

CHAPTER II

β*-CLOSED SETS IN TOPOLOGICAL SPACES

2.1 Introduction

For the first time the concept of generalized closed sets was considered by Levine in 1970 [30]. Since then several Topologists (14, 47,62) have contributed to the development of generalizations of closed sets in topological spaces. After the works of Levine on semi-open sets, various Mathematicians turned their attention to the generalizations of topology by considering semi-open sets instead of open sets. In 2002, M. Sheik John [51] introduced a class of sets known as ω-closed sets (also called ݃-closed sets) which is properly placed between the class of semi-closed sets and the class of generalized closed sets. The complement of an ω-closed set is called an ω-open set.

The concept of semi-preopen sets was defined by Andrijevic in 1986 [2] and are also known under the name β sets. In 1994, Julian Dontchev [14] introduced the notion of generalized semi-preopen sets (briefly gsp-open sets) via the concept of semi-preopen sets.

In this chapter by using ω-open sets, a new class of sets in topological spaces namely β*-closed set is introduced. It is explored that this class of sets is properly placed between the class of semi-closed sets and the class of gsp-closed sets. The complement of β*-closed set is called β*-open set. We prove that the class of β*-open sets form a topology under the condition that they are closed under finite intersection and finite union.
As an application of $\beta^*$-closed sets, five new spaces namely $T_{\beta^*}$, $\alpha T_{\beta^*}$, $pT_{\beta^*}$, $spT_{\beta^*}$, $sT_{\beta^*}$ are studied. Using these spaces some new characterizations have been obtained.

***************

2.2 $\beta^*$-closed sets and their basic properties

This section contains a new class of sets called $\beta^*$-closed sets and certain basic properties of them in topological spaces.

**Definition 2.2.1** A subset $A$ of $X$ is called a $\beta^*$-closed set if $spcl(A) \subseteq \text{int}(U)$ whenever $A \subseteq U$ and $U$ is $\omega$-open in $X$.

The class of all $\beta^*$-closed sets in $X$ is denoted by $\beta^*C(\tau)$. That is $\beta^*C(\tau) = \{A \subseteq X: A$ is $\beta^*$-closed in $X\}$.

First we prove that the class of $\beta^*$-closed sets properly lies between the class of semi-closed sets and the class of gsp-closed sets.

We shall start with a lemma.

**Lemma 2.2.2 [51]** A set $U$ is $\omega$-open if and only if $F \subseteq \text{int}(U)$ whenever $F$ is semi closed and $F \subseteq U$.

**Proposition 2.2.3** Every semi-closed set and hence $\alpha$-closed set and closed set are $\beta^*$-closed sets but not conversely.

**Proof:** Let $A \subseteq U$ be any semi-closed set and $U$ be $\omega$-open. Therefore by lemma 2.2.2, $A \subseteq \text{int}(U)$. Since $A$ is semi-closed, $A = scl(A) \subseteq \text{int}(U)$. Therefore $spcl(A) \subseteq scl(A) \subseteq \text{int}(U)$. Hence $A$ is $\beta^*$-closed.
Since every closed sets and $\alpha$-closed sets are semi-closed sets, it follows that they are $\beta^*$-closed sets. The converse is shown using the following example.

**Example 2.2.4** Let $X=\{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$ then the set $\{a, b\}$ is a $\beta^*$-closed set but none of semi-closed set, $\alpha$-closed set and closed set.

**Proposition 2.2.5** Every $\beta^*$-closed set is gsp-closed set but not conversely.

**Proof:** Let $A \subseteq \beta^*C(\tau)$ and $U$ be any open set such that $A \subseteq U$. Since every open set is $\omega$-open, we have $\text{spcl}(A) \subseteq \text{int}(U) = U$. Therefore $A$ is gsp-closed.

**Example 2.2.6** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. Then the set $\{a\}$ is gsp closed but not $\beta^*$-closed.

Thus the class of $\beta^*$-closed sets properly lies between the class of semi closed sets and the class of gsp-closed sets.

**Proposition 2.2.7** $\beta^*$-closed sets are independent of semi-preclosed sets and pre-semi closed sets.

**Example 2.2.8** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. Then the set $\{a, b, d\}$ is neither pre-semicolon closed nor semi-preclosed but it is $\beta^*$-closed. Also in the same topological space $X$, we see that the set $\{a\}$ is semi-pre closed and hence pre-semicolon closed but not $\beta^*$-closed.

**Proposition 2.2.9** Every $^*g$-closed set and hence $g^*$-closed set are $\beta^*$-closed but not conversely.

**Proof:** Let $A \subseteq U$ be a $^*g$-closed set and $U$ be an $\omega$-open set. Then $\text{cl}(A) \subseteq U$. Also $\text{cl}(A)$ is a closed set and hence a semi closed set. Hence by using
Lemma 2.2.2, $\text{cl}(A) \subseteq \text{int}(U)$. Therefore $\text{spcl}(A) \subseteq \text{cl}(A) \subseteq \text{int}(U)$. Hence $A$ is a $\beta^*$ closed set. Since every $g^*$-closed set is a $^*g$-closed set, it is $\beta^*$-closed.

**Example 2.2.10** Let $X = \{a, b, c\}$ and $\tau = \emptyset, \{a\}, \{b\}, \{a, b\}, X$. Then the set $\{b\}$ is $\beta^*$-closed but neither $g^*$-closed nor $^*g$-closed in $X$.

**Proposition 2.2.11** Every $g^\#s$-closed set and hence $g^\#$-closed sets are $\beta^*$-closed sets but not conversely.

**Proof:** Let $A \subseteq U$ be a $g^\#s$ closed set and $U$ be $\omega$-open. Since every $\omega$-open set is $\omega g$-open by Lemma 1.1.7 and $\text{scl}(A)$ is a semi closed set, we get the proof as in Proposition 2.2.9.

Since every $g^\#$-closed set is $g^\#s$-closed set, $g^\#$-closed set is also $\beta^*$-closed.

**Example 2.2.12** Let $X = \{a, b, c, d\}$ and $\tau = \emptyset, \{a, b\}, X$. Then the set $\{a, b, d\}$ is $\beta^*$ closed but neither $g^\#$-closed nor $g^\#s$-closed.

**Proposition 2.2.13** Every $\beta^*$-closed set is $\hat{\eta}^*$-closed set but not conversely.

**Proof:** Let $A \subseteq \beta^*C(\tau)$ and $U$ be an $\omega$-open set containing $A$. Then $\text{spcl}(A) \subseteq \text{int}(U) \subseteq U$ and hence $A$ is $\hat{\eta}^*$-closed.

**Example 2.2.14** Let $X$ and $\tau$ be defined as in Example 2.2.6. Then the set $\{a\}$ is $\hat{\eta}^*$-closed but not $\beta^*$-closed.

**Proposition 2.2.15** $\beta^*$-closed sets are independent of $g$-closed sets, $sg$-closed sets and $gs$-closed sets.

**Example 2.2.16** Let $X = \{a, b, c\}$ and $\tau = \emptyset, \{a\}, \{b, c\}, X$. Then the set $\{a, c\}$ is $g$-closed, $gs$ closed and $sg$-closed but not $\beta^*$-closed.

**Example 2.2.17** Let $X = \{a, b, c, d, e\}$, and $\tau = \emptyset, \{a, b\}, \{a, b, d\}, \{a, b, c, d\}, \{a, b, d, e\}, X$. Then the set $\{a, c\}$ is $\beta^*$-closed set but none of $g$-closed set, $sg$ closed set and $gs$ closed set.
\textbf{Proposition 2.2.18} \ $\beta^*$-closed sets are independent of \(rg\)-closed sets, \(gp\)-closed sets, \(gpr\)-closed sets and \(\omega\)-closed sets.

\textbf{Example 2.2.19} \ Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\). Then the set \(\{b\}\) is \(\beta^*\)-closed but none of \(gpr\)-closed, \(gp\)-closed, \(rg\)-closed and \(\omega\)-closed.

\textbf{Example 2.2.20} \ Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, \{b, c\}, X\}\). Then the set \(\{b\}\) is \(\omega\)-closed, \(rg\)-closed, \(gp\)-closed and \(gpr\)-closed in \(X\) but not \(\beta^*\)-closed in \(X\).

\textbf{Proposition 2.2.21} \ $\beta^*$-closed sets are independent of \(g^\#p\)-closed sets.

