5.2)
\]

Theorem 5.2.6. If $G$ is a $V$-mean graph then the vertices of $G$
can be arranged as $v_1, v_2, ..., v_p$ such that $q^2 - 2q \leq \sum_{k=1}^{p} kd(v_k) \leq 2qq_* - q^2 + 2q$.

Proof. Let $f$ be a $V$-mean labeling of a graph $G$. Let us denote the vertex that has the induced vertex label $k$, $1 \leq k \leq p$ as $v_k$. Observe that, $\sum_{v \in V} f^V(v)d(v)$ attains its maximum/minimum when each induced vertex label is obtained by rounding up/down and the first $q$ largest/smallest values of the set $\{0, 1, 2, ..., q_*\}$ are assigned as edge labels by $f$. This with Observation 5.2.5 completes the proof.

Corollary 5.2.7. Any 3-regular graph of order $2m$, $m \geq 4$ is not a $V$-mean graph.

Corollary 5.2.8. The ladder $L_n = P_n \times P_2$, $n \geq 7$ is not a $V$-mean graph.

$V$-mean labeling of ladders $L_3$ and $L_4$ are shown in Figure 5.1.

5.2.2 Classes of $V$-mean graphs

Theorem 5.2.9. If $n \geq 3$ then the path $P_n$ is $V$-mean graph.

Proof. Let $\{e_1, e_2, ..., e_{n-1}\}$ be the edge set of $P_n$ such that $e_i = v_iv_{i+1}$. We define $f : E \rightarrow \{0, 1, 2, ..., q_* = p\}$ as follows:

$$f(e_i) = \begin{cases} 
i, & \text{if } 1 \leq i \leq p - 2, \\
i + 1, & \text{if } i = p - 1 \end{cases}.$$
Then \( f^V(v_i) = i, \ 1 \leq i \leq n \). It can be easily verified that \( f \) is an injection and the set of induced edge labels is \( \{0, 1, 2, \ldots, n\} \). Thus \( P_n \) is \( V \)-mean.

A \( V \)-mean labeling of \( P_{10} \) is shown in Figure 5.3.

![Figure 5.3: A \( V \)-mean labeling of \( P_{10} \)](image)

**Theorem 5.2.10.** The Corona \( P_n \odot K^C_m \), where \( n \geq 2 \) and \( m \geq 1 \) is \( V \)-mean graph.

**Proof.** Let the vertex set and the edge set of \( G = P_n \odot K^C_m \) be as follows:

\[
V(G) = \{u_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}
\]

\[
E(G) = A \cup B \quad \text{where } A = \{e_i = u_iu_{i+1} : 1 \leq i \leq n-1\} \quad \text{and}
\]

\[
B = \{e_{ij} = u_iu_{ij} : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}. \quad \text{We observe that the order of } G \text{ is } (m+1)n \text{ and size is } (m+1)n-1. \]

The edges of \( G \) are labeled in three steps as follows:

**Step 1** The edges \( e_1 \) and \( e_{1j}, 1 \leq j \leq m \) are assigned distinct integers from 1 to \( (m+1) \) in such a way that \( e_1 \) receives the number \( \text{Round} \left( \frac{\sum_{j=1}^{m+1} j}{m+1} \right) \).

**Step 2** For each \( i, 2 \leq i \leq n-1 \), the edges \( e_i \) and \( e_{ij}, 1 \leq j \leq m \) are assigned distinct integers from \( (m+1)(i-1)+1 \) to \( (m+1)i \) in
such a way that $e_i$ receives the number $\text{Round}\left(\frac{f(e_{i-1})+\sum_{j=1}^{m+1}(m+1)(i-1)+j}{m+2}\right)$.

**Step 3** The edges $e_{nj}, 1 \leq j \leq m$ are assigned distinct integers from $(m+1)(n-1)+1$ to $(m+1)n$ in such a way that none of these edges receive the number $\text{Round}\left(\frac{f(e_{n-1})+\sum_{j=1}^{m+1}(m+1)(n-1)+j}{m+2}\right)$.

Then it can be easily verified that edges of $G$ receive distinct labels and the labels induced on the vertices are $1, 2, ..., (m+1)n$. Thus $G$ is $V$-mean.

Figure 5.4 displays a $V$-mean labeling of $P_5 \odot K_4^C$.

![Figure 5.4: A $V$-mean labeling of $P_5 \odot K_4^C$](image)

**Theorem 5.2.11.** The star graph $K_{1,n}$ is $V$-mean graph if and only if $n \equiv 0(\text{mod}2)$.

*Proof. Necessity:* Suppose $f$ be any $V$-mean labeling of $G = K_{1,n}$, where $n = 2m + 1$ for some $m \geq 1$. As no $V$-mean labeling assigns zero to a pendant edge, $f$ assigns $2m + 1$ distinct numbers from the set $\{1, 2, ..., 2m + 2\}$ to the edges of $G$ and let $x$ be the only integer left unassigned to any edge of $G$. Observe that, whatever be the
labels assigned to the edges of $G$, the label induced on the central vertex of $G$ will be either $m + 1$ or $m + 2$, according as $x \geq m + 2$ or $x \leq m + 1$. In both cases two induced vertex labels of $G$ will be identical. This contradiction proves necessity.

**Sufficiency:** Let $G = K_{1,n}$, $n = 2m$ for some $m \geq 1$. Then assignment of $2m$ distinct numbers except $m + 1$ from the set \{1, 2, ..., 2m + 1\} gives the desired $V$-mean labeling of $G$. \qed

A $V$-mean labeling of $K_{1,6}$ is shown in Figure 5.5.

