CHAPTER VI

β*-CLOSED SETS IN BITOPOLOGICAL SPACES

6.1 introduction

The triple \((X,\tau_1,\tau_2)\) where \(X\) is a non-empty set, \(\tau_1\) and \(\tau_2\) are any two arbitrary topologies is called a bitopological space. Kelly [25] initiated the systematic study of such spaces. Levine [29] introduced and studied the notion of generalized closed sets and \(T_{1/2}\) spaces in topological spaces. The notion of generalized closed sets and \(T_{1/2}\) spaces of a bitopological space were introduced and investigated by Fukutake [18]. Also several authors turned their attention to generalization of various concepts of topology by considering bitopological spaces.

The concepts of \(\beta^*\)-closed sets and \(\beta^*\)-open sets discussed in chapter II and some properties of \(\beta^*\)-continuous maps studied in chapter III are used in this chapter.

Section 2 deals with \(\beta^*\)-closed sets and \(\beta^*\)-open sets in bitopological spaces, section 3 concerns with application of \((\tau_i, \tau_j) - \beta^*\)-closed sets and section 4 deals with \(\beta^*\)-continuity, \(\beta^*\)-bicontinuity and \(\beta^*\)-S-bicontinuity and pairwise \(\beta^*\)-irresolute maps in bitopological spaces.

Throughout this chapter \((X,\tau_1,\tau_2)\) and \((Y,\sigma_1,\sigma_2)\) denote two non-empty bitopological spaces on which no separation axioms are assumed, unless otherwise mentioned and the fixed integers \(i,j,k,l,m,n \in \{1,2\}\).
6.2 - (τ_i, τ_j) – β*-closed set and (τ_i, τ_j) – β*-open set

In this section introduce (τ_i, τ_j) – β*-closed sets and (τ_i, τ_j) – β*-open sets in bitopological space are introduced and some of their basic properties are discussed.

**Definition 6.2.1** A subset A of a bitopological space (X, τ_1, τ_2) is said to be (τ_i, τ_j) – β*-closed if τ_j – spcl(A) ⊆ int(U) whenever A ⊆ U, and U is τ_i ω-open.

We denote the family of all (τ_i, τ_j) – β*-closed sets in (X, τ_1, τ_2) by β*(τ_i, τ_j).

**Remark 6.2.2** By setting τ_1=τ_2 in Definition 6.2.1, a (τ_i,τ_j) – β*-closed set is a β*-closed set.

**Remark 6.2.3** τ_j – β*-closed and (τ_i, τ_j) – β*-closed sets are in general independent. It is seen from the following examples.

**Example 6.2.4** Let X={a, b, c}, τ_1={φ, {a}, {b, c}, X}, τ_2 = {φ, {a}, {a, b}, X}. Then β*(τ_i, τ_j) = {φ, {a}, {b, c}, X}. Here the set {a, c} is τ_2 – β*-closed but not (τ_1,τ_2) – β*-closed.

**Example 6.2.5** Let X={a, b, c}, τ_1={φ, {a}, X}, τ_2 = {φ, {a}, {a, b}, X}.

Then β*(τ_i, τ_j) = P(X) - {a}. Here the set {a, b} is (τ_1,τ_2) – β* closed but not τ_2 - β*-closed.

**Proposition 6.2.6** If A is a τ_i-closed (resp. τ_j-α closed, τ_j – semi closed) subset of a bitopological space (X, τ_1, τ_2) and if (X, τ_i) is a T_ω space then the set A is (τ_i,τ_j) – β*-closed, but not conversely.

**Proof:** Let G be a τ_i-ω-open set such that A ⊆ G. Then by hypothesis, τ_j cl(A) ⊆ G (resp. τ_j acl(A) ⊆ G, τ_j scl(A) ⊆ G). From the result τ_j spcl(A) ⊆ τ_j cl(A),
(resp. \( \tau_j spcl(A) \subseteq \tau_j acl(A) \), \( \tau_j spcl(A) \subseteq \tau_j scl(A) \)) we get \( \tau_j spcl(A) \subseteq G = int(G) \) since \( G \) is open in \((X, \tau_1)\). Therefore \( A \) is \((\tau_i, \tau_j)\)-\( \beta^* \)-closed.

The converse is not true as we see in example 6.2.5

**Proposition 6.2.7** If \( A \) is a \( \tau_j \) open and semi-preclosed subset of a bitopological space \((X, \tau_1, \tau_2)\), then the set \( A \) is \((\tau_i, \tau_j)\)-\( \beta^* \)-closed.

**Proof:** Let \( G \) be a \( \tau_i \)-\( \omega \)-open set such that \( A \subseteq G \). Then by hypothesis, \( \tau_j spcl(A) = A = int(A) \subseteq int(G) \). Hence \( A \) is \((\tau_i, \tau_j)\)-\( \beta^* \)-closed.

The converse of the above Proposition need not be true. It is seen from the following example.

**Example 6.2.8** Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{\varnothing, \{a\}, X\} \), \( \tau_2 = \{\varnothing, \{a\}, \{a, b\}, X\} \). Then \( \beta^*(\tau_i, \tau_j) = P(X) - \{a\} \). The set \( A = \{a, c\} \) is neither open nor semi-preclosed in \( \tau_2 \).

**Proposition 6.2.9** If \( A \) is a \((\tau_i, \tau_j)\)-\( g^* \)-closed subset of a bitopological space \((X, \tau_1, \tau_2)\) and \((X, \tau_1)\) is a \( T_\omega \) space then \( A \) is \((\tau_i, \tau_j)\)-\( \beta^* \)-closed but not conversely.

**Proof:** Let \( A \subseteq G \) where \( G \) is \( \tau_i \)-\( \omega \)-open. By lemma 1.1.6 \( G \) is \( \tau_i \)-\( g \)-open. By hypothesis \( \tau_j cl(A) \subseteq G = int(G) \). Hence \( \tau_j spcl(A) \subseteq int(G) \) implies that \( A \) is \((\tau_i, \tau_j)\)-\( \beta^* \)-closed.

**Example 6.2.10** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\varnothing, \{a, b\}, X\} \), \( \tau_2 = \{\varnothing, \{a\}, X\} \). Then \( \beta^*(\tau_1, \tau_2) = \{\varnothing, \{c\}, \{b, c\}, \{a, c\}, X\} \) and \( g^*(\tau_1, \tau_2) = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\} \). Let \( A = \{a, c\} \). Then \( A \) is \((\tau_1, \tau_2)\)-\( \beta^* \)-closed but not \((\tau_1, \tau_2)\)-\( g^* \)-closed.

