CHAPTER IV

COMMON FIXED POINT THEOREMS FOR MAPPING UNDER ASYMPTOTIC REGULARITY AT A POINT
4.1. Let $T_1$ and $T_2$ be two self-maps of a metric space $(X, d)$ and \( \{x_n\} \) is a sequence in $X$ such that
\[
\lim_{n \to \infty} T_1 x_n = \lim_{n \to \infty} T_2 x_n = t \quad \text{for some } t \text{ in } X.
\]
Then $T_1, T_2$ is said to be a compatible pair if
\[
\lim_{n \to \infty} d(T_1 T_2 x_n , T_2 T_1 x_n) = 0
\]

Jungck [30] proved the following lemma for compatible mappings which was extended by many authors.

**Lemma.** If $T_1$ and $T_2$ are compatible self-maps of a metric space $(X, d)$ and
\[
\lim_{n \to \infty} T_1 x_n = \lim_{n \to \infty} T_2 x_n = t \quad \text{for some } t \text{ in } X
\]
then
\[
\lim_{n \to \infty} T_2 T_1 x_n = T_2 t \quad \text{if } T_2 \text{ is continuous}.
\]

Concerning the asymptotic regularity at a point in complete metric space, Sharma and Yuel [58] obtained a fixed point theorem for single mapping which was further generalized by Sharma and Sahu [56] for three mappings in the form of following :

**Theorem.** Let $T_1, T_2$ and $T_3$ be three self-maps of a complete metric space $(X, d)$ satisfying the following conditions:

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4.1.1 \[ d(T_3 x, T_3 y) \leq \frac{\alpha_1 d(T_2 y, T_3 y) [1 + d(T_1 x, T_3 y)]}{1 + d(T_1 x, T_2 y)} + \beta d(T_1 x, T_2 y) \]

for all \( x, y \in X \) where \( 0 \leq \alpha, \beta < 1 \)

(4.1.2) \( \{T_3, T_1\} \) is a compatible pair

(4.1.3) \( T_1 \) is continuous.

(4.1.4) \( d(x, T_2 x) \leq d(x, T_1 x) \) for all \( x \in X \)

(4.1.5) There exists an asymptotically \( T_3 \)-regular sequence w.r.t. both \( T_1 \) and \( T_2 \).

Then \( T_1, T_2 \) and \( T_3 \) have a unique common fixed point.

4.2. RESULTS. In this chapter we present the generalization of the above mentioned results in the form of following:

**Theorem 2.** Let \( T_1, T_2 \) and \( T_3 \) be three self-maps of a complete metric space \( (X, d) \) satisfying (4.1.2), (4.1.3), (4.1.4), (4.1.5) and

\[
\begin{align*}
(4.2.1) \quad d(T_3 x, T_3 y) & \leq \frac{\alpha_1 d(T_2 y, T_3 y) [1 + d(T_1 x, T_3 y)]}{1 + d(T_1 x, T_2 y)} + \frac{\alpha_2 d(T_1 x, T_3 y) [1 + d(T_2 y, T_3 y)]}{1 + d(T_3 x, T_3 y)} + \frac{\alpha_3 d(T_2 y, T_3 y) [1 + d(T_1 x, T_3 y)]}{1 + d(T_3 x, T_3 y)} \\
& + \frac{\beta d(T_1 x, T_2 y)}{1 + d(T_3 x, T_3 y)}
\end{align*}
\]
$$\alpha_2 d(T_x, T_x) [1 + d(T_{1n}, T_{3m})]$$
$$\alpha_3 d(T_{2m}, T_{3m}) [1 + d(T_{1n}, T_{3m})]$$
$$\alpha_4 d(T_{1n}, T_{2m})$$
$$\alpha_5 d(T_x, T_x)$$
$$\alpha_6 d(T_y, T_y)$$

for all $x, y \in X$, where $\alpha_1 \geq 0$, $\alpha_4 < 1$ and $\alpha_1 + \alpha_3 + \alpha_6 < 1$.

Then $T_1$, $T_2$ and $T_3$ have a unique common fixed point.