![Figure 5.5: A V-mean labeling of $K_{1,6}$](image)

**Theorem 5.2.12.** The graph $S(K_{1,n})$, obtained by subdividing every edge of $K_{1,n}$, is $V$-mean.

**Proof.** Let $V = \{u, v_i, w_i : 1 \leq i \leq n\}$ and $E = \{uv_i, v_iw_i : 1 \leq i \leq n\}$ be the vertex set and edge set of $S(K_{1,n})$ respectively. Then $G$ is of order $2n + 1$ and size $2n$.

**case 1:** $n$ is odd.

Let $n = 2m + 1$. Define $f : E \rightarrow \{0, 1, 2, ..., 4m + 3\}$ as follows:
\[
f(uv_i) = \begin{cases} 
2i & \text{if } 1 \leq i \leq m \\
2i + 1 & \text{if } m + 1 \leq i \leq n
\end{cases}
, \quad \text{and}
\]

\[
f(v_iw_i) = \begin{cases} 
2i - 1 & \text{if } 1 \leq i \leq m + 1 \\
2i & \text{if } m + 2 \leq i \leq n
\end{cases}.
\]

Then it is easy to verify that

\[
f^V(u) = 2m + 3, \quad f^V(v_i) = \begin{cases} 
2i & \text{if } 1 \leq i \leq m + 1 \\
2i + 1 & \text{if } m + 2 \leq i \leq n
\end{cases}, \quad \text{and}
\]

\[
f^V(w_i) = \begin{cases} 
2i - 1 & \text{if } 1 \leq i \leq m + 1 \\
2i & \text{if } m + 2 \leq i \leq n
\end{cases}.
\]

Hence \(f^V(V) = \{1, 2, 3, \ldots, 4m + 3\}\).

**case 2:** \(n\) is even.

Let \(n = 2m\). Define \(f : E \rightarrow \{0, 1, 2, \ldots, 4m + 1\}\) as follows:

\[
f(uv_i) = \begin{cases} 
2i & \text{if } 1 \leq i \leq m \\
2i - 1 & \text{if } i = m + 1 \\
2i + 1 & \text{if } m + 2 \leq i \leq n
\end{cases}, \quad \text{and}
\]
\[
f(v_iw_i) = \begin{cases} 
2i - 1 & \text{if } 1 \leq i \leq m \\
2i + 1 & \text{if } i = m + 1 \\
2i & \text{if } m + 2 \leq i \leq n
\end{cases}
\]

Then, it is easy to verify that \( f^V(u) = 2m + 1 \), \( f^V(v_i) = \begin{cases} 
2i & \text{if } 1 \leq i \leq m + 1 \\
2i + 1 & \text{if } m + 2 \leq i \leq n
\end{cases} \), \( f^V(w_i) = \begin{cases} 
2i - 1 & \text{if } 1 \leq i \leq m + 1 \\
2i + 1 & \text{if } i = m + 1 \\
2i & \text{if } m + 2 \leq i \leq n
\end{cases} \), and \( f^V(V) = \{1, 2, 3, \ldots, 4m + 1\} \).

Hence \( f^V(V) = \{1, 2, 3, \ldots, 4m + 1\} \).

Thus, \( S(K_{1,n}) \) is \( V \)-mean. \(\square\)

For example \( V \)-mean labeling of \( S(K_{1,5}) \) and \( S(K_{1,6}) \) are shown in Figure 5.6.

![Figure 5.6: V-mean labeling of \( S(K_{1,5}) \) and \( S(K_{1,6}) \)](image-url)
**Theorem 5.2.13.** If \( n \geq 3 \) then the cycle \( C_n \) is \( V \)-mean graph.

*Proof.* Let \( \{e_1, e_2, \ldots, e_n\} \) be the edge set of \( C_n \) such that \( e_i = v_i v_{i+1}, \) \( 1 \leq i \leq n-1, \) \( e_n = v_n v_1. \) Let \( \zeta = \left\lceil \frac{n}{2} \right\rceil - 1. \) The edges of \( C_n \) are labeled as follows: The numbers \( 0, 1, 2, \ldots, n \) except \( \zeta \) are arranged in an increasing sequence \( \alpha_1, \alpha_2, \ldots, \alpha_n \) and \( \alpha_k \) is assigned to \( e_k. \) Clearly the edges of \( C_n \) receive distinct labels and the vertex labels induced are \( 1, 2, \ldots, n. \) Thus \( C_n \) is \( V \)-mean graph. \( \square \)

Figure 5.7 displays \( V \)-mean labeling of cycles.

![Figure 5.7: V-mean labeling of \( C_9 \) and \( C_{10} \)](image)

**Theorem 5.2.14.** The crown \( C_n \odot K_1 \) is \( V \)-mean graph.
Proof. Let the vertex set and the edge set of $G = C_n \odot K_1$ be as follows: $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$, $E(G) = A \cup B$ where $A = \{e_i = u_iu_{i+1} : 1 \leq i \leq n - 1\} \cup \{e_n = u_nu_1\}$ and $B = \{e'_i = u_iv_i : 1 \leq i \leq n\}$. We observe that both the order and size of $G$ are equal to $2n$. For $3 \leq n \leq 5$, $V$-mean labeling of $G$ are shown in Figure 5.8. For $n \geq 6$, let $r = \left\lfloor \frac{n}{3} \right\rfloor$ and define $f : E(G) \longrightarrow \{0, 1, 2, \ldots, 2n\}$ as follows:

Case 1: $n \equiv 0(\text{mod } 3)$.

\[
\begin{align*}
    f(e_i) &= \begin{cases} 
        2i - 2 & \text{if } 1 \leq i \leq r - 1 \\
        2i & \text{if } i = r \\
        2i - 1 & \text{if } r + 1 \leq i \leq n 
    \end{cases}, \\
    f(e'_i) &= \begin{cases} 
        2i - 1 & \text{if } 1 \leq i \leq r \\
        2i & \text{if } r + 1 \leq i \leq n 
    \end{cases}.
\end{align*}
\]