**Definition 6.2.11[20]** A subset \( A \) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((\tau_i, \tau_j)\)-\( gspr \) (resp. \((\tau_i, \tau_j)\)-\( gpr \)) closed if \( \tau_j spcl(A) \subseteq U \) (resp. \( \tau_j pcl(A) \subseteq U \)) whenever \( A \subseteq U \) and \( U \) is \( \tau_i \)-regular open.
**Proposition 6.2.12** If $A$ is a $(\tau_i,\tau_j)$ $-\beta^*$-closed subset of a bitopological space $(X,\tau_1,\tau_2)$ then $A$ is $(\tau_i,\tau_j)$ $-\text{gspr}$-closed but not conversely.

**Proof:** Let $G$ be a $\tau_i$- regular open set such that $A \subset G$. Since $G$ is $\tau_i$-regular open, $G$ is $\tau_j$-open and hence $\tau_i$-$\omega$-open. By hypothesis $\tau_j$-$\text{spcl}(A) \subset \text{int}(G)=G$. Thus $A$ is $(\tau_i,\tau_j)$-$\text{gspr}$- closed.

**Example 6.2.13** Let $X_1$ and $\tau_2$ be defined as in example 6.2.10. Then the set $A = \{a, b\}$ is $(\tau_1,\tau_2)$ gspr closed but not $(\tau_1,\tau_2)$-$\beta^*$-closed, since $\text{gspr-}(\tau_i,\tau_j) = \{\varnothing, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$.

The following example shows that $(\tau_i,\tau_j)$-$\beta^*$-closed sets and $(\tau_i,\tau_j)$-$\text{gpr}$-closed sets are independent.

**Example 6.2.14** Let $(X,\tau_1,\tau_2)$ be defined as in example 6.2.10. Then $\text{gpr-}(\tau_1,\tau_2) = \{\varnothing, \{b\}, \{c\}, \{b, c\}, X\}$. Let $A = \{b\}$. Then $A$ is $(\tau_1,\tau_2)$-$\text{gpr}$-closed but not $(\tau_1,\tau_2)$-$\beta^*$-closed. Also $B = \{a, c\}$ is $(\tau_1,\tau_2)$-$\beta^*$-closed but not $(\tau_1,\tau_2)$-$\text{gpr}$-closed.

**Definition 6.2.15** A subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is called $(\tau_i,\tau_j)$-$\text{gsp}$ closed if $\tau_j$-$\text{spcl}(A) \subset U$ whenever $A \subset U$ and $U$ is open in $\tau_i$.

The collection of all $(\tau_i,\tau_j)$-$\text{gsp}$ closed sets are denoted by $\text{GSPC}(\tau_1,\tau_2)$.

**Proposition 6.2.16** If $A$ is a $(\tau_i,\tau_j)$-$\beta^*$-closed subset of a bitopological space $(X,\tau_1,\tau_2)$ then $A$ is $(\tau_i,\tau_j)$-$\text{gsp}$ closed but not conversely.

**Proof:** Let $G$ be a $\tau_i$ open set such the $A \subset G$. Hence $G$ is a $\tau_i$-$\omega$-open set. By hypothesis $\tau_j$-$\text{spcl}(A) \subset \text{int}(G) \subset G$. Hence $A$ is $(\tau_i,\tau_j)$-$\text{gsp}$ closed.

**Example 6.2.17** Let $(X,\tau_1,\tau_2)$ be the bitopological space, defined as in example 6.2.10. Then $\text{GSPC}(\tau_1,\tau_2) = \{\varnothing, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Let $A = \{b\}$. Then $A$ is $(\tau_1,\tau_2)$-$\text{gsp}$ closed but not $(\tau_1,\tau_2)$-$\beta^*$-closed.
**Proposition 6.2.18** If $A$ is a $(\tau_i, \tau_j)$-$\beta^*$-closed subset of a bitopological space $(X, \tau_1, \tau_2)$ then $A$ is $(\tau_i, \tau_j)$-$\hat{\eta}^*$-closed but not conversely.

**Proof:** Let $G$ be a $\tau_i$-$\omega$-open set such that $A \subseteq G$. By hypothesis $\tau_j \text{spcl}(A) \subseteq \text{int}(G) \subseteq G$. Hence $A$ is $(\tau_i, \tau_j)$-$\hat{\eta}^*$-closed.

**Example 6.2.19** $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, X\}$. Then $\hat{\eta}^*(\tau_1, \tau_2) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$, $\beta^*(\tau_1, \tau_2) = \{\emptyset, \{b, c\}, X\}$. Let $A = \{b\}$. Then $A$ is $(\tau_1, \tau_2)$-$\hat{\eta}^*$-closed but not $(\tau_1, \tau_2)$-$\beta^*$-closed.

The following example shows that $(\tau_i, \tau_j)$-$\beta^*$-closed sets and $(\tau_i, \tau_j)$-rg closed set are independent.

**Example 6.2.20** Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$ and $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then $\text{rg}-(\tau_1, \tau_2) = \{\emptyset, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, X\}$, $\beta^*(\tau_1, \tau_2) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$. We see that the set $\{a, b\}$ is $(\tau_1, \tau_2)$-rg closed but not $(\tau_1, \tau_2)$-$\beta^*$-closed. Also $\{b\}$ is $(\tau_1, \tau_2)$-$\beta^*$-closed but not $(\tau_1, \tau_2)$-rg closed.

The above findings are given below in pictorial representation as follows.

![Fig(vi)](image-url)
Remark 6.2.21 The following example shows that the intersection (resp.union) of two \((\tau_i,\tau_j)\)-\(\beta^*\)-closed sets is not \((\tau_i,\tau_j)\)-\(\beta^*\)-closed.

Example 6.2.22 Let \(X=\{a, b, c, d\}\), \(\tau_1=\{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}\) and \(\tau_2=\{\varnothing, \{a, b\}, X\}\). Then \(\beta^*(\tau_1,\tau_2)=P(X)-\{a, b\}\). Here \(\{a,b,c\}\) and \(\{a, b, d\}\) are \((\tau_1,\tau_2)\)-\(\beta^*\)-closed sets but \(\{a, b, c\}\cap\{a, b, d\}=\{a, b\}\) is not \((\tau_1,\tau_2)\)-\(\beta^*\)-closed. Also \(\{a\}\) and \(\{b\}\) are \((\tau_1,\tau_2)\)-\(\beta^*\)-closed. But \(\{a\}\cup\{b\}=\{a, b\}\) is not \((\tau_1,\tau_2)\)-\(\beta^*\)-closed.