\textbf{Example 2.2.22} \ Let \(X\) be defined as in example 2.2.12. Then the set \(\{a\}\) is \(g^\#p\)-closed but not \(\beta^*\)-closed in \(X\) and \(\{a, b, c\}\) is \(\beta^*\)-closed but not \(g^\#p\)-closed in \(X\).

\textbf{Remark 2.2.23} \ From the above discussions and known results we have the following implications

\[\begin{array}{ccc}
\text{Closed} & \downarrow & \text{\(g^\#\) closed set} \\
\downarrow & & \downarrow \\
\text{\(a\) - closed} & & \text{\(g^\#s\) closed set} \\
\downarrow & & \downarrow \\
\text{Semi closed set} & & \text{\(\beta^*\) closed} \\
& & \downarrow \\
\text{\(g^*\)-closed} & & \text{\(g^{sp}\) closed} \\
& & \downarrow \\
\text{*g closed} & & \text{\(\eta^*\)-closed}
\end{array}\]

Fig. (i)
Remark 2.2.24  The union and intersection of any two $\beta^*$-closed sets are not $\beta^*$-closed.

Example 2.2.25  Let $X=\{a, b, c, d\}$ and $\tau = \emptyset, \{a\}, X$. Then the sets $A=\{a, b\}$ and $B=\{a, c\}$ are $\beta^*$-closed in $X$ but $A \cap B = \{a\}$ is not $\beta^*$-closed in $X$.

Example 2.2.26  Let $X=\{a, b, c\}$ and $\tau = \emptyset, \{a\}, \{b\}, \{a, b\}, X$. Then the sets $A=\{a\}$ and $B=\{b\}$ are $\beta^*$-closed but $A \cup B = \{a, b\}$ is not $\beta^*$-closed.

Definition 2.2.27  Let $X$ be a topological space and $A \subseteq X$ and $x \in X$. Then $x$ is said to be a semi-prelimit point of $A$ if every semi-preopen set containing $x$ contains a point of $A$ different from $x$.

Definition 2.2.28  Let $X$ be a topological space and $A \subseteq X$. The set of all semi-pre limit point of $A$ is said to be semi-pre derived set of $A$ and is denoted by $D_{sp}[A]$. 

Fig (ii)
**Theorem 2.2.29**  If \( D[A] \subseteq D_{sp}[A] \) for each subset \( A \) of a space \( X \), then the union of two \( \beta^* \)-closed set is \( \beta^* \)-closed.

**Proof:**  Let \( A \) and \( B \) be \( \beta^* \) closed subsets of \( X \) and \( U \) be an \( \omega \)-open set with \( A \cup B \subseteq U \). Then \( spcl(A) \subseteq int(U) \) and \( spcl(B) \subseteq int(U) \). Since for each subset of \( X \), we have \( D_{sp}[A] \subseteq D[A] \), we get \( cl(A) = spcl(A) \) and \( cl(B) = spcl(B) \). Therefore \( cl(A \cup B) = cl(A) \cup cl(B) = spcl(A) \cup spcl(B) \subseteq int(U) \). But \( spcl(A \cup B) \subseteq cl(A \cup B) \). So \( spcl(A \cup B) \subseteq int(U) \) and hence \( A \cup B \) is \( \beta^* \)-closed.

**Proposition 2.2.30**  Let \( A \) be a \( \beta^* \)-closed set in \( X \). Then \( spcl(A)-A \) does not contains any non empty \( \omega \)-closed set. But the converse is not true.

**Proof:**  Suppose that \( A \) is \( \beta^* \)-closed and \( F \) is \( \omega \)-closed with \( F \subseteq spcl(A)-A \). Then \( A \subseteq F^c \) and so \( spcl(A) \subseteq int(F^c) \subseteq F^c \). Therefore \( F \subseteq (spcl(A))^c \) which implies \( F = \emptyset \).

**Example 2.2.31**  Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\} \). Then the set \( A = \{a, b\} \) is not \( \beta^* \) closed but \( spcl(A)-A = \emptyset \) which contains no \( \omega \)-closed set.

**Proposition 2.2.32**  Let \( A \) and \( B \) be any two subsets of a space \( X \). If \( A \) is \( \beta^* \)-closed such that \( A \subseteq B \subseteq spcl(A) \), then \( B \) is \( \beta^* \)-closed.

**Proof:**  Let \( U \) be an \( \omega \)-open set of \( X \) such that \( B \subseteq U \). Then \( A \subseteq U \) and \( A \) is \( \beta^* \) closed implies \( spcl(A) \subseteq int(U) \). Therefore \( spcl(B) \subseteq spcl(spcl(A)) = spcl(A) \subseteq int(U) \) using lemma 1.1.6. Thus \( B \) is \( \beta^* \)-closed.

**Proposition 2.2.33**  If a set \( A \) of \( X \) is \( \omega \)-open and \( \beta^* \)-closed then \( A \) is semi- preclosed in \( X \).

**Proof:**  Let \( A \) be \( \omega \)-open and \( \beta^* \)-closed. Then since \( A \subseteq A \), \( spcl(A) \subseteq int(A) \subseteq A \). But \( A \subseteq spcl(A) \). Hence \( A = spcl(A) \). So \( A \) is semi-preclosed.
Theorem 2.2.34  Every open and semi-preclosed subset of X is β*-closed but not conversely.

Proof: Let A be an open and semi-preclosed subset of X and A ⊆ U and U be ω-open in X. Therefore spcl(A) = A = int (A) ⊆ int(U). Hence A is β*-closed.

Example 2.2.35 Let X = {a, b, c, d} and τ = {∅, {a}, X}. Then the set A = {a, c} is β*-closed but neither open nor semi-preclosed.

Theorem 2.2.36 In a T₁-space X, β*-closed sets are semi-preclosed.

Proof: Let A be a β*-closed set in a T₁ space X. Let x ∈ spcl(A)-A. Since X is a T₁ space {x} is a closed set in X. But by Proposition 2.2.30 spcl(A)-A contains no nonempty ω-closed sets and hence closed sets. Therefore {x} = ∅. Hence spcl(A)-A = ∅ which implies A is semi-preclosed.

Theorem 2.2.37 Let A be an open subset of a topological space X. Then the following are true.

(a) If A is ω-closed then A is β*-closed.
(b) If A is gsp-closed then A is β*-closed.
(c) If A is gp-closed then A is β*-closed.
(d) If A is regular closed then A is β*-closed.
(e) If A is β*-closed then A is regular open.

Proof: (a) By lemma 1.1.8, A is semi-preclosed. Therefore by Theorem 2.2.34, A is β*-closed.

(b) By lemma 1.1.9, A is semi-preclosed. Therefore by theorem 2.2.34, A is β*-closed.

(c) Let A ⊆ U be a gp-closed set and U be an ω-open. Therefore A = int(A) ⊆ int(U) and int(U) is open. Hence pcl(A) ⊆ int(U) which implies spcl(A) ⊆ pcl(A) ⊆ int(U) and therefore A is β*-closed.
(d) Since \( A = \text{cl}(\text{int}(A)) \), \( \text{int}(\text{cl}(\text{int}(A))) \subseteq \text{cl}(\text{int}(A)) \subseteq A \). Therefore \( A \) is semi-preclosed. Hence by Theorem 2.2.34, \( A \) is \( \beta^* \)-closed.

(e) Since every \( \beta^* \)-closed set is gsp-closed set by Proposition 2.2.5, \( A \) is regular open using lemma 1.1.14.

**Theorem 2.2.38** Let \( X \) be a topological space. Let \( F \subseteq A \subseteq X \) and \( A \) be open in \( X \).

(i) If \( F \) is \( \beta^* \)-closed in \( A \) and \( A \) is closed in \( X \), then \( F \) is \( \beta^* \)-closed in \( X \).

(ii) If \( F \) is \( \beta^* \)-closed in \( X \), then \( F \) is \( \beta^* \)-closed in \( A \).