Then $f^V(u_i) = \begin{cases} 
2r & \text{if } i = 1 \\
2i - 2 & \text{if } 2 \leq i \leq r \\
2i - 1 & \text{if } r + 1 \leq i \leq n
\end{cases}$, and

\[
\begin{align*}
f^V(v_i) = f(e'_i) &= \begin{cases} 
2i - 1 & \text{if } 1 \leq i \leq r \\
2i & \text{if } r + 1 \leq i \leq n
\end{cases}.
\end{align*}
\]
Case 2: $n \equiv (\text{mod } 3)$

$$f(e_i) = \begin{cases} 
2i - 2 & \text{if } 1 \leq i \leq r \\
2i - 1 & \text{if } r + 1 \leq i \leq n
\end{cases}$$

$$f(e'_i) = \begin{cases} 
2i - 1 & \text{if } 1 \leq i \leq r \\
2i & \text{if } r + 1 \leq i \leq n \\
2r + 1 & \text{if } i = 1
\end{cases}$$

Then $f^V(u_i) = \begin{cases} 
2i - 2 & \text{if } 2 \leq i \leq r + 1, \text{ and} \\
2i - 1 & \text{if } r + 2 \leq i \leq n
\end{cases}$

$$f^V(v_i) = f(e'_i) = \begin{cases} 
2i - 1 & \text{if } 1 \leq i \leq r \\
2i & \text{if } r + 1 \leq i \leq n
\end{cases}$$

Clearly, in both cases, $f$ is an injection and the set of induced vertex labels is $\{1, 2, ..., 2n\}$. Thus the crown $C_n \odot K_1$ is $V$-mean.  

$V$-mean labeling of some crowns are shown in Figure 5.9.

**Theorem 5.2.15.** A dragon graph is $V$-mean.

**Proof.** Let $G$ be the dragon consisting of the path $P_m : v_1v_2...v_m$ and the cycle $C_n : u_1u_2...u_n$. Let $v_m$ be identified with $u_n$ and $r = \left\lceil \frac{n}{2} \right\rceil$. 

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Let \( e_i = v_i v_{i+1}, 1 \leq i \leq m - 1, \) \( e'_i = u_i u_{i+1}, 1 \leq i \leq n - 1 \) and \( e'_n = u_n u_1 \) be the edges of \( G \). Observe that both the order and size of \( G \) are equal to \( m + n - 1 \).

The edges of \( G \) are labeled as follows:

For \( 1 \leq i \leq m - 1 \), the integer \( i \) is assigned to the edge \( e_i \). The odd and even integers from 1 to \( n \) are respectively arranged in increasing sequences \( \alpha_1, \alpha_2, ..., \alpha_r \) and \( \beta_1, \beta_2, ..., \beta_{n-r} \) and \( m - 1 + \alpha_k \) is assigned to \( e'_k \) and \( m - 1 + \beta_k \) is assigned to \( e'_{n-k+1} \). Clearly the edges of \( G \) receive distinct labels from \( \{0, 1, 2, ..., m + n - 1\} \) and the vertex labels induced are \( 1, 2, ..., m + n - 1 \) as illustrated in Table 5.1. Thus
<table>
<thead>
<tr>
<th>vertex</th>
<th>induced edge label</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_i, 1 \leq i \leq m-1$</td>
<td>$i$</td>
</tr>
<tr>
<td>$u_k, 1 \leq k \leq n-r$</td>
<td>$m - 1 + \beta_k$</td>
</tr>
<tr>
<td>$u_{n-k+1}, 1 \leq i \leq r$</td>
<td>$m - 1 + \alpha_k$</td>
</tr>
</tbody>
</table>

Table 5.1: Induced vertex labels

$G$ is $V$-mean. \hfill $\square$

For example $V$-mean labelings of dragons obtained from $P_5$ and $C_7$ and $P_6$ and $C_7$ are shown in Figure 5.10.

![Figure 5.10: V-mean labelings of dragons](image)

**Theorem 5.2.16.** Let $G$ be the graph obtained by identifying one vertex of the cycle $C_3$ with the central vertex of $K_{1,n}$. Then $G$ is $V$-mean.

**Proof.** Let $u_1, u_2, u_3$ be the consecutive vertices of $C_3$. Let $V(K_{1,n}) =$
\{w, w_1, w_2, ..., w_m\} with \(d(w) = m\) and \(u_1\) be identified with \(w\). Then \(G\) is of order \(n + 3\) and size \(n + 2\). Let \(r = \lceil \frac{n}{2} \rceil\). Define \(f : E(G) \rightarrow \{0, 1, 2, ..., n + 3\}\) as follows:

\[
f(wu_2) = r + 2, \quad f(wu_3) = r + 4, \quad f(u_2u_3) = r + 3, \quad \text{and}
\]

\[
f(ww_i) = \begin{cases} 
i & \text{if } 1 \leq i \leq r + 1 \\
i + 3 & \text{if } r + 2 \leq i \leq n\end{cases}.
\]

Then, it follows easily that

\[
f^V(w) = r + 2, \quad f^V(u_2) = r + 3, \quad f^V(u_3) = r + 4, \quad \text{and}
\]

\[
f^V(w_i) = \begin{cases} 
i & \text{if } 1 \leq i \leq r + 1 \\
i + 3 & \text{if } r + 2 \leq i \leq n\end{cases}.
\]

Hence \(f^V(V(G)) = \{1, 2, 3, ..., n + 3\}\). Thus \(G\) is \(V\)-mean. \qed

For example \(V\)-mean labeling of the graphs obtained from \(K_{1,5}\) and \(K_{1,6}\) are shown in Figure 5.11.