Proposition 6.2.23 Let \(A\) be a subset of bitopological space \((X, \tau_1, \tau_2)\). If \(A\) is \((\tau_i,\tau_j)\)-\(\beta^*\)-closed, then \(\tau_j\text{spcl}(A)-A\) does not contain any non-empty \(\tau_i\)-\(\omega\)-closed set.

Proof: Let \(A\) be a \((\tau_i,\tau_j)\)-\(\beta^*\)-closed set and \(F\) be a \(\tau_i\)-\(\omega\)-closed set contained in \(\tau_j\text{spcl}(A)-A\). Since \(A\in\beta^*(\tau_i,\tau_j)\), \(\tau_j\text{spcl}(A)\subseteq\text{int}(F^c)\subseteq F^c\). Consequently, \(F\subseteq (\tau_j\text{spcl}(A))^c\). But \(F\subseteq \tau_j\text{spcl}(A)\). Hence \(F\) must be empty.

Remark 6.2.24 The converse of Proposition-6.2.23 is not true. It is evident from the following example.

Example 6.2.25 Let \(X=\{a, b, c, d\}\), \(\tau_1=\{\varnothing, \{a, b, c\}, X\}\) and \(\tau_2=\{\varnothing, \{a\}, X\}\). Then \(\beta^*(\tau_1,\tau_2)=\{\varnothing, \{b\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}\). Let \(A=\{b\}\). Then \(\tau_2\text{spcl}(A)-A = b - b = \varnothing\), but \(A\) is not \((\tau_1,\tau_2)\)-\(\beta^*\)-closed.

Proposition 6.2.26 If \(A\) is \((\tau_i,\tau_j)\)-\(\beta^*\)-closed and \(A\subseteq B\subseteq \tau_j\text{spcl}(A)\), Then \(B\) is \((\tau_i,\tau_j)\)-\(\beta^*\)-closed.

Proof: Let \(U\) be a \(\tau_i\)-\(\omega\)-open set of \(X\) such that \(B\subseteq U\). Then \(A\subseteq U\). Since \(A\) is \((\tau_i,\tau_j)\)-\(\beta^*\)-closed, we get \(\tau_j\text{spcl}(A)\subseteq\text{int}(U)\). Now \(\tau_j\text{spcl}(B)\subseteq\tau_j\text{spcl}(\text{spcl}(\tau_j\text{spcl}(A)))\). Using lemma 1.1.6. Thus \(B\) is \((\tau_i,\tau_j)\)-\(\beta^*\)-closed.
Proposition 6.2.27  If $A$ is $\tau_i$-ω-open and $(\tau_i,\tau_j)$-$\beta^*$-closed, then $A$ is $\tau_j$-semi-preclosed

**Proof:** Since $A \subseteq A$. Since $A$ is $\tau_i$-ω-open and $(\tau_i,\tau_j)$-$\beta^*$-closed, we have $\tau_j$-$\text{spcl}(A) \subseteq \text{int}(A) \subseteq A$. Therefore $A$ is $\tau_j$ semi-preclosed.

Proposition 6.2.28  For each $x$ of $(X,\tau_1,\tau_2),\{x\}$ is $\tau_i$-ω-closed or $\{x\}^c$ is $(\tau_i,\tau_j)$-$\beta^*$-closed.

**Proof:** Suppose $\{x\}$ is not $\tau_i$-ω-closed. Since $\{x\}^c$ is not $\tau_i$-ω-open, the only $\tau_i$-ω-open set containing $\{x\}^c$ is $X$. But $\tau_j$-$\text{spcl}(\{x\}^c) \subseteq X = \text{int} X$. Therefore $\{x\}^c$ is $(\tau_i,\tau_j)$-$\beta^*$-closed.

Proposition 6.2.29  If $\tau_1 \subset \tau_2$ in $(X,\tau_1,\tau_2)$ then $\beta^*(\tau_2,\tau_1) \supset \beta^*(\tau_1,\tau_2)$.

**Proof:** Let $A$ be a $(\tau_2,\tau_1)$-$\beta^*$-closed set and $G$ be a $\tau_1$-ω-open set containing $A$. Since $\tau_1 \subset \tau_2$, $G$ is a $\tau_2$-ω-open set and since $A$ is a $(\tau_2,\tau_1)$-$\beta^*$-closed set $\tau_1$-$\text{spcl}(A) \subseteq \text{int}(G)$. But $\tau_2$-$\text{spcl}(A) \subseteq \tau_1$-$\text{spcl}(A)$. Therefore $\tau_2$-$\text{spcl}(A) \subseteq \text{int}(G)$, Thus $A$ is a $(\tau_1,\tau_2)$-$\beta^*$-closed set.

We now introduce $(\tau_i,\tau_j)$-$\beta^*$-open sets in bitopological spaces as follows.

**Definition 6.2.30**  A subset $A$ in a bitopological space $(X,\tau_1,\tau_2)$ is called $(\tau_i,\tau_j)$-$\beta^*$-open if $A^c$ is $(\tau_i,\tau_j)$-$\beta^*$-closed in $(X,\tau_1,\tau_2)$.

**Remark 6.2.31**  The union(intersection) of any two $(\tau_i,\tau_j)$-$\beta^*$-open set is not $(\tau_i,\tau_j)$-$\beta^*$-open.

**Example 6.2.32**  Let $X = \{a, b, c, d\}$, $\tau_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\varnothing, \{a, b\}, X\}$. Then $(\tau_1,\tau_2)$-$\beta^*$-open sets are $P(X) - \{c, d\}$. We see that $\{c\} \cup \{d\} = \{c, d\}$ and $\{a, c, d\} \cap \{b, c, d\} = \{c, d\}$ are not $(\tau_1,\tau_2)$-$\beta^*$-open sets though individually they are $\beta^*$-open sets.
Theorem 6.2.33 A subset $A$ of $(X,\tau_1,\tau_2)$ is $(\tau_i,\tau_j)$-$\beta$*-open if and only if $\text{cl}(F) \subset \tau_j$-spint$(A)$ whenever $F$ is $\tau_i$-$\omega$-closed and $F \subset A$.