**Proof:** (a) Let \( U \) be an \( \omega \)-open set in \( X \) such that \( F \subseteq U \). Then \( F \subseteq U \cap A \) and \( U \cap A \) is \( \omega \)-open in \( X \), by lemma 1.1.11. Since \( U \cap A \subseteq A \subseteq X \), \( U \cap A \) is \( \omega \)-open in \( X \) and \( A \) is closed in \( X \), \( U \cap A \) is \( \omega \)-open in \( X \) by lemma 1.1.10. Given \( F \) is \( \beta^* \)-closed relative to \( A \). Hence \( \text{spcl}_A(F) \subseteq \text{int}(U \cap A) \). Since \( \text{spcl}_A(F) = \text{spcl}(F) \cap A \), we have \( \text{spcl}(F) \cap A \subseteq \text{int}(U \cap A) \). Also \( A \) is closed and hence semi pre closed. Thus \( \text{spcl}(A) = A \). Since \( \text{spcl}(F) \subseteq \text{spcl}(A) = A \) we get \( \text{spcl}(F) \cap A = \text{spcl}(F) \). Hence \( \text{spcl}(F) \subseteq \text{int}(U \cap A) \subseteq \text{int}(U) \cap \text{int}(A) \subseteq \text{int}(U) \). Hence \( F \) is \( \beta^* \)-closed relative to \( X \).

(b) Let \( F \subseteq V \), \( V \) be \( \omega \)-open in \( A \) and \( A \) be open in \( X \). Then by lemma 1.1.12, \( V \) is \( \omega \)-open in \( X \). Hence by hypothesis \( \text{spcl}(F) \subseteq \text{int}(V) \). Thus \( \text{spcl}(F) \cap A \subseteq \text{int}(V) \cap A \) which implies \( \text{spcl}_A(F) \subseteq \text{int}(V) \). Hence \( F \) is \( \beta^* \)-closed relative to \( A \).

**Proposition 2.2.39** For each \( x \in X \), either \( \{x\} \) is \( \omega \)-closed or \( \{x\}^c \) is \( \beta^* \)-closed in \( X \).

**Proof:** Suppose that \( \{x\} \) is not \( \omega \)-closed in \( X \). Then \( \{x\}^c \) is not \( \omega \)-open and the only \( \omega \)-open set containing \( \{x\}^c \) is the space itself. Also \( \text{spcl}(\{x\}^c) \subseteq X = \text{int} X \) and so \( \{x\}^c \) is \( \beta^* \)-closed.
**Theorem 2.2.40**  Let A be $\beta^*$-closed in X. Then A is semi-preclosed if and only if $spcl(A)-A$ is $\omega$-closed.

**Proof:** Let A be a semi-preclosed set. Then $spcl(A) = A$ and so $spcl(A)-A = \emptyset$ which is $\omega$-closed. Conversely let $spcl(A)-A$ be $\omega$-closed. Since A is $\beta^*$-closed, by Theorem 2.2.29 $spcl(A)-A$ contains no nonempty $\omega$-closed set. Hence $spcl(A)-A = \emptyset$ which implies $spcl(A) = A$ and so A is semi-preclosed.

**Definition 2.2.41** A space X is called a sp$\omega$-space if the intersection of every semi-preclosed set of X with an $\omega$-closed set of X is $\omega$-closed.

**Example 2.2.42** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then $(X, \tau)$ is a sp$\omega$-space.

**Example 2.2.43** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. The set $\{a\}$ and $\{b\}$ are semi-preclosed and $\{a, b\}$ is $\omega$-closed. Then $\{a\} \cap \{a, b\} = \{a\}$ and $\{b\} \cap \{a, b\} = \{b\}$ are not $\omega$-closed. Therefore X is not a sp$\omega$-space.

**Theorem 2.2.44** For an open subset A of a sp$\omega$-space X the following are equivalent

1. A is $\beta^*$-closed.
2. $\omega cl(\{x\}) \cap A \neq \emptyset$ for each $x \in spcl(A)$.
3. $spcl(A)-A$ contains no nonempty $\omega$-closed set.

**Proof:** (i)$\Rightarrow$(ii) Let $A \subseteq X$ be $\beta^*$-closed and let $x \in spcl(A)$ and $\omega cl(\{x\}) \cap A = \emptyset$. Then $A \subseteq (\omega cl(\{x\}))^c$ and $(\omega cl(\{x\}))^c$ is $\omega$-open. By assumption, $spcl(A) \subseteq int((\omega cl(\{x\}))^c) \subseteq (\omega cl(\{x\}))^c$ which is a contradiction to $x \in spcl(A)$.
(ii)⇒(iii) Let $F \subseteq \text{spcl}(A) - A$ be $\omega$-closed. If there is a $x \in F$, then $x \in \text{spcl}(A)$ and so by assumption $\emptyset \neq \omega \text{cl}(\{x\}) \cap A \subseteq F \cap A \subseteq (\text{spcl}(A) - A) \cap A = \emptyset$, a contradiction. Therefore $F = \emptyset$.

(iii)⇒(i) Let $\text{spcl}(A) - A$ contains no non empty $\omega$-closed set. Let $A \subseteq U$ and $U$ be $\omega$-open in $X$. If $\text{spcl}(A)$ is not contained in $\text{int}(U)$, then $\text{spcl}(A) \cap (\text{int} U)^c \neq \emptyset$. Since the space is a sp$\omega$-space, $\text{spcl}(A) \cap (\text{int} U)^c$ is an $\omega$-closed subset of $\text{spcl}(A) - A$ which is non empty. This is a contradiction. Therefore $A$ is $\beta^*$-closed.

*************

2.3 $\beta^*$-closure

In this section, we define $\beta^*$-closure of a set and we prove that $\beta^*$-closure is a Kuratowski Closure Operator on $X$ under certain condition.

**Definition 2.3.1** For every set $E \subseteq X$, we define the $\beta^*$-closure of $E$ to be the intersection of all $\beta^*$-closed sets containing $E$. In symbols, $\beta^*\text{cl}(E) = \cap \{A : E \subseteq A, A \in \beta^*\text{c}(\tau)\}$

**Lemma 2.3.2** For any $E \subseteq X$, $E \subseteq \beta^*\text{cl}(E) \subseteq \text{cl}(E)$.

**Proof:** Follows from Proposition 2.2.3.

**Lemma 2.3.3** If $A \subseteq B$ then $\beta^*\text{cl}(A) \subseteq \beta^*\text{cl}(B)$.

**Proof:** follows from Definition 2.3.1.

**Remark 2.3.4** $\beta^*$-closure of a set need not be $\beta^*$-closed.

**Example 2.3.5** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. Here $\beta^*\text{cl}(\{a\}) = \{a\}$ is not $\beta^*$-closed.
Lemma 2.3.6  If E is β*-closed then β*cl(E)=E but not conversely.

Proof: From definition 2.3.1, the proof follows. In Example 2.3.5 β*cl({a}) = {a} but {a} is not β*-closed.

Theorem 2.3.7 If β*cl(τ) is closed under finite union and intersection, then β*-closure is a Kuratowski closure operator on X.

Proof: (i) Since ∅ and X are β*-closed, by lemma 2.3.6, we get,

β*cl(∅) = ∅, β*cl(X) = X.

(ii) E ⊆ β*cl(E) by definition 2.3.1.

(iii) Suppose E and F are two subsets of X, then by lemma 2.3.3 we get

β*cl(E) ⊆ β*cl(E ∪ F) and β*cl(F) ⊆ β*cl(E ∪ F). Hence

β*cl(E) ∪ β*cl(F) ⊆ β*cl(E ∪ F). Conversely if x ∉ β*cl(E) ∪ β*cl(F), then there exist A, B belong to β*cl(τ) such that E ⊆ A, x ∉ A and F ⊆ B, x ∉ B. Hence E ∪ F ⊆ A ∪ B and x ∉ A ∪ B. By hypothesis A ∪ B is β*-closed. Thus x ∉ β*cl(E ∪ F). Hence β*cl(E ∪ F) ⊆ β*cl(E) ∪ β*cl(F). Therefore

β*cl(E ∪ F) = β*cl(E) ∪ β*cl(F).

(iv) Let E be a subset of X and A be a β*-closed set containing E. Then by Definition 2.3.1, β*cl(E) ⊆ A and β*cl(β*cl(E)) ⊆ A. Since β*cl(β*cl(E)) ⊆ A we have, by assumption β*cl(β*cl(E)) ⊆ ∩ {A: E ⊆ A, A ∈ β*C(τ)} = β*cl(E). By lemma 2.3.2, β*cl(E) ⊆ β*cl(β*cl(E)) and therefore β*cl(E) = β*cl(β*cl(E)). Hence β*-closure is a Kuratowski closure operator on X.