**Theorem 5.2.17.** The graph \(C_n(3, 1)\) is \(V\)-mean.

**Proof.** Let \(G = C_n(3, 1)\). Let \(V(G) = \{v_1, v_2, ..., v_n\}\) and \(E(G) = \{v_1v_3, v_nv_1, v_iv_{i+1} : 1 \leq i \leq n - 1\}\). Then \(G\) is of order \(n\) and size \(n + 1\). Let \(r = \lceil \frac{n}{2} \rceil\). The edges of \(G\) are assigned labels as follows: Define \(f : E(G) \rightarrow \{0, 1, 2, ..., n + 1\}\) as follows: The integers 0, 1, 2, 3 are respectively assigned to the edges \(v_1v_2, v_2v_3, v_1v_3, \) and \(v_nv_1\).
The odd and even integers of \( \{4, 5, 6, \ldots, n\} \) are respectively arranged in increasing sequences \( \alpha_1, \alpha_2, \ldots, \alpha_{r-2} \) and \( \beta_1, \beta_2, \ldots, \beta_{n-r-1} \) and \( \alpha_k \) is assigned to \( v_{k+2}v_{k+3} \), and \( \beta_k \) is assigned to \( v_{n-k}v_{n-k+1} \).

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Induced edge label</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 )</td>
<td>2</td>
</tr>
<tr>
<td>( v_2 )</td>
<td>1</td>
</tr>
<tr>
<td>( v_3 )</td>
<td>3</td>
</tr>
<tr>
<td>( v_n )</td>
<td>( \beta_1 = 4 )</td>
</tr>
<tr>
<td>( v_{k+3}, 1 \leq k \leq n - r - 2 )</td>
<td>( \beta_{k+1} )</td>
</tr>
<tr>
<td>( v_{n-k}, 1 \leq k \leq r - 2 )</td>
<td>( \alpha_k )</td>
</tr>
</tbody>
</table>

Table 5.2: Induced vertex labels

Clearly the assignment is an injection and the set of induced vertex labels is \( \{1, 2, \ldots, n\} \), as illustrated in Table 5.2. Thus \( G \) is \( V \)-mean.

For example \( V \)-mean labeling of \( C_8(3, 1) \) and \( C_9(3, 1) \) are shown.
Figure 5.12: V-mean labeling of $C_n(3,1)$

Theorem 5.2.18. If $m \in \{n, n+1, n+2\}$, the graph $G$ consisting of two cycles $C_n$ and $C_m$ connected by a bridge is V-mean.

Proof. Let $G$ be the graph consisting of two cycles $C_n : v_1v_2...v_n$ and $C_m : u_1u_2...u_m$ and $e_0 = v_nv_1$ be the bridge connecting them. Then $G$ is of order $m+n$ and size $m+n+1$. Let $e_i = v_iv_{i+1}$, $1 \leq i \leq n-1$ and $e_n = v_nv_1$ and $e_i' = u_iu_{i+1}$, $1 \leq i \leq m-1$, and $e_m' = u_mu_1$. Let $r = \lceil \frac{n}{2} \rceil$. Define $f : E(G) \longrightarrow \{0, 1, 2, ..., m+n+1\}$ as follows:
\[ f(e_i) = \begin{cases} 
1 & \text{if } i = 0 \\
2i + 1 & \text{if } 1 \leq i \leq r - 1, \\
2(n - i) & \text{if } r \leq i \leq n 
\end{cases} \]

\[ f(e'_i) = n + i \text{ if } 1 \leq i \leq m. \]

Then

\[ f(V(v_i)) = \begin{cases} 
2i & \text{if } 1 \leq i \leq r - 1, \\
2(n - i) + 1 & \text{if } r \leq i \leq n 
\end{cases} \]

\[ f(V(u_i)) = n + i \text{ if } 1 \leq i \leq m. \]

Clearly \( f \) is an injection and the set of induced vertex labels is \( \{1, 2, ..., n + m\} \). Hence the theorem. \( \square \)

A \( V \)-mean labeling of the graph obtained from \( C_7 \) and \( C_9 \) is shown in Figure 5.13.

**Theorem 5.2.19.** The graph \( C_n \cup C_m \) is \( V \)-mean.

**Proof.** Let \( \{e_1, e_2, ..., e_n\} \) be the edge set of \( C_n \) such that \( e_i = v_iv_{i+1}, \) \( 1 \leq i \leq n - 1, \) \( e_n = v_nv_1 \) and \( \{e'_1, e'_2, ..., e'_m\} \) be the edge set of \( C_m \) such that \( e_i = u_iu_{i+1}, 1 \leq i \leq m - 1, e'_m = u_mu_1. \) Then the graph
Figure 5.13: A V-mean labeling of a graph obtained from $C_7$ and $C_9$

$G = C_n \cup C_m$ is of order and size both equal to $m + n$. Let $m \geq n$.

Define $f : E(G) \rightarrow \{0, 1, 2, \ldots, m + n\}$ as follows:

$$f(e_i) = \begin{cases} 
  i - 1 & \text{if } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
  i & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n - 1, \\
  n + 1 & \text{if } i = n
\end{cases}$$

$$f(e'_i) = \begin{cases} 
  n + 2i + 1 & \text{if } 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor - 1 \\
  n + 2(m - i) & \text{if } \left\lfloor \frac{m}{2} \right\rfloor \leq i \leq m
\end{cases}$$
Then

\[
f^V(v_i) = \begin{cases} 
\lfloor \frac{n}{2} \rfloor + 1 & \text{if } i = 1 \\
i - 1 & \text{if } 2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1 \\
i & \text{if } \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n 
\end{cases}
\]

\[
f^V(u_i) = \begin{cases} 
n + 2i & \text{if } 1 \leq i \leq \lfloor \frac{m}{2} \rfloor \\
n + 2(m - i) + 1 & \text{if } \lfloor \frac{m}{2} \rfloor + 1 \leq i \leq m 
\end{cases}
\]

Clearly \( f \) is an injection and the set of induced vertex labels is \( \{1, 2, \ldots, n + m\} \). Hence the theorem. \( \square \)

A \( V \)-mean labeling of \( C_8 \cup C_{12} \) is shown in Figure 5.14.