Proof: Suppose that $A$ is $(\tau_i,\tau_j)$-$\beta$*-open and $F \subset A$, where $F$ is $\tau_i$-$\omega$-closed. Then $A^c \subset F^c$ and $F^c$ is $\tau_i$-$\omega$-open in $(X,\tau_1,\tau_2)$. Hence we get $\tau_i$spcl$(A^c) \subset \text{int}(F^c)$=\left((\text{int}(U))^c\right)^c=\text{int}(U)$. Thus $\tau_i$-spcl$(A^c)$ is $(\tau_i,\tau_j)$-$\beta$*-closed. Conversely, suppose $A^c \subset U$ and $U$ is $\tau_i$-$\omega$-open. Then $U^c \subset A$ and $U^c$ is $\tau_i$-$\omega$-closed. Therefore by hypothesis, $\text{cl}(U^c) \subset \tau_j$-spint $(A)$ implies $(\tau_j$-spint$(A))^c \subset (\text{cl}(U^c))^c=\text{int}(U)$ implies $A^c$ is $(\tau_i,\tau_j)$-$\beta$*-open. Thus $A$ is $(\tau_i,\tau_j)$-$\beta$*-open.

Proposition 6.2.34 If $\tau_j$-spint $(A) \subset B \subset A$ and $A$ is $(\tau_i,\tau_j)$-$\beta$*-open, then $B$ is $(\tau_i,\tau_j)$-$\beta$*-open.

Proof: Suppose that $\tau_j$-spint$(A) \subset B \subset A$ and $A$ is $(\tau_i,\tau_j)$-$\beta$*-open. Then $A^c \subset B^c \subset \tau_j$-spcl$(A^c)$ and $A^c$ is $(\tau_i,\tau_j)$-$\beta$*-closed. By Proposition 6.2.26, $B^c$ is $(\tau_i,\tau_j)$-$\beta$*-closed. Thus $B$ is $(\tau_i,\tau_j)$-$\beta$*-open.

Proposition 6.2.35 If a subset $A$ of a bitopological space $(X,\tau_1,\tau_2)$ is $(\tau_i,\tau_j)$-$\beta$*-closed then $\tau_j$-spcl$(A)$-A is $(\tau_i,\tau_j)$-$\beta$*-open.

Proof: Suppose that $A$ is $(\tau_i,\tau_j)$-$\beta$*-closed. Let $F \subset \text{spcl}(A)$-A where $F$ is a $\tau_i$-$\omega$-closed set. By Proposition 6.2.23, $F = \phi$. Therefore, $\text{cl}(F) \subset \tau_j$-spint $(\tau_j$spcl$(A)$-A) and so by Theorem 6.2.33, $\tau_j$-spcl$(A)$-A is $(\tau_i,\tau_j)$-$\beta$*-open.

Remark 6.2.36 The converse of Proposition 6.2.35 does not hold. The subset $A=\{c\}$ of $(X,\tau_1, \tau_2)$ in Example 6.2.25 is not $(\tau_i,\tau_j)$-$\beta$*-closed. However $\tau_2$-spcl$(A)$-A=$\{a, c\}$-\{c\}=$\{a\}$ is $(\tau_i,\tau_j)$-$\beta$*-open in $(X,\tau_1, \tau_2)$.
6.3 Applications

As applications of \((\tau_i, \tau_j)-\beta^*-\text{closed sets}\), two new bitopological spaces namely \((\tau_i, \tau_j)-T_{\beta^*}\) and \((\tau_i, \tau_j)-spT_{\beta^*}\) spaces have been introduced and obtained a characterization for the bitopological space \((\tau_i, \tau_j)-\text{semi-pre } T_{1/2}\).

**Definition 6.3.1** A bitopological space \((X, \tau_1, \tau_2)\) is said to be a \((\tau_i, \tau_j)-T_{\beta^*}\)-space if every \((\tau_i, \tau_j)-\beta^*\)-closed set is \(\tau_j\)-closed.

**Proposition 6.3.2** If a bitopological space \((X, \tau_1, \tau_2)\) is \((\tau_i, \tau_j)-T_{\beta^*}\)-space, then \(\{x\}\) is \(\tau_i\)-\(\omega\)-closed or \(\tau_j\)-open for each \(x \in X\).

**Proof:** Suppose \(\{x\}\) is not \(\tau_i\)-\(\omega\)-closed. Since \(\{x\}\) is not \(\tau_i\)-\(\omega\)-open, the only \(\tau_i\)-\(\omega\)-open set containing \(\{x\}\) is \(X\). Therefore \(\tau_j\)-spcl(\(\{x\}\)) \(\subseteq X = \text{int } X\) implies \(\{x\}\) is \((\tau_i, \tau_j)-\beta^*\)-closed. Since \((X, \tau_1, \tau_2)\) is a \((\tau_i, \tau_j)-T_{\beta^*}\)-space, \(\{x\}\) is \(\tau_j\)-closed. Thus \(\{x\}\) is \(\tau_j\)-open.

**Remark 6.3.3** The following example supports the fact that the converse of proposition -6.3.2 is not true.

**Example 6.3.4** Let \(X = \{a, b, c\}, \tau_1 = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}\) and \(\tau_2 = \{\varnothing, \{a\}, \{b, c\}, X\}\). Then \(\beta^*(\tau_1, \tau_2) = P(X)\). Here \(\{b\}\) and \(\{c\}\) are \(\tau_j\)-\(\omega\)-closed and \(\{a\}\) is \(\tau_2\)-open. But \((X, \tau_1, \tau_2)\) is not \((\tau_i, \tau_j)-T_{\beta^*}\)-space since \(\{c, a\}\) is \((\tau_1, \tau_2)-\beta^*\)-closed but not \(\tau_2\)-closed.

**Proposition 6.3.5** Every \((\tau_i, \tau_j)-T_{\eta^*}\) space is \((\tau_i, \tau_j)-T_{\beta^*}\)-space but not conversely.