Definition 2.3.8  Let τ_β = the topology on X generated by β*-closure in the usual manner. That is τ_β = {U: β*cl(U^c) = U^c}. 
Proposition 2.3.9 If $\beta^*c(\tau)$ is closed under finite union and intersection, then $\tau_{\beta^*}$ is a topology on $X$.

Proof: By theorem 2.3.7, $\beta^*$-closure satisfies the Kuratowski axioms. Hence $\tau_{\beta^*}$ is a topology on $X$.

***************

2.4 $\beta^*$-open sets

In this section, first we define $\beta^*$-open sets and $\beta^*$-interior in topological spaces and then obtain certain characterizations of these sets.

J.Antony Rex Rodrigo[3] has studied the topological properties of $\hat{\eta}^*$-derived, $\hat{\eta}^*$-border, $\hat{\eta}^*$-frontier and $\hat{\eta}^*$-exterior of a set using the concept of $\hat{\eta}^*$-open following M.Caldas, S.Jafari and T.Noiri. By the same technique the concept of $\beta^*$-derived, $\beta^*$-border, $\beta^*$-frontier and $\beta^*$-exterior of a set using the concept of $\beta^*$-open sets are introduced.

Definition 2.4.1 A subset $A$ in $X$ is called $\beta^*$-open in $X$ if $A^c$ is $\beta^*$-closed in $X$. We denote the family of all $\beta^*$-open sets in $X$ by $\beta^*O(\tau)$.

The following propositions are the analogue of Propositions 2.2.3, 2.2.5, 2.2.9, 2.2.11, 2.2.13.

Proposition 2.4.2 Every open (resp. $\alpha$-open, semi-open) set is $\beta^*$-open.

Proposition 2.4.3 Every $\beta^*$-open set is gsp-open.

Proposition 2.4.4 Every $*g$-open set (resp. $g^*$-open set) is $\beta^*$ open.

Proposition 2.4.5 Every $g#^*$-open set (resp. $g^#$-open set) is $\beta^*$ open.
**Proposition 2.4.6**  Every $\beta^*$-open set is $\check{\beta}^*$-open.

**Remark 2.4.7**  The union (intersection) of any two $\beta^*$-open set is not $\beta^*$-open.

**Theorem 2.4.8**  A subset $A$ of a topological space $X$ is $\beta^*$-open if and only if $\text{cl}(F) \subseteq \text{spint}(A)$ whenever $A \supseteq F$ and $F$ is $\omega$-closed in $X$.

**Proof:**  Suppose $A$ is $\beta^*$-open in $X$ and $A \supseteq F$ where $F$ is $\omega$-closed in $X$. Then $A^c \subseteq F^c$ where $F^c$ is $\omega$-open in $X$. Hence, we get $\text{spcl}(A^c) \subseteq \text{int}(F^c)$ implies $(\text{spint}(A))^c \subseteq (\text{cl}(F))^c$. Thus we have $\text{cl}(F) \subseteq \text{spint}(A)$.

Conversely, suppose that $A^c \subseteq U$ and $U$ is $\omega$-open in $X$. Then $A \supseteq U^c$ and $U^c$ is $\omega$-closed. By hypothesis $\text{spint}(A) \supseteq \text{cl}(U^c) = (\text{int}(U))^c$. Therefore $(\text{spint}(A))^c \subseteq \text{int}(U)$ which implies $\text{spcl}(A^c) \subseteq \text{int}(U)$. Hence $A^c$ is $\beta^*$-closed.

**Proposition 2.4.9**  If $\text{spint}(A) \subseteq B \subseteq A$ and if $A$ is $\beta^*$-open, then $B$ is $\beta^*$-open.

**Proof:**  Suppose $\text{spint}(A) \subseteq B \subseteq A$ and $A$ is $\beta^*$-open, $A^c \subseteq B^c \subseteq \text{spcl}(A^c)$ and since $A^c$ is $\beta^*$closed by Proposition 2.2.32, $B^c$ is $\beta^*$-closed. Hence $B$ is $\beta^*$-open.

**Proposition 2.4.10**  Let $A \subseteq Y \subseteq X$ where $A$ is $\beta^*$-open relative to $X$ and $Y$ is open in $X$. Then $A$ is $\beta^*$-open relative to $Y$.

**Proof:**  Let $A^c \subseteq V$ where $V$ is $\omega$-open in $Y$. By hypothesis and by lemma 1.1.12, $V$ is $\omega$-open in $X$. Since $A^c$ is $\beta^*$-closed relative to $X$ we get $\text{spcl}(A^c) \subseteq \text{int}(V)$ implies $\text{spcl}(A^c) \cap Y \subseteq \text{int}(V) \cap Y = \text{int}(V)$. Now we obtain $\text{spcl}_{Y}(A^c) \subseteq \text{int}(V)$ which implies $A^c$ is $\beta^*$-closed relative to $Y$. Thus $A$ is $\beta^*$-open relative to $Y$. 

28
**Theorem 2.4.11** The union of an arbitrary collection of pairwise separated \(\beta^*-\)open sets is again \(\beta^*-\)open.

**Proof:** Let \(A\) and \(B\) be any two separated \(\beta^*\) open subset of \(X\). Let \(F\) be an \(\omega\)-closed set such that \(F\subset A \cup B\), since \(A\) and \(B\) are separated sets \(\text{cl}(A) \cap B = A \cap \text{cl}(B) = \emptyset\). Now \(F \cap \text{cl}(A) \subset (A \cup B) \cap \text{cl}(A) \subset A \emptyset = A\). Similarly \(F \cap \text{cl}(B) \subset B\). Also \(\text{cl}(A)\) is closed and hence \(\omega\)-closed [58] and \(F\) is \(\omega\)-closed. Therefore by lemma 1.1.11, \(F \cap \text{cl}(A)\) is \(\omega\)-closed. Therefore by Theorem 2.4.8 \(\text{cl}(F \cap \text{cl}(A)) \subset \text{spint}(B)\) and \(\text{cl}(F \cap \text{cl}(B)) \subset \text{spint}(B)\). Now \(\text{cl}(F) = \text{cl}(F \cap (A \cup B)) = \text{cl}[(F \cap A) \cup (F \cap B)] \subset \text{cl}[(F \cap \text{cl}(A)) \cup (F \cap \text{cl}(B))] \subset \text{cl}(F \cap \text{cl}(A)) \cup \text{cl}(F \cap \text{cl}(B)) \subset \text{spint}(A) \cup \text{spint}(B) \subset \text{spint}(A \cup B)\). Therefore \(A \cup B\) is \(\beta^*\)-open.

**Proposition 2.4.12** If a set \(A\) is \(\beta^*\)-closed, then \(\text{spcl}(A) - A\) is \(\beta^*\)-open.

**Proof:** Suppose \(A\) is \(\beta^*\)-closed. Let \(F \subset \text{spcl}(A) - A\) and \(F\) be \(\omega\)-closed. By Proposition 2.2.29 \(F = \emptyset\). Therefore \(\text{cl}(F) \subset \text{spint}(\text{spcl}(A) - A)\) and by theorem 2.4.8, \(\text{spcl}(A) - A\) is \(\beta^*\)-open.

The converse of Proposition 2.4.12 is not true. It is seen from the following example.

**Example 2.4.13** Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, \{b, c\}, X\}\). Let \(A = \{a, b\}\). Then \(\text{spcl}(A) - A = \emptyset\) which is \(\beta^*\)-open but \(A\) is not \(\beta^*\)-closed.

**Proposition 2.4.14** Let \(A\) be a subset of a topological space \(X\). For any \(x \in X\), \(x \in \beta^*\text{cl}(A)\) if and only if \(U \cap A \neq \emptyset\) for every \(\beta^*\)-open set \(U\) containing \(x\).

**Proof:** **Necessity:** Suppose that \(x \in \beta^*\text{cl}(A)\). Let \(U\) be a \(\beta^*\)-open set containing \(x\) such that \(A \cap U = \emptyset\) and so \(A \subset U^c\). But \(U^c\) is a \(\beta^*\)-closed set and hence \(\beta^*\text{cl}(A) \subset U^c\). Since \(x \notin U^c\) we obtain \(x \notin \beta^*\text{cl}(A)\) which is contrary to the hypothesis.