![Figure 5.14: A V-mean labeling of \( C_8 \cup C_{12} \)](image_url)

**Theorem 5.2.20.** If \( m \in \{3, 4\} \), the graph \( G \) obtained by identify-
ing one vertex of the cycle $C_m$ with a vertex of $C_n$ is $V$-mean.

**Proof. case 1: $m = 3$.**

Let $G$ be the graph consisting of two cycles $C_3 : v_1v_2v_3v_1$ and $C_n : v_3v_4...v_{n+2}v_3$. Let $r = \lceil \frac{n}{2} \rceil$. Define $f : E(G) \rightarrow \{0, 1, 2, ..., n + 3\}$ as follows: The integers 0, 1, 2, 3, 4 are assigned respectively to the edges $v_1v_2, v_3v_1, v_{n+2}v_3, v_2v_3, v_3v_4$. The odd and even integers of $\{5, 6, 7, ..., n+2\}$ are arranged respectively in increasing sequences $\alpha_1, \alpha_2, ..., \alpha_{r-1}$ and $\beta_1, \beta_2, ..., \beta_{n-r-1}$ and $\alpha_k$ is assigned to $v_{k+3}v_{k+4}$, and $\beta_k$ is assigned to $v_{n+2-k}v_{n+3-k}$.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Induced edge label</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_k$, $k = 1, 2, 3$</td>
<td>$k$</td>
</tr>
<tr>
<td>$v_{n+2}$</td>
<td>4</td>
</tr>
<tr>
<td>$v_4$</td>
<td>$\alpha_1 (= 5)$</td>
</tr>
<tr>
<td>$v_{k+4}$, $1 \leq k \leq n - r - 1$</td>
<td>$\beta_k$</td>
</tr>
<tr>
<td>$v_{n-k+2}$, $1 \leq k \leq r - 2$</td>
<td>$\alpha_{k+1}$</td>
</tr>
</tbody>
</table>

Table 5.3: Induced vertex labels

Clearly the assignment is an injection and the set of induced vertex labels is $\{1, 2, ..., n + 2\}$, as illustrated in Table 5.3. Thus $G$ is $V$-mean.

**case 2: $m = 4$.**

Let $G$ be the graph consisting of two cycles $C_4 : v_1v_2v_3v_4v_1$ and $C_n : v_4v_5...v_{n+3}v_4$. Let $r = \lceil \frac{n}{2} \rceil$. Define $f : E(G) \rightarrow \{0, 1, 2, ..., n + 4\}$ as follows: The integers 0, 1, 2, 3, 4, 5, 6 are assigned respectively to the edges $v_1v_2, v_2v_3, v_3v_4, v_{n+3}v_4$, and $v_{n-3}v_4$. The odd and even integers of $\{7, 8, 9, ..., n+4\}$ are arranged respectively in increasing sequences $\alpha_1, \alpha_2, ..., \alpha_{r-1}$ and $\beta_1, \beta_2, ..., \beta_{n-r-1}$ and $\alpha_k$ is assigned to $v_{k+3}v_{k+4}$, and $\beta_k$ is assigned to $v_{n+2-k}v_{n+3-k}$.
$v_4 v_5, v_4 v_1, v_5 v_6$. The odd and even integers of $\{7, 8, ..., n + 3\}$ are arranged respectively in increasing sequences $\alpha_1, \alpha_2, ..., \alpha_{r-1}$ and $\beta_1, \beta_2, ..., \beta_{n-r-2}$ and $\alpha_k$ is assigned to $v_{k+3} v_{k+4}$, and $\beta_k$ is assigned to $v_{n+2-k} v_{n+3-k}$.

![Figure 5.15: V-mean labeling of graphs obtained from $C_n$](image)

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Induced edge label</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>3</td>
</tr>
<tr>
<td>$v_2$</td>
<td>1</td>
</tr>
<tr>
<td>$v_3$</td>
<td>2</td>
</tr>
<tr>
<td>$v_4$</td>
<td>4</td>
</tr>
<tr>
<td>$v_5$</td>
<td>5</td>
</tr>
<tr>
<td>$v_{n+3}$</td>
<td>6</td>
</tr>
<tr>
<td>$v_6$</td>
<td>$\alpha_1 (= 7)$</td>
</tr>
<tr>
<td>$v_{k+6}, 1 \leq k \leq n - r - 2$</td>
<td>$\beta_k$</td>
</tr>
<tr>
<td>$v_{n-k+3}, 1 \leq k \leq r - 2$</td>
<td>$\alpha_{k+1}$</td>
</tr>
</tbody>
</table>

Table 5.4: Induced vertex labels

Clearly the assignment is an injection and the set of induced
vertex labels is \{1, 2, ..., n + 3\}, as illustrated in Table 5.4. Thus \(G\) is \(V\)-mean.

\[\square\]

For example \(V\)-mean labeling of graphs obtained from \(C_3\) and \(C_8\) and \(C_4\) and \(C_8\) are shown in Figure 5.15.

### 5.2.3 Some disconnected \(V\)-mean graphs

In this section we present a method to construct disconnected \(V\)-mean graphs from \(V\)-mean graphs.