**Proof:** Let \((X, \tau_1, \tau_2)\) be a \((\tau_i, \tau_j)-T_{\eta^*}\) space and \(A\) be a \((\tau_i, \tau_j)-\beta^*\)-closed set. Then \(A\) is \((\tau_1, \tau_2)-\eta^*\)-closed by proposition 5.2.9. By hypothesis \(A\) is \(\tau_j\)-closed. Hence \((X, \tau_1, \tau_2)\) is a \((\tau_i, \tau_j)-T_{\beta^*}\)-space.
Example 6.3.6 If \((X, \tau_1, \tau_2)\) is the space as defined in Example 6.2.19, then the space is \((\tau_i, \tau_j)\) \(T_{\beta^*}\)-space but it is not a \((\tau_i, \tau_j)\) \(T_{\eta^*}\)-space, since \(\{c\}\) is \((\tau_1, \tau_2)\)-\(\eta^*\)-closed but it is not \(\tau_2\)-closed.

Definition 6.3.7 A bitopological space \((X, \tau_1, \tau_2)\) is said to be \((\tau_i, \tau_j)\)-\(T_{gsp}\)-space if every \((\tau_i, \tau_j)\)-gsp-closed set is \(\tau_j\)-closed.

Proposition 6.3.8 Every \((\tau_i, \tau_j)\)-\(T_{gsp}\)-space is \((\tau_i, \tau_j)\)-\(T_{\beta^*}\)-space but not conversely.

Proof: Let \((X, \tau_1, \tau_2)\) be a \((\tau_i, \tau_j)\)-\(T_{gsp}\)-space and \(A\) be a \((\tau_i, \tau_j)\)-\(\beta^*\)-closed set. Then by Proposition-6.2.16, \(A\) is \((\tau_i, \tau_j)\)-gsp-closed. By hypothesis \(A\) is \(\tau_j\)-closed. Hence \((X, \tau_1, \tau_2)\) is a \((\tau_i, \tau_j)\)-\(T_{\beta^*}\)-space.

Example 6.3.9 Let \((X, \tau_1, \tau_2)\) be the space in Example 6.2.19. Then \(gsp-(\tau_i, \tau_j) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}\). Then \(\{a, c\}\) is not \(\tau_2\)-closed. Therefore the space is not a \((\tau_i, \tau_j)T_{gsp}\) space. But the space is a \((\tau_i, \tau_j)\)-\(T_{\beta^*}\)-space.

Definition 6.3.10 A bitopological space \((X, \tau_1, \tau_2)\) is said to be \((\tau_i, \tau_j)\)-\(T_{gspr}\)-space if every \((\tau_i, \tau_j)\)-gspr closed set is \(\tau_j\)-closed.

Proposition 6.3.11 Every \((\tau_i, \tau_j)\)-\(T_{gspr}\)-space is \((\tau_i, \tau_j)\)-\(T_{\beta^*}\) space but not conversely.

Proof: Let \((X, \tau_1, \tau_2)\) be a \((\tau_i, \tau_j)\)-\(T_{gspr}\)-space and \(A\) be a \((\tau_i, \tau_j)\)-\(\beta^*\)-closed set. Then by Proposition-6.2.12 \(A\) is \((\tau_i, \tau_2)\)-gspr-closed. By hypothesis \(A\) is \(\tau_j\)-closed. Therefore \((X, \tau_1, \tau_2)\) is a \((\tau_i, \tau_j)\)-\(T_{\beta^*}\)-space.

Example 6.3.12 Let \(X=\{a, b, c\}\), \(\tau_1=\{\emptyset, \{a\}, \{b, c\}, X\}\) and \(\tau_2=\{\emptyset, \{a\}, X\}\). Then \(\beta^*(\tau_1, \tau_2) = \{\emptyset, \{b, c\}, X\}\) and \(gspr-(\tau_1, \tau_2) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}\).
{c, a}, X}. The set \{a, b\} is not \(\tau_2\)-closed. Therefore the space \((X, \tau_1, \tau_2)\) is not a \((\tau_i, \tau_j)\) T_{gsp}-space but it is a \((\tau_i, \tau_j)\) T_{\beta^*}-space.

**Definition 6.3.13** A bitopological space \((X, \tau_1, \tau_2)\) is said to be

(i) \((\tau_i, \tau_j)\)-sp\(\beta^*\) space if every \((\tau_1, \tau_2)\)-\(\beta^*\)-closed set is \(\tau_j\)-semi pre closed.

(ii) \((\tau_i, \tau_j)\)-semi-pre-\(T_{1/2}\) if every \((\tau_1, \tau_2)\)-gsp-closed set is \(\tau_j\)-semi-pre closed.

**Proposition 6.3.14** Every \((\tau_i, \tau_j)\)-\(T_{\beta^*}\) space is a \((\tau_i, \tau_j)\)-sp\(\beta^*\) space, but not conversely.

**Proof:** Let \(A\) be a \((\tau_1, \tau_2)\)-\(\beta^*\)-closed set. Since \((X, \tau_1, \tau_2)\) is a \((\tau_i, \tau_j)\)-\(T_{\beta^*}\) space, \(A\) is \(\tau_j\)-closed. Then \(A\) is \(\tau_j\)-semi-preclosed which implies that \((X, \tau_1, \tau_2)\) is a \((\tau_i, \tau_j)\)-sp\(\beta^*\) space.

**Example 6.3.15** Let \((X, \tau_1, \tau_2)\) be the space as defined in Example 6.3.4. We see that it is not a \((\tau_i, \tau_j)\)\(T_{\beta^*}\) space but it is a \((\tau_i, \tau_j)\)-sp\(\beta^*\) space.

**Theorem 6.3.16** A bitopological space \((X, \tau_1, \tau_2)\) is a \((\tau_i, \tau_j)\)-sp\(\beta^*\) space if and only if \(\{x\}\) is either \(\tau_j\) semi-preopen or \(\tau_i\)-\(\omega\)-closed for each \(x \in X\).

**Proof:** Let \(x \in X\) and suppose that \(\{x\}\) is not \(\tau_i\)-\(\omega\)-closed. Since \(\{x\}^c\) is not \(\tau_i\)-\(\omega\)-open, the only \(\tau_i\)-\(\omega\) open set containing \(\{x\}^c\) is \(X\). Therefore \(\tau_j\)-spcl \(\{x\}^c\) \(\subset X = \text{int } X\) which implies \(\{x\}^c\) is \((\tau_1, \tau_2)\)-\(\beta^*\)-closed. Since \((X, \tau_1, \tau_2)\) is a \((\tau_i, \tau_j)\)-sp\(\beta^*\) space, \(\{x\}^c\) is \(\tau_j\)-semi-preclosed. Thus \(\{x\}\) is \(\tau_j\)-semi-preopen.