**Sufficiency:** Suppose that every \(\beta^*\)-open set of \(X\) containing \(x\) intersects \(A\). If \(x \notin \beta^*\text{cl}(A)\), then there exists a \(\beta^*\)-closed set \(F\) of \(X\) such that \(A \subset F\) and
\(x \in F\). Therefore \(x \in F^c\) and \(F^c\) is a \(\beta^*\)-open set. But \(F^c \cap A = \emptyset\). This is contrary to the hypothesis.

**Definition 2.4.15** For any \(A \subseteq X\), \(\beta^*\text{int}(A)\) is defined as the union of all \(\beta^*\)-open set contained in \(A\). That is \(\beta^*\text{int}(A) = \bigcup \{U: U \subseteq A \text{ and } U \in \beta^*\text{O}(\tau)\}\).

**Proposition 2.4.16** For any \(A \subseteq X\), \(\text{int}(A) \subseteq \beta^*\text{int}(A)\).

**Proof:** Follows from Proposition 2.4.2.

**Proposition 2.4.17** For any two subsets \(A_1\) and \(A_2\) of \(X\).

a) If \(A_1 \subseteq A_2\), then \(\beta^*\text{int}(A_1) \subseteq \beta^*\text{int}(A_2)\).

b) \(\beta^*\text{int}(A_1 \cup A_2) \supseteq \beta^*\text{int}(A_1) \cup \beta^*\text{int}(A_2)\).

**Proposition 2.4.18** If \(A\) is \(\beta^*\)-open then \(A = \beta^*\text{int}(A)\).

**Remark 2.4.19** Converse of Proposition 2.4.18 is not true. It is seen by the following example.

**Example 2.4.20** Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, X\}\) then for the set \(A = \{b, c\}\), \(\beta^*\text{int}(A) = A\) but \(\{b, c\}\) is not \(\beta^*\)-open.

**Proposition 2.4.21** Let \(A\) be a subset of a space \(X\). Then the following are true

(i) \((\beta^*\text{int}(A))^c = \beta^*\text{cl}(A^c)\)

(ii) \((\beta^*\text{int}(A)) = (\beta^*\text{cl}(A^c))^c\)

(iii) \(\beta^*\text{cl}(A) = (\beta^*\text{int}(A^c))^c\)

**Proof:** (i) Let \(x \in (\beta^*\text{int}(A))^c\). Then \(x \notin \beta^*\text{int}(A)\). That is every \(\beta^*\)-open set \(U\) containing \(x\) is such that \(U \not\subseteq A\). Thus every \(\beta^*\)-open set \(U\) containing \(x\) is such that \(U \cap A^c \neq \emptyset\). By Proposition 2.4.14, \(x \in \beta^*\text{cl}(A^c)\) and therefore
(β*int(A))\(^c\) ⊆ β*cl(A\(^c\)). Conversely, let x ∈ β*cl(A\(^c\)). Then by Proposition 2.4.14, every β*-open set U containing x is such that U∩A\(^c\) ≠ ∅. By definition 2.4.15, x∉ β*int(A). Hence x∈(β*int(A))\(^c\) and so β*cl(A\(^c\)) ⊆ (β*int(A))\(^c\). Hence (β*int(A))\(^c\) = β*cl(A\(^c\)).

(ii) Follows by taking complements in (i).

(iii) Follows by replacing A by A\(^c\) in (i).

**Proposition 2.4.22** For a subset A of a topological space X, the following conditions are equivalent.

(i) β*O(τ) is closed under any union.

(ii) A is β*-closed if and only if β*cl(A) = A.

(iii) A is β*-open if and only if β*int(A) = A.

**Proof:** (i) ⇒ (ii): Let A be a β* closed set. Then by the definition of β*-closure we get β*cl(A) = A. Conversely, assume β*cl(A) = A. For each x∈ A\(^c\) and x∉ β*cl(A), by Proposition 2.4.14, there exists a β* open set G\(_x\) containing x such that G\(_x\)∩A=∅ and hence x∈G\(_x\)⊆ A\(^c\). Therefore we obtain A\(^c\)=∪\(_{x∈A\(^c\)}\) G\(_x\). By (i) A\(^c\) is β*-open and hence A is β* closed.

(ii)⇒ (iii): Follows by (ii) and Proposition 2.4.21.

(iii)⇒ (i): Let \{U\(_\alpha\) / α ∈ Λ\} be a family of β*-open sets of X. Put U=∪\(_\alpha\) U\(_\alpha\). For each x ∈ U, there exists α(x)∈ Λ such that x∈ U\(_{α(x)}\)⊆ U. Since U\(_{α(x)}\) is β*-open, x∈ β*int(U) and so U=β*int(U). By (iii), U is β*-open. Thus β*O(τ) is closed under any union.

**Proposition 2.4.23** In a topological space X, assume that β*O(τ) is closed under any union. Then β*cl(A) is a β*-closed set for every subset A of X.

**Proof:** Since β*cl(A) = β*cl(β*cl(A)) and by Proposition 2.4.22, β*cl(A) is a β*-closed set.
Definition 2.4.24 For any $A \subseteq X$, $\beta \ker(A)$ is defined as the intersection of all $\beta$-open sets containing $A$. In notation $\beta \ker(A) = \cap \{U/A \subseteq U, U \in \beta \mathcal{O}(\tau)\}$.

Example 2.4.25 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$. Here $\beta \mathcal{O}(\tau) = \mathcal{P}(X) - \{\{b\}, \{b, c\}\}$. Let $A = \{b,c\}$ then $\beta \ker A = X$ and $B = \{a\}$, then $\beta \ker B = \{a\}$.

Definition 2.4.26 A subset $A$ of a topological space $X$ is a $U^*$-set if $A = \beta \ker(A)$.

Example 2.4.27 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, c\}, X\}$. Here $\{a\}, \{c\}, \{a, b\}, \{a, c\}$ are $U^*$-sets. The set $\{b, c\}$ is not a $U^*$-set.

Lemma 2.4.28 For subsets $A, B$ and $A_\alpha(\alpha \in \Lambda)$ of a topological space $X$, the following hold,

(i) $A \subseteq \beta \ker(A)$.

(ii) If $A \subseteq B$, then $\beta \ker(A) \subseteq \beta \ker(B)$.

(iii) $\beta \ker(\beta \ker(A)) = \beta \ker(A)$.

(iv) If $A$ is $\beta$-open then $A = \beta \ker(A)$.

(v) $\beta \ker(\cup \{A_\alpha/\alpha \in \Lambda\} \supseteq \cup \{\beta \ker(A_\alpha)/\alpha \in \Lambda\}$

(vi) $\beta \ ker(\cap \{A_\alpha/\alpha \in \Lambda\}) \subseteq \cap \{\beta \ker(A_\alpha)/\alpha \in \Lambda\}$.

Proof: (i) Clearly follows from Definition 2.4.24.

(ii) Suppose $x \notin \beta \ker(B)$, then there exists a subset $U \in \beta \mathcal{O}(\tau)$ such that $U \supseteq B$ with $x \notin U$. Since $A \subseteq B$, $x \notin \beta \ker(A)$. Thus $\beta \ker(A) \subseteq \beta \ker(B)$.

(iii) Follows from (i) and Definition 2.4.24.

(iv) By definition 2.4.24 and since $A \subseteq \beta \mathcal{O}(\tau)$, we have $\beta \ker(A) \subseteq A$. Using (i) we get $A = \beta \ker(A)$.

(v) For each $\alpha \in \Lambda$, $\beta \ker(A_\alpha) \subseteq \beta \ker(\cup_{\alpha \in \Lambda} A_\alpha)$. Therefore we obtain $\cup_{\alpha \in \Lambda} \beta \ker(A_\alpha) \subseteq \beta \ker(\cup_{\alpha \in \Lambda} A_\alpha)$. 

32
(vi) Suppose that $x \notin \bigcap \{ \beta^* \ker(A_{\alpha})/\alpha \in \Lambda \}$ then there exists an $\alpha_0 \in \Lambda$, such that $x \notin \beta^* \ker(A_{\alpha_0})$ and there exists a $\beta^*$-open set $U$ such that $x \notin U$ and $A_{\alpha_0} \subseteq U$. We have $\bigcap_{\alpha \in \Lambda} A_{\alpha} \subseteq A_{\alpha_0} \subseteq U$ and $x \notin U$. Therefore $x \notin \beta^* \ker\{ \bigcap A_{\alpha}/\alpha \in \Lambda \}$. Hence $\bigcap \{ \beta^* \ker(A_{\alpha})/\alpha \in \Lambda \}=\beta^* \ker(\bigcap \{ A_{\alpha}/\alpha \in \Lambda \})$.