The following observation is obvious from the definition of \(V\)-mean labeling.

**Observation 5.2.21.** If \(f\) is any \(V\)-mean labeling of a \((p, q)\) graph \(G\), then \(f(e) \geq p\) for some edge \(e \in E(G)\). In particular, if \(p \geq q\) then \(f(e) \leq p\) for every edge \(e \in E(G)\) and hence \(f(e) = p\) for some edge \(e \in E(G)\).

**Notation 5.2.22.** We call a \(V\)-mean labeling \(f\) of a graph \(G(p, q)\) as type-A, if \(f(e) \leq p\) for every edge \(e \in E(G)\), type-B if \(f(e) \geq 1\) for every edge \(e \in E(G)\), and type-AB if \(1 \leq f(e) \leq p\) for every edge \(e \in E(G)\). For \(S \in \{A, B, AB\}\), we call \(G\) as \(V\)-mean graph of type-\(S\) if it has a \(V\)-mean labeling \(f\) of type-\(S\).

**Remark 5.2.23.** We observe that the \(V\)-mean graphs presented so far in this section can be classified as given in Table 5.5.
<table>
<thead>
<tr>
<th>S.N</th>
<th>V-mean Graph</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$C_n$</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>$C_n \odot K_1^C$</td>
<td>A</td>
</tr>
<tr>
<td>3</td>
<td>$C_n(3,1)$</td>
<td>A</td>
</tr>
<tr>
<td>4</td>
<td>The graph consisting of two cycles $C_n$ and $C_m$ connected by a bridge</td>
<td>A</td>
</tr>
<tr>
<td>5</td>
<td>The graph $C_n \cup C_m$ where $m \in {n, n+1, n+2}$</td>
<td>A</td>
</tr>
<tr>
<td>6</td>
<td>The graph obtained by identifying one vertex of the cycle $C_m$ with a vertex of $C_n$ when $m = 3$ or $4$</td>
<td>A</td>
</tr>
<tr>
<td>7</td>
<td>$P_n$ where $n \geq 3$</td>
<td>AB</td>
</tr>
<tr>
<td>8</td>
<td>$P_n \odot K_1^C_m$ where $n \geq 2$</td>
<td>AB</td>
</tr>
<tr>
<td>9</td>
<td>$K_{1,n}$ if and only if $n \equiv 0 \text{ (mod 2)}$</td>
<td>AB</td>
</tr>
<tr>
<td>10</td>
<td>The graph $S(K_{1,n})$, obtained by subdividing every edge of $K_{1,n}$</td>
<td>AB</td>
</tr>
<tr>
<td>11</td>
<td>Dragon graph</td>
<td>AB</td>
</tr>
<tr>
<td>12</td>
<td>The graph obtained by identifying one vertex of cycle $C_m$ with the central vertex of $K_{1,n}$ when $m = 3$ or $4$</td>
<td>AB</td>
</tr>
</tbody>
</table>

Table 5.5: V-mean graphs

Let $f$ be a V-mean labeling of $G(p_1, q_1)$ and $g$ be a V-mean labeling of $H(p_2, q_2)$. Observe that the order of $G \cup H$ is $p = p_1 + p_2$ and size is $q = q_1 + q_2$. Define $h : E(G \cup H) \rightarrow \{0, 1, 2, ..., q_1 \}$ as follows: $h(e) = \begin{cases} f(e) & \text{if } e \in E(G) \\ g(e) + p_1 & \text{if } e \in E(H) \end{cases}$.

Suppose $f$ is of type-A and $g$ is of type-$B$. Then $f(e) \leq p_1$ for every edge $e \in E(G)$ and $g(e) \geq 1$ for every edge $e \in E(H)$. As $f$ and $g$ are injective functions, $f(e) \leq p_1$ for every edge $e \in E(G)$ and $g(e) + p_1 \geq p_1 + 1$ for every edge $e \in E(H)$, $h$ is injective.

Suppose $f$ is of type-A and $g$ is of type-$AB$. Then $f(e) \leq p_1$ for every edge $e \in E(G)$ and $1 \leq g(e) \leq p_2$ for every edge $e \in E(H)$. As $f$ and $g$ are injective functions, $f(e) \leq p_1$ for every edge $e \in E(G)$.
and \( p_1 + 1 \leq g(e) + p_1 \leq p_1 + p_2 \) for every edge \( e \in E(H) \), \( h \) is injective and \( h(e) \leq p \) for every edge \( e \in E(G \cup H) \).

Suppose \( f \) is of type-\( AB \) and \( g \) is of type-\( B \). Then \( 1 \leq f(e) \leq p_1 \) for every edge \( e \in E(G) \) and \( g(e) \geq 1 \) for every edge \( e \in E(H) \). As \( f \) and \( g \) are injective functions, \( 1 \leq f(e) \leq p_1 \) for every edge \( e \in E(G) \) and \( p_1 + 1 \leq g(e) + p_1 \) for every edge \( e \in E(H) \), \( h \) is injective and \( h(e) \geq 1 \) for every edge \( e \in E(G \cup H) \).

Suppose, both \( f \) and \( g \) are of type-\( AB \). Then \( 1 \leq f(e) \leq p_1 \) for every edge \( e \in E(G) \) and \( 1 \leq g(e) \leq p_2 \) for every edge \( e \in E(H) \). As, \( f \) and \( g \) are injective functions, \( 1 \leq f(e) \leq p_1 \) for every edge \( e \in E(G) \) and \( p_1 + 1 \leq g(e) + p_1 \leq p_1 + p_2 \) for every edge \( e \in E(H) \), \( h \) is injective and \( 1 \leq h(e) \leq p \) for every edge \( e \in E(G \cup H) \).