Conversely, let \(A\) be a \((\tau_i, \tau_j)\)-\(\beta^*\)-closed set. For any \(x \in \tau_j\)-spcl\(\{A\}\); \(\{x\}\) is \(\tau_j\) semi-pre open or \(\tau_i\)-\(\omega\)-closed by assumption.
Case(i): Suppose that \( \{x\} \) is \( \tau_i \)-\( \omega \)-closed. If \( x \notin A \), then \( \tau_j \)-spcl(A)-A contains the \( \tau_i \)-\( \omega \)-closed set \( \{x\} \). But A is \( (\tau_i, \tau_j) \)-\( \beta^* \)-closed. This is a contraction to Proposition, 6.2.23. Thus \( x \in A \).

Case(ii): Suppose that \( \{x\} \) is \( \tau_j \)-semi preopen. Since \( x \in \tau_j \)-spcl(A), \( \{x\} \cap A \neq \emptyset \). Thus \( x \in A \).

In both the cases \( \tau_j \)-spcl(A)\( \subseteq A \). Hence, A is \( \tau_j \)-semi pre closed implies that \( (X, \tau_1, \tau_2) \) is a \( (\tau_i, \tau_j) \)-sp\( T_{\beta^*} \) space.

**Proposition 6.3.17** Every \( (\tau_i, \tau_j) \)-semi-pre-\( T_{1/2} \)-space is \( (\tau_i, \tau_j) \)-sp\( T_{\beta^*} \) space but not conversely.

**Proof:** Let \( (X, \tau_1, \tau_2) \) be a \( (\tau_i, \tau_j) \)-semi-pre-\( T_{1/2} \)-space and F be a \( (\tau_i, \tau_j) \)-\( \beta^* \)-closed. Then F is \( (\tau_i, \tau_j) \)-gsp-closed by Proposition 6.2.16. Also F is \( \tau_j \)-semi preclosed by hypothesis. Thus \( (X, \tau_1, \tau_2) \) is \( (\tau_i, \tau_j) \)-sp\( T_{\beta^*} \) space.

**Example 6.3.18** Let \( (X, \tau_1, \tau_2) \) be the space defined as in example 6.2.10. Here \( \beta^*(\tau_1, \tau_2) = \{\emptyset, \{c\}, \{b, c\}, \{a, c\}, X\} \) and \( \text{GSPC-(} \tau_1, \tau_2) = \{\emptyset, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\} \). Then \( (X, \tau_1, \tau_2) \) is a \( (\tau_1, \tau_2) \)-gsp-closed set but not \( (\tau_i, \tau_j) \)-semi-pre-\( T_{1/2} \) space since \( \{a, c\} \) is a \( (\tau_1, \tau_2) \)-gsp-closed set but it is not \( \tau_2 \)-semi-pre closed.

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6.4 \( \beta^*(\tau_i, \tau_j) \)-\( \sigma_k \) continuous maps

This section contains the concepts of \( \beta^*(\tau_i, \tau_j) \)-\( \sigma_k \)-continuity, \( \beta^* \)-bicontinuity, \( \beta^* \)-s-bicontinuity and pairwise \( \beta^* \)-irresolute maps in bitopological space. Further the properties of these maps have been studied.
**Definition 6.4.1** A map $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\beta^*(\tau_i, \tau_j)$$-\sigma_k$-continuous maps if the inverse image of every $\sigma_k$-closed set is $(\tau_i, \tau_j)^{-}\beta^*$-closed.

**Remark 6.4.2** If $\tau_1=\tau_2=\tau$ and $\sigma_1=\sigma_2=\sigma$ in Definition 6.4.1, then the $\beta^*(\tau_i, \tau_j)$$-\sigma_k$-continuous maps coincide with $\beta^*$-continuous maps in topological spaces.

**Proposition 6.4.3** If a map $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\tau_j^{-}\sigma_k$-continuous and if $(X, \tau)$ is $T_0$-space then it is $\beta^*(\tau_i, \tau_j)$$-\sigma_k$-continuous.

**Proof:** Let $V$ be a $\sigma_k$-closed set. Then $f^{-1}(V)$ is $\tau_j$-closed and by Proposition 6.2.6, $f^{-1}(V)$ is $(\tau_i, \tau_j)^{-}\beta^*$-closed in $(X, \tau_1, \tau_2)$. Therefore, $f$ is $\beta^*(\tau_i, \tau_j)$$-\sigma_k$-continuous.

**Remark 6.4.4** The converse of the Proposition 6.4.3 is not true by the following example.

**Example 6.4.5** Let $X=Y=\{a, b, c\}$, $\tau_1=\{\emptyset, \{a\}, X\}$, $\tau_2=\{\emptyset, \{b\}, X\}$, $\sigma_1=\{\emptyset, \{a\}, \{a, b\}, Y\}$ and $\sigma_2=\{\emptyset, \{b\}, \{b, c\}, Y\}$. Here $\beta^*(\tau_1, \tau_2)=P(X)$. Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be the identity map. Clearly $f$ is $\beta^*(\tau_1, \tau_2)-\sigma_2$ continuous but not $\tau_1^{-}\sigma_2$-continuous, since $\{a, c\}$ is $\sigma_2$-closed but $f^{-1}(\{a, c\})=\{a, c\}$ is not $\tau_1$-closed.

**Definition 6.4.6** A map $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called GSP $(\tau_i, \tau_j)$$-\sigma_k$-continuous, if the inverse image of every $\sigma_k$-closed set is a $(\tau_i, \tau_j)$-gsp-closed set.

**Proposition 6.4.7** If a map $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\beta^*(\tau_i, \tau_j)$$-\sigma_k$-continuous, then it is GSP$(\tau_i, \tau_j)$$-\sigma_k$-continuous.
**Proof:** Let $V$ be a $\sigma_k$-closed set, then $f^{-1}(V)$ is $(\tau_i, \tau_j)^{-}\beta^*$ closed in $(X, \tau_1, \tau_2)$. By Proposition 6.2.16, $f^{-1}(V)$ is $(\tau_i, \tau_j)$-gsp-closed and so $f$ is GSP$(\tau_i, \tau_j^-\sigma_k$-continuous.