**PROOF.** For any integers $m, n$, using condition (4.2.1), we have

$$d(T_{3n} x, T_{3m} x) \leq$$
$$\frac{\alpha_2 d(T_x, T_x) [1 + d(T_{1n}, T_{3m})]}{1 + d(T_{1n}, T_{2m})}$$
$$\frac{\alpha_3 d(T_{2m}, T_{3m}) [1 + d(T_{1n}, T_{3m})]}{1 + d(T_{3n}, T_{3m})}$$
$$\frac{\alpha_4 d(T_{1n}, T_{2m}) + \alpha_5 d(T_x, T_x) + \alpha_6 d(T_y, T_y)}{1 + d(T_{1n}, T_{3m})}$$

$$\alpha_1 d(T_{2m}, T_{3m}) [1 + d(T_{1n}, T_{3m})]$$
$$\alpha_2 d(T_{1n}, T_{3n}) [1 + d(T_{2m}, T_{3n})]$$
$$\frac{\alpha_3 d(T_{3n}, T_{3m}) [1 + d(T_{1n}, T_{3m})]}{1 + d(T_{3n}, T_{3m})}$$
$$\frac{\alpha_4 d(T_{1n}, T_{2m}) + \alpha_5 d(T_x, T_x) + \alpha_6 d(T_y, T_y)}{1 + d(T_{1n}, T_{3m})}$$

$$d(T_{3n} x, T_{3m} x) \leq$$
\[ + a \sum_{1 \leq i < j \leq \infty} \left\{ d(T_{1n}, T_{3n}) + d(T_{2n}, T_{3n}) + d(T_{3n}, T_{2m}) \right\} \]

\[ + a d(T_{1n}, T_{3n}) + a d(T_{2n}, T_{3n}) \]

Taking the limit as \( m, n \to \infty \) and using (4.1.5), we get

\[ \lim_{m, n \to \infty} d(T_{x}, T_{x}) \leq a \lim_{m, n \to \infty} d(T_{x}, T_{x}) \]

\[ \Rightarrow \lim_{m, n \to \infty} d(T_{3n}, T_{x}) = 0 \]

which implies that \( \{T_{3n}\} \) is a Cauchy sequence and hence by completeness of \( X \), there is a \( z \) in \( X \) such that \( T_{3n} \to z \).

Since

\[ d(T_{1n}, z) \leq d(T_{1n}, T_{3n}) + d(T_{3n}, z) \]

\[ \to 0 \text{ as } n \to \infty, \text{ so } T_{1n} \to z. \]

Similarly

\[ T_{2n} \to z. \]

Using conditions (4.1.2), (4.1.3) and lemma, we obtain

\[ T_{2n} \to T_{z} \text{ and } T_{3n} \to T_{z}. \]

Further condition (4.2.1) gives

\[ d(T_{3n}, T_{2n}) \leq \frac{a d(T_{x}, T_{x}) [1 + d(T_{x}, T_{3n})]}{1 + d(T_{x}, T_{x})} \]
\[ \alpha_2 d(T_x, T_T x) \left[ 1 + d(T_x, T_T x) \right] \]
\[ \frac{a_d(T_x, T_T x) \left[ 1 + d(T_x, T_T x) \right]}{1 + d(T_{3n}, T_{3n})} \]
\[ l + d(T_{3n}, T_{3n}) \]
\[ \frac{a_d(T_x, T_T x) \left[ 1 + d(T_x, T_T x) \right]}{1 + d(T_{3n}, T_{3n})} \]
\[ \frac{2}{l + d(T_{3n}, T_{3n})} \]
\[ + a_d(T_x, T_T x) + a_d(T_x, T_T x) + a_d(T_x, T_T x) \]

Now taking the limit as \( n \to \infty \), we obtain

\[ d(T_{1z}, z) \leq \frac{1}{4} d(T_{1z}, z) \]

a contradiction. Hence \( T_{1z} = z \). From (4.1.4) we deduce that

\[ d(z, T_{2z}) \leq d(z, T_{1z}) = 0 \]

i.e. \( T_{2z} = z = T_{1z} \) (4.2.2)