The set of induced edge labels of \( G \cup H \) in all four cases is as follows: \( h^V(V(G \cup H)) = \{ f^V(v) : v \in V(G) \} \cup \{ p_1 + g^V(u) : u \in V(H) \} \)
\[
= \{1, 2, ..., p_1\} \cup \{p_1 + 1, p_1 + 2, ..., p_1 + p_2\}
= \{1, 2, ..., p_1 + p_2\}.
\]
Thus we have the following four theorems.

**Theorem 5.2.24.** If \( G(p_1, q_1) \) is a \( V \)-mean graph of type-\( A \) and \( H(p_2, q_2) \) is a \( V \)-mean graph of type-\( B \), then \( G \cup H \) is \( V \)-mean.

**Theorem 5.2.25.** If \( G(p_1, q_1) \) is a \( V \)-mean graph of type-\( A \) and \( H(p_2, q_2) \) is a \( V \)-mean graph of type-\( AB \), then \( G \cup H \) is \( V \)-mean graph of type-\( A \).
**Theorem 5.2.26.** If $G(p_1, q_1)$ is a $V$-mean graph of type-AB and $H(p_2, q_2)$ is a $V$-mean graph of type-B, then $G \cup H$ is $V$-mean graph of type-B.

**Theorem 5.2.27.** If both $G(p_1, q_1)$ and $H(p_2, q_2)$ are $V$-mean graphs of type-AB then the graph $G \cup H$ is $V$-mean graph of type-AB.

**Corollary 5.2.28.** Let $G$ be a tree or a unicyclic graph or a two regular graph. If $G$ is $V$-mean and $H$ is a $V$-mean graph of type-B, then $G \cup H$ is $V$-mean.

**Corollary 5.2.29.** If $G(p, q)$ is $V$-mean graph of type-AB then, the graph $mG$ is $V$-mean graph type-AB.

**Corollary 5.2.30.** If both $G(p_1, q_1)$ and $H(p_2, q_2)$ are $V$-mean graphs of type-AB, then the graph $mG \cup nH$ is $V$-mean graph of type-AB.

**Corollary 5.2.31.** If $G(p_1, q_1)$ is a $V$-mean graph of type-A and $H(p_2, q_2)$ is a $V$-mean graph of type-AB, then $G \cup mH$ is $V$-mean graph of type-A.

---

**Figure 5.16:** V-mean labeling of $C_{10} \cup P_4 \cup P_5$
It is interesting to note that a number of disconnected V-mean graphs can be obtained by applying Theorem 5.2.24 through Corollary 5.2.31 on V-mean graphs listed in Table 5.5. For example, the graph $\bigcup_{i=1}^{k} P_{n_i}$, where $n_i \geq 3$, the graph $mP_n$ where $n \geq 3$, the graph $C_n \cup \left( \bigcup_{i=1}^{k} P_{n_i} \right)$, where $n_i \geq 3$, the graph $C_n \cup kP_m$ where $m \geq 3$, the graph $C_n \cup C_m \cup \left( \bigcup_{i=1}^{k} P_{n_i} \right)$, where $n_i \geq 3$, the graph $C_n \cup C_m \cup kP_t$ where $t \geq 3$ are some of such graphs. To illustrate this a V-mean labeling of $C_{10} \cup P_4 \cup P_5$ and a V-mean labeling of $(C_8 \cup C_{12}) \cup K_{1,8}$ are given in Figure 5.16 and Figure 5.17 respectively.

### 5.3 Triangular Mean Labeling Of graphs

#### 5.3.1 Triangular mean labeling

**Definition 5.3.1.** Let $G$ be a graph with $p$ vertices and $q$ edges. A Triangular mean labeling is an injective function $f$ from $V$ to the set
Figure 5.18: Some triangular mean graphs

\{0, 1, 2, ..., T_q\}, where \( T_q \) is the \( q^{th} \) triangular number, that induces for each edge \( uv \) the label \( f^*(uv) = \left\lceil \frac{f(u) + f(v)}{2} \right\rceil \) such that the set of edge labels is \( \{T_1, T_2, ..., T_q\} \). A graph which admits such a labeling is called a triangular mean graph.

Some triangular mean graphs are shown in Figure 5.18 and some non-triangular mean graphs are shown in Figure 5.19.

**Theorem 5.3.2.** The path \( P_n \), where \( n \geq 2 \) is a triangular mean graph.

**Proof.** Let \( P_n : v_1v_2...v_n \) be a path.

**case 1:** \( n \) is odd.

Define \( f : V(P_n) \rightarrow \{0, 1, 2, ..., T_{n-1}\} \) by

\[
f(v_i) = \begin{cases} 
0 & \text{if } i = 1 \\
T_i - 1 & \text{if } i \text{ is even} \\
T_{i-1} & \text{if } i \text{ is odd and } i \neq 1.
\end{cases}
\]
Clearly $f$ is injective and for $2 \leq i \leq n$, $f^*(v_{i-1}v_i) = T_{i-1}$.

case 2: $n$ is even.

Define $f : V(P_n) \rightarrow \{0, 1, 2, ..., T_{n-1}\}$ by

$$f(v_i) = \begin{cases} 
T_i - 1 & \text{if } i \text{ is odd} \\
T_{i-1} & \text{if } i \text{ is even}.
\end{cases}$$

Clearly $f$ is injective and for $2 \leq i \leq n$, $f^*(v_{i-1}v_i) = T_{i-1}$.

In both cases, $f^*(E(P_n)) = \{T_1, T_2, ..., T_{n-1}\}$. Hence $P_n$ is triangular mean.

For example, triangular mean labelings of $P_7$ and $P_8$ are shown in Figure 5.20.