**Remark 6.4.8** The converse of Proposition-6.4.7 need not be true. It is evident from the following example.

**Example 6.4.9** Let $X=Y=\{a, b, c\}$, $\tau_1=\{\varnothing, \{a\}, \{b, c\}, X\}$, $\tau_2=\{\varnothing, \{a\}, X\}$, $\sigma_1=\{\varnothing, \{a\}, \{a, b\}, Y\}$, $\sigma_2=\{\varnothing, \{b\}, Y\}$. Then $\beta^*(\tau_1, \tau_2)=\{\varnothing, \{b\}, X\}$ and GSP$(\tau_1, \tau_2)=P(X)\setminus\{a\}$. Let $f: (X, \tau_1, \tau_2)\to (Y, \sigma_1, \sigma_2)$ be the identity map. Then $f$ is GSP$(\tau_1, \tau_2)$-$\sigma_2$-continuous, but $f$ is not $\beta^*(\tau_1, \tau_2)$-$\sigma_2$-continuous, since $\{a, c\}$ is $\sigma_2$-closed in $(Y, \sigma_1, \sigma_2)$ but $f^{-1}(\{a, c\})=\{a, c\}$ is not $(\tau_1, \tau_2)^{-}\beta^*$-closed in $(X, \tau_1, \tau_2)$.

**Proposition 6.4.10** If a map $f: (X, \tau_1, \tau_2)\to (Y, \sigma_1, \sigma_2)$ is $D^*(\tau_i, \tau_j)$-$\sigma_k$-continuous and if $(X, \tau_i)$ is $T_{\omega}$ space then it is $\beta^*(\tau_i, \tau_j)$-$\sigma_k$-continuous.

**Proof:** Let $V$ be a $\sigma_k$-closed set. Then $f^{-1}(V)$ is $(\tau_i, \tau_j)^{-}\beta^*$-closed in $(X, \tau_1, \tau_2)$. By Proposition -6.2.9, $f^{-1}(V)$ is $(\tau_i, \tau_j)^{-}\beta^*$-closed and so $f$ is $\beta^*(\tau_i, \tau_j)$-$\sigma_k$-continuous.

The converse of Proposition 6.4.10 is not true. It is seen from the following example.

**Example 6.4.11** Let $X= Y=\{a, b, c\}$, $\tau_1=\{\varnothing, \{a\}, X\}$, $\tau_2=\{\varnothing, \{b\}, X\}$, $\sigma_1=\{\varnothing, \{b\}, \{a\}, Y\}$, $\sigma_2=\{\varnothing, \{b\}, \{c\}, \{b, c\}, Y\}$ Then $D^*(\tau_1, \tau_2)=\{\varnothing, \{a, c\}, \{b, c\}, X\}$ and $\beta^*(\tau_1, \tau_2)=P(X)$. Let $f: (X, \tau_1, \tau_2)\to (Y, \sigma_1, \sigma_2)$ be the identity map. Clearly $f$ is $\beta^*(\tau_1, \tau_2)$-$\sigma_2$-continuous but not $D^*(\tau_1, \tau_2)$-$\sigma_2$-continuous, since $\{a, b\}$ is $\sigma_2$-closed in $Y$ but $f^{-1}(\{a, b\})=\{a, b\}$ is not $(\tau_1, \tau_2)^{-}\beta^*$-closed in $X$. 

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Remark 6.4.12  We have the following diagram from the above findings, assuming that \((X, \tau_i)\) is a \(T_\omega\) space.

\[
\begin{array}{c}
D^*(\tau_i, \tau_j)-\sigma_k\text{-continuous} \\
\downarrow \\
\tau_j-\sigma_k\text{-continuity} \\
\Rightarrow \\
\beta^*(\tau_i, \tau_j)-\sigma_k\text{-continuous} \\
\downarrow \\
GSP(\tau_i, \tau_j)-\sigma_k\text{-continuous}
\end{array}
\]

Fig(vii)

Definition 6.4.13  A map \(f: (X, \tau_1, \tau_2)\rightarrow(Y, \sigma_1, \sigma_2)\) is called \(\beta^*\)-bi-continuous if \(f\) is \(\beta^*(\tau_1, \tau_2)-\sigma_2\)-continuous and \(\beta^*(\tau_2, \tau_1)-\sigma_1\)-continuous.

Definition 6.4.14  A map \(f: (X, \tau_1, \tau_2)\rightarrow(Y, \sigma_1, \sigma_2)\) is called \(\beta^*\)-strongly-bi-continuous (briefly \(\beta^*\)-s-bicontinuous) if \(f\) is \(\beta^*\)-bi-continuous, \(\beta^*(\tau_2, \tau_1)-\sigma_2\)-continuous and \(\beta^*(\tau_1, \tau_2)-\sigma_1\)-continuous.

Proposition 6.4.15  Let \(f: (X, \tau_1, \tau_2)\rightarrow(Y, \sigma_1, \sigma_2)\) be a map, and \((X, \tau_1)\) be a \(T_\omega\)-space. Then

(i) if \(f\) is bi-continuous, then \(f\) is \(\beta^*\)-bi-continuous.

(ii) if \(f\) is s-bi continuous, then \(f\) is \(\beta^*\)-s-bicontinuous.

Proof:  (i) Let \(f: (X, \tau_1, \tau_2)\rightarrow(Y, \sigma_1, \sigma_2)\) be bi-continuous. Then \(f\) is \(\tau_1-\sigma_1\)-continuous and \(\tau_2-\sigma_2\)-continuous and so by Proposition 6.4.3, \(f\) is \(\beta^*(\tau_2, \tau_1)-\sigma_1\)-continuous and \(\beta^*(\tau_1, \tau_2)-\sigma_2\)-continuous. Therefore \(f\) is \(\beta^*\)-bi-continuous.

(ii) similar to (i)
The converses of the Proposition 6.4.15 are not true. It is evident from the following example

**Example 6.4.16** Let \( X = \{a, b, c\} \), \( \tau_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\} \), \( \tau_2 = \{\varnothing, \{a\}, \{a, b\}, X\} \); \( Y = \{p, q\} \), \( \sigma_1 = \{\varnothing, \{q\}, Y\} \), \( \sigma_2 = \{\varnothing, \{p\}, Y\} \). Then \( \beta^*(\tau_1, \tau_2) = P(X) - \{a, b\} \). Define \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) by \( f(a) = f(c) = q \) and \( f(b) = p \). Then \( f \) is \( \beta^* \)-bi-continuous but not bi-continuous, since \( f \) is not \( \tau_1 - \sigma_1 \)-continuous. Also we see that \( f \) is \( \beta^* \)-s-bi-continuous but not s-bi continuous, since \( f \) is not \( \tau_2 - \sigma_1 \) continuous, that is \( f^{-1}(\{p\}) = \{b\} \) is not \( \tau_2 \)-closed.