Again by (4.2.1) and (4.2.2), we have

\[ d(T_{31n}, T_{3z}) \leq \frac{2}{l + d(T_{1n}, T_{2z})} \]
\[ + \frac{a_d(T_x, T_T x) \left[ 1 + d(T_x, T_T x) \right]}{l + d(T_{T_T x}, T_{z})} \]
\[ + \frac{2}{l + d(T_{T_T x}, T_{z})} \]
\[
\alpha d(T z, T z) \left[ \frac{1 + d(T x, T z)}{1 + d(T x, T z)} \right] + \frac{2}{4} \alpha d(T x, T z) + \frac{2}{6} \alpha d(T x, T x) + \frac{2}{2} \alpha d(T z, T z)
\]

Taking \( n \to \infty \), we get

\[
d(T z, T z) \leq (\alpha_1 + \alpha_2 + \alpha_3) d(T z, T z).
\]

This implies that \( T z = T z \) as \( \alpha_1 + \alpha_2 + \alpha_3 < 1 \). Thus \( z \) is a common fixed point of \( T_1 \), \( T_2 \) and \( T_3 \).

For uniqueness, if possible let \( y \neq z \) be another common fixed point of \( T_1 \), \( T_2 \) and \( T_3 \) then condition (4.2.1) gives

\[
d(y, z) = d(T y, T z) \leq \frac{\alpha_1 d(T z, T z) \left[ 1 + d(T y, T y) \right] + \frac{2}{4} \alpha d(T y, T y) + \frac{2}{5} \alpha d(T y, T y) + \frac{2}{6} \alpha d(T y, T y) + \frac{2}{3} \alpha d(T y, T y)}{1 + d(T y, T y)}
\]

\[
\leq \alpha_4 d(y, z)
\]
a contradiction. Hence \( y = z \) and this completes the proof of the theorem.

**REMARK 1.** A result analogous to theorem 2 can be formulated supposing \( T_2 \) continuous, 

\[
\text{\textquoteleft\textquoteleft} d(x, T_1 x) \leq d(x, T_2 x) \text{\textquoteright\textquoteright}
\]

for all \( x \) in \( X \) and requiring the pair \( \{ T_3, T_2 \} \) to be compatible.

**REMARK 2.** Assumption of condition (4.1.4) is necessary in theorem 2. The following example shows that \( z \) is not a common fixed point of \( T_1, T_2 \), and \( T_3 \). Although all the conditions of theorem 2 are satisfied except condition (4.1.4).

**EXAMPLE 1.** Consider \( X = [0, 1] \) with the usual metric \( d \) and define

\[
T_1 x = \frac{x}{4} \quad \text{for all } x \text{ in } X
\]

\[
T_2 x = \begin{cases} 
3/4, & \text{if } x = 0 \\
\frac{x}{2}, & \text{if } x \neq 0
\end{cases}
\]

\[
T_3 x = \begin{cases} 
\frac{1}{3}, & \text{if } x = 0 \\
\frac{x}{5}, & \text{if } x \neq 0
\end{cases}
\]

Then we have

\[
d(T_3 T_0, T T_0) = d(T_3 0, T T_0) = d(\frac{1}{3}, \frac{1}{12})
\]

\[
= \frac{1}{4} < \frac{1}{3} = d(T_0 T_0, T_0)
\]

and

\[
T_3 T_1 x = T T_3 x = \frac{x}{20} \quad \text{for any } x \neq 0.
\]

Thus \( \{ T_3, T_1 \} \) is a weakly commuting pair and hence compatible.