**Theorem 5.3.3.** Combs are triangular mean graphs.

**Proof.** A graph obtained by joining a single pendant edge to each
vertex of a path is called a Comb. Let $G$ be a Comb obtained from the path $P_n = v_1v_2v_3...v_n$ by joining a vertex $u_i$ to $v_i$, $1 \leq i \leq n$. We observe that $G$ has $2n$ vertices and $2n - 1$ edges. Define $f : V(G) \to \{0, 1, 2, ..., T_{2n-1}\}$ by $f(v_i) = T_{2i-1} - 1$, $1 \leq i \leq n$ and $f(u_i) = T_{2i-1}$. It is easy to very that $f$ is injective, $f^*(v_{i-1}v_i) = T_{2i-2}$, $2 \leq i \leq n$ and $f^*(v_iu_i) = T_{2i-1}, 1 \leq i \leq n$. Hence $f(E(G)) = \{T_1, T_2, ..., T_n\}$. Hence Combs are triangular mean.

For example, a triangular mean labelings of a comb for $n = 7$ is shown Figure 5.21.

**Theorem 5.3.4.** The graph $mP_n$ is triangular mean.

*Proof.* The graph $mP_n$ consists of $m$ copies of the path $P_n$. It is of order $mn$ and size $m(n - 1)$. Let $u_{ij}$ denotes the $j^{th}$ vertex of the
\(^{i}\)th copy of \(mP_n\). Define \(N_{ij} = T_{(i-1)(n-1)+j}\).

**Case 1:** \(n\) is even.

Define \(f : V(mP_n) \rightarrow \{0, 1, 2, ..., T_{m(n-1)}\}\) by

\[
    f(u_{ij}) = \begin{cases} 
        N_{ij} - 1 & \text{if } j \text{ is odd} \\
        N_{ij-1} & \text{if } j \text{ is even}
    \end{cases}
\]

**Case 2:** \(n\) is odd.

Define \(f : V(mP_n) \rightarrow \{0, 1, 2, ..., T_{m(n-1)}\}\) by

\[
    f(u_{ij}) = \begin{cases} 
        N_{ij} - 1 & \text{if } j \text{ is odd and } j \neq n \\
        N_{ij-1} & \text{if } j \text{ is even} \\
        N_{(i+1)1} - 2 & \text{if } j = n
    \end{cases}
\]

Clearly, in both cases \(f\) is injective and \(f^*(u_{ij}u_{ij}) = N_{ij-1}, 1 \leq i \leq m, 2 \leq j \leq n\). Also, \(f^*(E(mP_n)) = \{N_{ij-1} : 1 \leq i \leq m, 2 \leq j \leq n \}\)

\[
= \{T_{(i-1)(n-1)+j-1} : 1 \leq i \leq m, 2 \leq j \leq n \} \\
= \{T_1, T_2, ..., T_{m(n-1)}\}
\]

Thus, \(mP_n\) is triangular mean. \(\square\)

For example, a triangular mean labeling of 5\(P_4\) is shown in Figure 5.22.

![Figure 5.22: A Triangular mean labeling of 5\(P_4\)](image-url)
Theorem 5.3.5. A graph $G$ in which every edge lies on a triangle is not triangular mean.

Proof. Let $G$ be a triangular mean graph in which every edge lies on a triangle. To get $T_q$ as induced edge label, there must be two adjacent vertices $u$ and $v$ such that $f(u) = T_q - 1$ and $f(v) = T_q$. Let $uvw$ be a triangle in $G$. Then $|f^*(uw) - f^*(vw)| \leq 1$. This is a contradiction to the fact that the difference between any two triangular numbers is at least two. Thus $G$ is not triangular mean. \hfill \Box

Corollary 5.3.6. The complete graph $K_n$ where $n \geq 3$, the wheel $W_n$ and the triangular cactus are not triangular mean.

Theorem 5.3.7. If $n \geq 3$, $K_{1,n}$ is not triangular mean.

Proof. Let $n \geq 3$. Suppose that $K_{1,n}$ is triangular mean. Let $u$ be the central vertex and $v_1, v_2, ..., v_n$ be the other vertices of $K_{1,n}$ adjacent to $u$. To get $T_n$ as induced edge label, we must have $T_n - 1$ and $T_n$ as the labels of adjacent vertices. Therefore $f(u) = T_n - 1$ or $f(u) = T_n$. In either case no edge has the label less than $T_k$, where $k$ is the largest positive integer such that $T_k \leq \lceil \frac{T_n - 1}{2} \rceil$. As $n \geq 3$, $T_1$ cannot be an edge label. This contradiction proves the theorem. \hfill \Box

Theorem 5.3.8. If $n \geq 2$, the graph $G = K_2 \odot K_n^C$ is not triangular mean.
Proof. Let $V(K_2) = \{u,v\}$ and $u_i, 1 \leq i \leq n; v_i, 1 \leq i \leq n$ be the vertices adjacent to $u$ and $v$ respectively. We observe that $G$ is of order $2n + 2$ and size $2n + 1$. Suppose $G$ is triangular mean. Then $T_{2n+1}$ must be an induced edge label. Since every edge of $G$ is incident at $u$ or $v$, one of these two vertices must receives $T_{2n+1}$ or $T_{2n+1} - 1$ as its vertex label. With out loss of generality, let it be $u$. In both cases, where $u$ receives either $T_{2n+1}$ or $T_{2n+1} - 1$, all the $n + 1$ edges incident at $u$ have induced edge labels greater than or equal to $T_k$, where $k$ is the smallest positive integer such that $T_k \geq \left\lceil \frac{T_{2n+1} - 1}{2} \right\rceil$. But there are at most $n$ triangular numbers in the interval $[T_k, T_{2n+1}]$. This contradiction proves the theorem. \[\square\]