**Remark 2.4.29** In (v) and (vi) of Lemma 2.4.28, the equality does not necessarily hold as shown by the following example.

**Example 2.4.30** Let $X=\{a, b, c, d\}$ and $\tau = \{ \emptyset, \{a\}, X\}$. Let $A = \{b\}$ and $B = \{c, d\}$. Here $\beta^* \ker(A) = \{b\}$ and $\beta^* \ker(B)=\{c, d\}$. $\beta^* \ker(A)U \beta^* \ker B = \{b\} \cup \{c, d\} = \{b, c, d\}$. But $\beta^* \ker(AU B) = \beta^* \ker(\{b, c, d\}) = X$.

Let $X=\{a, b, c\}$ and $\tau = \{ \emptyset, \{a\}, X\}$. Let $P=\{a, b\}$ and $Q=\{b, c\}$. Here $\beta^* \ker(P \cap Q) = \beta^* \ker(\{b\}) = \{b\}$. But $\beta^* \ker(P) \cap \beta^* \ker(Q) = \{a, b\} \cap X = \{a, b\}$.

**Remark 2.4.31** From (iii) of Lemma 2.4.28 it is clear that $\beta^* \ker(A)$ is a U*-set and every open set is a U*-set.

**Lemma 2.4.32** Let $A_{\alpha}(\alpha \in \Lambda)$ be a subset of a topological space $X$. If $A_{\alpha}$ is a U*-set then $\bigcap_{\alpha \in \Lambda} A_{\alpha}$ is a U*-set.

**Proof:** $\beta^* \ker(\bigcap_{\alpha \in \Lambda} A_{\alpha}) \subseteq \bigcap_{\alpha \in \Lambda} \beta^* \ker(A_{\alpha})$, by lemma 2.4.28. Since each $A_{\alpha}$ is a U*-set, we get $\beta^* \ker(\bigcap_{\alpha \in \Lambda} A_{\alpha}) \subseteq (\bigcap_{\alpha \in \Lambda} A_{\alpha})$. Again by (i) of lemma 2.4.28, $(\bigcap_{\alpha \in \Lambda} A_{\alpha}) \subseteq \beta^* \ker(\bigcap_{\alpha \in \Lambda} A_{\alpha})$. Thus $\beta^* \ker(\bigcap_{\alpha \in \Lambda} A_{\alpha}) = \bigcap_{\alpha \in \Lambda} A_{\alpha}$ which implies $\bigcap_{\alpha \in \Lambda} A_{\alpha}$ is a U*-set.

**Definition 2.4.33** A subset $A$ of a topological space $X$ is said to be U*-closed if $A= L \cap F$ where $L$ is a U*-set and $F$ is a closed set of $X$. 

33
Remark 2.4.34  It is clear that every U*-set and closed sets are U*-closed.

Theorem 2.4.35  For a subset A of a topological space X, the following conditions are equivalent.

(i)  A is U*-closed
(ii)  A = L∩cl(A) where L is a U*-set.
(iii) A = β*ker(A) ∩ cl(A).

Proof: (i)⇒(ii): Let A = L∩F where L is a U*-set and F is a closed set. Since A⊂F, cl(A)⊂ F and A⊂L∩cl(A)⊂ L∩F = A. Therefore, we obtain L∩cl(A) = A.

(ii)⇒(iii): Let A=L∩cl(A) where L is a U*-set. Since A⊂L, we have β*ker(A)⊂β*ker(L) = L. Therefore β*ker(A)∩cl(A)⊂ L∩cl(A) = A. Hence A = β*ker(A) ∩ cl(A).

(iii)⇒ (i): Since β*ker(A) is a U*-set, the proof follows.

Definition 2.4.36  Let A be a subset of a space X. A point x ∈ X is said to be a β*limit point of A, if for each β*-open set U containing x, U∩ (A-{x}) ≠∅. The set of all β*-limit point of A is called a β*-derived set of A and is denoted by D_{β*}(A).

Theorem 2.4.37  For subsets A, B of a space X, the following statements hold

a)  D_{β*}(A) ⊆ D(A) where D(A) is the derived set of A.

b)  If A⊂ B, then D_{β*}(A) ⊆ D_{β*}(B).

c)  D_{β*}(A)∪ D_{β*}(B)⊂ D_{β*}(A∪B) and D_{β*}(A∩B)⊂ D_{β*}(A)∩ D_{β*}(B).

d)  D_{β*}(D_{β*}(A))-A⊂ D_{β*}(A).

e)  D_{β*}(A∪D_{β*}(A))⊂ A∪D_{β*}(A).
Proof: (i) Since every open set is $\beta^*$-open, the proof follows.

(ii) Follows from definition 2.4.36.

(iii) Follows by (ii).

(iv) If $x \in D_{\beta^*}(D_{\beta^*}(A)) - A$ and $U$ is a $\beta^*$-open set containing $x$, then $U \cap (D_{\beta^*}(A) - \{x\}) \neq \emptyset$. Let $y \in U \cap (D_{\beta^*}(A) - \{x\})$. Then since $y \in D_{\beta^*}(A)$ and $y \in U$, $U \cap (A - \{y\}) \neq \emptyset$. Let $z \in U \cap (A - \{y\})$. Then $z \neq x$ for $z \in A$ and $x \notin A$. Hence $U \cap (A - \{x\}) \neq \emptyset$. Therefore, $x \in D_{\beta^*}(A)$.

(v) Let $x \in D_{\beta^*}(A \cup D_{\beta^*}(A)) - A$. If $x \in A$, the result is obvious. So let $x \in D_{\beta^*}(A \cup D_{\beta^*}(A)) - A$, then for a $\beta^*$-open set $U$ containing $x$, $U \cap ((A \cup D_{\beta^*}(A)) - \{x\}) \neq \emptyset$. Thus $U \cap (A - \{x\}) \neq \emptyset$ or $U \cap (D_{\beta^*}(A) - \{x\}) \neq \emptyset$. By the same argument in (iv), it follows that $U \cap (A - \{x\}) \neq \emptyset$. Hence $x \in D_{\beta^*}(A)$. Therefore in either case $D_{\beta^*}(A \cup D_{\beta^*}(A)) \subseteq A \cup D_{\beta^*}(A)$.

Remark 2.4.38 In general, the converse of (i) is not true.

Proposition 2.4.39 $D_{\beta^*}(A \cup B) \neq D_{\beta^*}(A) \cup D_{\beta^*}(B)$.

Example 2.4.40 Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. Then $\beta^*O(\tau) = P(X) - \{a\}, \{b\}, \{a, b\}$. Let $A = \{a, b, d\}$ and $B = \{c\}$. Then $D_{\beta^*}(A \cup B) = \{a, b\}$ and $D_{\beta^*}(A) = \emptyset$, $D_{\beta^*}(B) = \emptyset$.

Theorem 2.4.41 For any subset $A$ of a space $X$, $\beta^*cl(A) = A \cup D_{\beta^*}(A)$.

Proof: Since $D_{\beta^*}(A) \subseteq \beta^*cl(A)$, $A \cup D_{\beta^*}(A) \subseteq \beta^*cl(A)$. On the other hand, let $x \in \beta^*cl(A)$. If $x \in A$, then the proof is complete. If $x \notin A$, each $\beta^*$-open set $U$
containing x intersects A at a point distinct from x, so \( x \in D_\beta^*(A) \). Thus \( \beta^*\text{cl}(A) \subseteq D_\beta^*(A) \cup A \) and hence the theorem.

**Definition 2.4.42** Let A be a subset of a space X. Then the \( \beta^* \)-border of A is defined as \( b_\beta^*(A) = A - \beta^*\text{int}(A) \).

**Theorem 2.4.43** For a subset A of a space X, the following statements hold.

1. \( b_\beta^*(A) \subseteq b(A) \) where \( b(A) \) denote the border of A.
2. \( A = \beta^*\text{int}(A) \cup b_\beta^*(A) \).
3. \( \beta^*\text{int}(A) \cap b_\beta^*(A) = \emptyset \).
4. If A is \( \beta^* \)-open then \( b_\beta^*(A) = \emptyset \).
5. \( \beta^*\text{int}(b_\beta^*(A)) = \emptyset \).
6. \( b_\beta^*(b_\beta^*(A)) = b_\beta^*(A) \).
7. \( b_\beta^*(A) = A - \beta^*\text{int}(A) = A - (\beta^*\text{cl}(A^c))^c = A \cap \beta^*\text{cl}(A^c) \).