The above findings may be shown using the following diagram. Assume that \( (X, \tau_i) \) is a \( T_\omega \) space.

![Diagram](https://via.placeholder.com/150)

**Fig (viii)**

**Definition 6.4.17** A map \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is said to be pairwise \( \beta^* \)-irresolute if the inverse image of each \( (\sigma_k, \sigma_m) \)-\( \beta^* \)-closed set of \( (Y, \sigma_1, \sigma_2) \) is \( (\tau_i, \tau_j) \)-\( \beta^* \)-closed in \( (X, \tau_1, \tau_2) \).

**Proposition 6.4.18** If \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is pairwise \( \beta^* \)-irresolute, and if \( (Y, \sigma_i) \) is a \( T_\omega \) space, then \( f \) is \( \beta^*(\tau_i, \tau_j) \)-\( \sigma_m \)-continuous.

**Proof:** Let \( F \) be any \( \sigma_m \)-closed set in \( (Y, \sigma_1, \sigma_2) \). Then \( F \) is \( (\sigma_k, \sigma_m) \)-\( \beta^* \)-closed by Proposition 6.2.6. By hypothesis \( f^{-1}(F) \) is \( (\tau_i, \tau_j) \)-\( \beta^* \)-closed in \( X \).
Remark 6.4.19  The following example shows that the converse of Proposition 6.4.19 is not true.

Example 6.4.20  Let $X = \{a, b, c\} = Y$, $\tau_1 = \{\emptyset, \{a, b\}, X\}$ and $\tau_2 = \{\emptyset, \{b\}, X\}$, $\sigma_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\sigma_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then $\beta^*(\tau_1, \tau_2) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$, $\beta^*(\sigma_1, \sigma_2) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{c, a\}, Y\}$. Define $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then $f$ is $\beta^*(\tau_1, \tau_2)$-continuous but $f$ is not pairwise $\beta^*$-irresolute since for the $(\sigma_1, \sigma_2)$-closed set $\{b\}$, $f^{-1}(\{b\}) = \{a\}$ is not $(\tau_1, \tau_2)$-$\beta^*$-closed in $(X, \tau_1, \tau_2)$.

Proposition 6.4.21  If $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ and $g: (Y, \sigma_1, \sigma_2) \to (Z, \delta_1, \delta_2)$ are two pairwise $\beta^*$-irresolute maps, then their composition is also pairwise $\beta^*$-irresolute.

Proof: Let $A$ be a $(\delta_1, \delta_2)$-$\beta^*$-closed set in $(Z, \delta_1, \delta_2)$. Since $g$ is pairwise $\beta^*$-irresolute, $g^{-1}(A)$ is $(\sigma_k, \sigma_m)$-$\beta^*$-closed in $(Y, \sigma_1, \sigma_2)$. Again by hypothesis $f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$ is $(\tau_i, \tau_j)$-$\beta^*$-closed in $(X, \tau_1, \tau_2)$ and so $g \circ f$ is pairwise $\beta^*$-irresolute.

Proposition 6.4.22  A map $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called

(i) $\tau_i$-$\sigma_m$-\(\omega\)-open if the image $f(U)$ is $\omega$-open in $\sigma_m$ for every $\omega$-open set $U$ in $\tau_i$.

(ii) $\tau_i$-$\sigma_k$-\(\omega\) continuous if for every $\sigma_k$-$\omega$-open set $U$ in $(Y, \sigma_1, \sigma_2)$ the inverse image $f^{-1}(U)$ is $\tau_i$-$\omega$-open in $(X, \tau_1, \tau_2)$ and

(iii) $\tau_j$-$\sigma_m$-$\beta$-continuous if for every $\sigma_m$-semi preclosed set $F$ in $(Y, \sigma_1, \sigma_2)$, the inverse image $f^{-1}(F)$ is $\tau_j$-semi preclosed in $(X, \tau_1, \tau_2)$. 
**Proposition 6.4.23** If \( f: (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) is bijective, \( \tau_i-\sigma_k \)-continuous, \( \tau_i-\sigma_k \)-\( \omega \)-open and \( \tau_j-\sigma_m \)-\( \beta \) continuous, then \( f \) is pairwise \( \beta^* \)-irresolute.

**Proof:** Let \( A \) be \( (\sigma_k, \sigma_m) \)-\( \beta^* \)-closed in \( (Y, \sigma_1, \sigma_2) \) and \( U \) be any \( \tau_i-\omega \)-open set in \( (X, \tau_1, \tau_2) \) such that \( f^{-1}(A) \subseteq U \). Then \( A \subseteq f(U) \). Since \( f \) is \( \tau_i-\sigma_k \)-\( \omega \)-open, \( f(U) \) is \( \sigma_k \)-\( \omega \)-open in \( (Y, \sigma_1, \sigma_2) \). Since \( A \) is \( (\sigma_k, \sigma_m) \)-\( \beta^* \)-closed in \( (Y, \sigma_1, \sigma_2) \) and \( f \) is \( \tau_i-\sigma_k \)-continuous, \( \sigma_m\text{-spcl}(A) \subseteq \text{int}(f(U)), f^{-1}(\sigma_m\text{-spcl}(A)) \subseteq f^{-1}(\text{int}(f(U))) = \text{int} U \). Also \( f \) is \( \tau_j-\sigma_m \)-\( \beta \) continuous. Therefore \( f^{-1}(\sigma_m\text{-spcl}(A)) \) is \( \tau_j \) semi pre closed in \( (X, \tau_1, \tau_2) \). Hence \( \tau_j\text{-spcl}(f^{-1}(\sigma_m\text{-spcl}(A))) \subseteq \text{int}(U) \). So \( \tau_j\text{-spcl}(f^{-1}(A)) \subseteq \tau_j\text{-spcl}(f^{-1}(\sigma_m\text{-spcl}(A))) \subseteq \text{int}(U) \) implies \( f^{-1}(A) \) is \( (\tau_i, \tau_j) \)-\( \beta^* \)-closed in \( (X, \tau_1, \tau_2) \).