**Proof:** (i),(ii) and (iii) are obvious from the definitions of \( \beta^* \)-interior of A and \( \beta^* \)-border of A where A is any subset of X.

iv) If A is \( \beta^* \)-open, then \( A = \beta^*\text{int}(A) \). Hence the result follows.

v) If \( x \in \beta^*\text{int}(b_\beta^*(A)) \), then \( x \in b_\beta^*(A) \). Now \( b_\beta^*(A) \subseteq A \) implies \( \beta^*\text{int}(b_\beta^*(A)) \subseteq \beta^*\text{int}(A) \). Hence \( x \in \beta^*\text{int}(A) \) which is a contradiction to \( x \in b_\beta^*(A) \). Thus \( \beta^*\text{int}(b_\beta^*(A)) = \emptyset \).

vi) \( b_\beta^*(b_\beta^*(A)) = b_\beta^*(A - \beta^*\text{int}(A)) = (A - \beta^*\text{int}(A)) - \beta^*\text{int}(A - \beta^*\text{int}(A)) \)

which is \( b_\beta^*(A) - \emptyset \), by (v). Hence \( b_\beta^*(b_\beta^*(A)) = b_\beta^*(A) \).

vii) \( b_\beta^*(A) = A - \beta^*\text{int}(A) = A - (\beta^*\text{cl}(A^c))^c = A \cap \beta^*\text{cl}(A^c) \).

**Remark 2.4.44** In general, the converse of (i) is not true. For example, let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, \{a, b\}, X\} \). Then \( \beta^*\text{O}(\tau) = \{\emptyset, \{a\}, \{b\}, \).
\{a, b\}, \{a, c\}, X\}. Let \(A = \{a, c\}\), then \(b_\beta^*(A) = \{a, c\} - \{a, c\} = \emptyset\) and \(b(A) = \{a, c\} - \{a\} = \{c\}\). Therefore \(b(A) \subset b_\beta^*(A)\).

**Definition 2.4.45** Let \(A\) be a subset of a space \(X\). Then \(\beta^*\)-frontier of \(A\) is defined as \(\text{Fr}_{\beta^*}(A) = \beta^*\text{cl}(A) - \beta^*\text{int}(A)\).

**Theorem 2.4.46** For a subset \(A\) of a space \(X\), the following statements hold

a) \(\text{Fr}_{\beta^*}(A) \subseteq \text{Fr}(A)\), where \(\text{Fr}(A)\) denotes the frontier of \(A\).

b) \(\beta^*\text{cl}(A) = \beta^*\text{int}(A) \cup \text{Fr}_{\beta^*}(A)\)

c) \(\beta^*\text{int}(A) \cap \text{Fr}_{\beta^*}(A) = \emptyset\).

d) \(b_\beta(A) \subset \text{Fr}_{\beta^*}(A)\)

e) \(\text{Fr}_{\beta^*}(A) = b_\beta(A) \cup D_{\beta^*}(A)\)

f) If \(A\) is \(\beta^*\)-open, then \(\text{Fr}_{\beta^*}(A) = D_{\beta^*}(A)\)

g) \(\text{Fr}_{\beta^*}(A) = \beta^*\text{cl}(A) \cap \beta^*\text{cl}(A^c)\)

h) \(\text{Fr}_{\beta^*}(A) = \text{Fr}_{\beta^*}(A^c)\)

i) \(\text{Fr}_{\beta^*}(\beta^*\text{int}(A)) \subset \text{Fr}_{\beta^*}(A)\).

j) \(\text{Fr}_{\beta^*}(\beta^*\text{cl}(A)) \subset \text{Fr}_{\beta^*}(A)\)

**Proof:**

(i) Since every open set is \(\beta^*\)-open we get the proof.

(ii) \(\beta^*\text{int}(A) \cup \text{Fr}_{\beta^*}(A) = \beta^*\text{int}(A) \cup (\beta^*\text{cl}(A) - \beta^*\text{int}(A)) = \beta^*\text{cl}(A)\).

(iii) \(\beta^*\text{int}(A) \cap \text{Fr}_{\beta^*}(A) = \beta^*\text{int}(A) \cap (\beta^*\text{cl}(A) - \beta^*\text{int}(A)) = \emptyset\).

(iv) Obvious from the definition.

(v) Since \(\beta^*\text{int}(A) \cup \text{Fr}_{\beta^*}(A) = \beta^*\text{int}(A) \cup b_\beta(A) \cup D_{\beta^*}(A)\) is obvious from the definition, we get \(\text{Fr}_{\beta^*}(A) = b_\beta(A) \cup D_{\beta^*}(A)\).

(vi) If \(A\) is \(\beta^*\)-open, then \(b_\beta(A) = \emptyset\), then by (v) \(\text{Fr}_{\beta^*}(A) = D_{\beta^*}(A)\).

(vii) \(\text{Fr}_{\beta^*}(A) = \beta^*\text{cl}(A) - \beta^*\text{int}(A) = \beta^*\text{cl}(A) - (\beta^*\text{cl}(A^c))^c = \beta^*\text{cl}(A) \cap \beta^*\text{cl}(A^c)\).
(viii) Follows from (vii).
(ix) Obvious.

(x) Fr_{\beta^*}(\beta^*\text{cl}(A)) = \beta^*\text{cl}(\beta^*\text{cl}(A)) - \beta^*\text{int}(\beta^*\text{cl}(A)) \subseteq \beta^*\text{cl}(A) - \beta^*\text{int}(A) = \text{Fr}_{\beta^*}(A).

In general the converse of (i) of theorem 2.4.46 is not true.

**Example 2.4.47** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then $\beta^*\text{cl}(\tau) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$. Let $A = \{a, b\}$. Then $\beta^*\text{cl}(A) - \beta^*\text{int}(A) = \text{Fr}_{\beta^*}(A) = X - \{a, b\} = \{c\}$. But $\text{cl}(A) - \text{int}(A) = \text{Fr}(A) = X - \{a\} = \{b, c\}$. Therefore $\text{Fr}(A) \subset \text{Fr}_{\beta^*}(A)$.

**Definition 2.4.48** $\beta^*\text{Ext}(A) = \beta^*\text{int}(A^c)$ is said to be the $\beta^*$ exterior of $A$.

**Theorem 2.4.49** For a subset $A$ of a space $X$, the following statements hold

a) $\text{Ext}(A) \subseteq \beta^*\text{Ext}(A)$ where $\text{Ext}(A)$ denote the exterior of $A$.
b) $\beta^*\text{Ext}(A) = \beta^*\text{int}(A^c) = (\beta^*\text{cl}(A))^c$.
c) $\beta^*\text{Ext}(\beta^*\text{Ext}(A)) = \beta^*\text{int}(\beta^*\text{cl}(A))$
d) If $A \subset B$, then $\beta^*\text{Ext}(A) \supseteq \beta^*\text{Ext}(B)$.
e) $\beta^*\text{Ext}(A \cup B) \subset \beta^*\text{Ext}(A) \cup \beta^*\text{Ext}(B)$.
f) $\beta^*\text{Ext}(A \cap B) \supseteq \beta^*\text{Ext}(A) \cap \beta^*\text{Ext}(B)$.
g) $\beta^*\text{Ext}(X) = \emptyset$.
h) $\beta^*\text{Ext}(\emptyset) = X$.
i) $\beta^*\text{int}(A) \subseteq \beta^*\text{Ext}(\beta^*\text{Ext}(A))$.

**Proof:** (i) & (ii) follows from definition 2.4.48.

(iii) $\beta^*\text{Ext}(\beta^*\text{Ext}(A)) = \beta^*\text{Ext}(\beta^*\text{int}(A^c)) = \beta^*\text{Ext}(\beta^*\text{cl}(A))^c = \beta^*\text{int}(\beta^*\text{cl}(A))$.  

38
(iv) If \( A \subseteq B \), then \( A^c \supseteq B^c \). Hence \( \beta^*\text{int}(A^c) \supseteq \beta^*\text{int}(B^c) \) and so \\
\( \beta^*\text{Ext}(A) \supseteq \beta^*\text{Ext}(B) \).

(v) and (vi) follows from (iv).

(vii) and (viii) follows from 2.4.48.

(ix) \( \beta^*\text{int}(A) \subseteq \beta^*\text{int}(\beta^*\text{cl}(A)) = \beta^*\text{int}(\beta^*\text{int}(A^c))^c = \beta^*\text{int}(\beta^*\text{Ext}(A))^c = \beta^*\text{Ext}(\beta^*\text{Ext}(A)). \)

**Proposition 2.4.50** In general equality does not hold in (i), (v) and (vi) of Theorem 2.4.49.

**Example 2.4.51** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\} \). Then \( \beta^*\text{O}(\tau) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \). If \( A = \{a\}, B = \{b\} \) and \( C = \{c\} \) then \( \beta^*\text{Ext}(A) = \{b\} \), \( \beta^*\text{Ext}(B) = \{a\} \) and \( \text{Ext}(A) = \emptyset \). Also \( \beta^*\text{Ext}(A \cup B) = \emptyset, \beta^*\text{Ext}(A \cap B) = X \). Therefore \\
\( \beta^*\text{Ext}(A) \nsubseteq \text{Ext}(A), \ \beta^*\text{Ext}(A) \cup \beta^*\text{Ext}(B) \nsubseteq \beta^*\text{Ext}(A \cup B) \) and \( \beta^*\text{Ext}(A \cap B) \nsubseteq \beta^*\text{Ext}(A) \cap \beta^*\text{Ext}(B). \)

**************

### 2.5 Application

As application of \( \beta^* \)-closed sets, five new topological spaces namely \( T_{\beta^*} \)-spaces, \( \Lambda T_{\beta^*} \)-spaces, \( pT_{\beta^*} \)-spaces, \( spT_{\beta^*} \)-spaces and \( sT_{\beta^*} \)-spaces are introduced.

**Definitions 2.5.1** A space \( X \) is called a \( T_{\beta^*} \) space if every \( \beta^* \)-closed set is closed.

**Example 2.5.2** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\} \). Then \( X \) is a \( T_{\beta^*} \) space.
Proposition 2.5.3 In a topological space $X$, for each $x \in X$, \{x\} is either $\omega$-closed or its complement is $\beta^*$-closed.

Proof: Suppose that \{x\} is not $\omega$-closed in $X$. Then $X$-\{x\} is not $\omega$-open. The only $\omega$-open set containing $X$-\{x\} is $X$. Hence $\text{spcl}(X$-\{x\})$\subset X = \text{int} X$. Therefore $X$-\{x\} is $\beta^*$-closed.

Proposition 2.5.4 If $X$ is a $T_{\beta^*}$-space, then every singleton of $X$ is either $\omega$-closed or open.

Proof: By the hypothesis and by Proposition 2.5.3, the proof follows.

The converse of Proposition 2.5.4 is not true.

Example 2.5.5 Let $X = \{a, b, c\}$ and $\tau= \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then the space $X$ satisfies the conclusion of Proposition 2.5.4, but $X$ is not a $T_{\beta^*}$ space since \{b\} is $\beta^*$-closed but not closed.

Proposition 2.5.6 Every $T_{\tilde{\eta}^*}$-space is $T_{\beta^*}$-space.

Proof: Let $A$ be any $\beta^*$ closed set in a $T_{\tilde{\eta}^*}$-space $X$. Since every $\beta^*$closed set is an $\tilde{\eta}^*$-closed set and $X$ is a $T_{\tilde{\eta}^*}$-space, $A$ is closed. Hence $X$ is a $T_{\beta^*}$ space.

The converse of Proposition 2.5.6 need not be true.

Example 2.5.7 Let $X= \{a, b, c\}$ and $\tau= \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $X$ is a $T_{\beta^*}$ space but not a $T_{\tilde{\eta}^*}$-space since the set $A = \{a, c\}$ is $\tilde{\eta}^*$-closed but not closed in $X$.

Definition 2.5.8 A space $X$ is called a $T_{gsp}$-pace if every gsp-closed set is closed.
**Proposition 2.5.9** Every $T_{gsp}$ space is a $T_{\beta^*}$ space but not conversely.

**Proof:** Let $X$ be a $T_{gsp}$ - space. Let $A$ be any $\beta^*$-closed set in $X$. Since every $\beta^*$ closed set is a gsp closed set and $X$ is a $T_{gsp}$ - space, $A$ is closed.

**Example 2.5.10** Let $X= \{a, b, c\}$ and $\tau= \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $X$ is a $T_{\beta^*}$ space but not a $T_{gsp}$-space since, the set $B=\{b\}$ is gsp-closed but not closed.

**Definition 2.5.11** A space $X$ is called a $*_g T_{\beta^*}$ if every $\beta^*$-closed set is $*g$-closed in $X$.

**Example 2.5.12** Let $X= \{a, b, c\}$ and $\tau= \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then $X$ is a $*_g T_{\beta^*}$ space.

**Definition 2.5.13** A space $X$ is called a $\alpha T_{\beta^*}$ (resp. $p T_{\beta^*}, sp T_{\beta^*}, s T_{\beta^*}, A T_{\beta^*}$) if every $\beta^*$ closed set is $\alpha$-closed (resp. pre-closed, semi-preclosed, semi-closed and $\hat{\eta}^*$ closed).

**Proposition 2.5.14** Every $T_{\beta^*}$ space is $\alpha T_{\beta^*}, p T_{\beta^*}, sp T_{\beta^*}, s T_{\beta^*}$ and $A T_{\beta^*}$ space.

**Proof:** Since every $\beta^*$-closed set is closed in a $T_{\beta^*}$ space and every closed set is $\alpha$-closed, pre-closed, semi-preclosed, semi closed and $\hat{\eta}^*$ -closed, the proof follows.

**Theorem 2.5.15** In a topological space $X$, the following conditions are equivalent

(i) $X$ is a $sp T_{\beta^*}$ space.

(ii) Every singleton of $X$ is either $\omega$-closed or semi-preopen.
**Proof:** Let \( x \in X \) and suppose that \( \{x\} \) is not \( \omega \)-closed in \( X \). Then \( \{x\}^c \) is not \( \omega \)-open in \( X \). Since \( X \) is the only \( \omega \)-open set containing \( \{x\}^c \), and \( \text{spcl}(\{x\}^c) \subset X = \text{int} X \), \( \{x\}^c \) is \( \beta^* \)-closed. By (i) \( \{x\}^c \) is semi pre closed and hence \( \{x\} \) is semi-preopen.

Conversely, let \( A \) be a \( \beta^* \)-closed set in \( X \). Clearly \( A \subset \text{spcl}(A) \). Let \( x \in \text{spcl}(A) \), by (ii), \( \{x\} \) is either \( \omega \)-closed or semi-preopen. Then there are two cases.

**Case (i):** Suppose that \( \{x\} \) is \( \omega \)-closed and if \( x \in A \), then \( \text{spcl}(A) \) contains the \( \omega \)-closed set \( \{x\} \). But \( A \) is \( \beta^* \)-closed. This is a contradiction to Proposition 2.2.30. Thus \( x \not\in A \).

**Case (ii):** Suppose that \( \{x\} \) is semi-preopen, since \( x \in \text{spcl}(A) \); \( \{x\} \cap A \neq \emptyset \). So \( x \in A \). Hence in both the cases, \( x \in \text{spcl}(A) \) implies \( x \in A \). Hence \( \text{spcl}(A) \subset A \) which implies \( A \) is semi-preclosed.

**Proposition 2.5.16** If \( X \) is a \( \ast gT_{\beta^*} \) space, then every singleton of \( X \) is \( \omega \)-closed or \( \ast g \) open.

**Proof:** Suppose that \( \{x\} \) is not \( \omega \)-closed in \( X \). Then \( \{x\}^c \) is not \( \omega \)-open and the only \( \omega \)-open set containing \( \{x\}^c \) is the space \( X \) itself. Therefore \( \text{spcl}(\{x\}^c) \subset X = \text{int} X \) and so \( \{x\}^c \) is \( \beta^* \)-closed. By hypothesis, \( \{x\}^c \) is \( \ast g \) closed. Therefore \( \{x\} \) is \( \ast g \)-open.