Let \( \{ x_n \} \) be a sequence converging to zero such that \( x_n \neq 0 \) for any positive integer \( n \). Then
\[
\lim_{n \to \infty} d(T^3 x_n, T^2 x_n) = \lim_{n \to \infty} \frac{x_n}{20} = 0 \\
\lim_{n \to \infty} d(T^3 x_n, T^2 x_n) = \lim_{n \to \infty} \frac{x_n}{4} = 0.
\]

So \([x_n]\) is asymptotically \(T_3\)-regular sequence w.r.t. \(T_1\) and \(T_2\). Also \(T_1\) is continuous for all \(x\) in \(X\). To verify condition (4.2.1), we have:

**Case I.** When \(x \neq 0\), \(y \neq 0\) then condition (4.2.1) reduces to

\[
\frac{x-y}{5} \leq \frac{1}{2} \left[1 + \frac{x}{4} - \frac{x}{5}\right] + \frac{1}{4} \left[1 + \frac{y}{4} - \frac{y}{5}\right] + \frac{1}{5} \left[1 + \frac{x}{5} - \frac{y}{5}\right] + \frac{1}{6} \left[1 + \frac{x}{2} - \frac{y}{5}\right].
\]

(1) When \(y > x\), then \(2y > x\). Taking \(a_4 = \frac{1}{12}\), \(a_5 = a_6 = \frac{11}{12}\), \(a_1 = a_2 = a_3 = 0\), we get

\[
\frac{y-x}{5} \leq \frac{1}{12} (2y-x) + \frac{11}{20} \left(\frac{x}{20}\right) + \frac{11}{12} \left(\frac{3y}{10}\right)
\]

iff \(48y - 43x \leq 106y - 9x\)

which is true.

(ii) When \(x > y\) then taking \(a_5 \geq 4\), \(a_1 = 0\), \(1 = 1, 2, 3, 4\), \(0 \leq a_6 < 1\), we have
\[-75-\]

\[
\frac{x-y}{5} \leq \frac{x}{5} + \frac{3y}{10}.
\]

**Case II.** When \(x = 0, y \neq 0\) then condition (4.2.1) become

\[
\left| \frac{1}{3} - \frac{y}{5} \right| \leq \frac{2a_1y}{5[1+\frac{y}{2}]} + \frac{\alpha_2[1+\frac{y}{2} - \frac{1}{3}]}{3[1+\frac{1}{3} - \frac{y}{5}]} \]

\[
\frac{\alpha_3}{2} \left[ \frac{y}{2} - \frac{5}{3} \right] \left[ 1 + \frac{1}{3} - \frac{y}{5} \right] + \frac{\alpha_4}{2} + \frac{\alpha_5}{3} + \frac{\alpha_6}{10}.
\]

(i) If \(\frac{1}{3} > \frac{y}{5}\) then taking \(\alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = \frac{4}{5}\),
\(\alpha_5 = 3, \alpha_6 = \frac{2}{3}\), we obtain
\[
\frac{1}{3} - \frac{y}{5} \leq \frac{2}{5}y + 1 + \frac{y}{5} = \frac{3y}{5} + 1.
\]

(ii) If \(\frac{1}{3} < \frac{y}{5}\) then taking \(\alpha_1 = \alpha_2 = \alpha_3 = 0, \alpha_4 = \frac{4}{5}, \alpha_5 = 1\),
\(\alpha_6 = \frac{2}{3}\), we get
\[
\frac{y}{5} - \frac{1}{3} \leq \frac{2}{5}y + \frac{1}{3} + \frac{y}{5} = \frac{3y}{5} + \frac{1}{3}.
\]

**Case III.** When \(x \neq 0, y = 0\) then condition (4.2.1) reduces to

\[
\left| \frac{x}{5} - \frac{1}{3} \right| \leq \frac{\alpha_1 \left[ \frac{3}{4} - \frac{1}{3} \right] \left[ 1 + \frac{x}{4} - \frac{x}{5} \right]}{1 + \left| \frac{x}{4} - \frac{3}{4} \right|} + \frac{\alpha_2 \left[ \frac{x}{4} - \frac{x}{5} - \frac{1}{3} \right]}{1 + \frac{x}{5} - \frac{1}{3} !}
\]
\[ \frac{\alpha_1 \frac{3}{4} - \frac{1}{3} \left[ \frac{1+\frac{x}{4} - \frac{1}{4}}{3} \right]}{1 + \frac{x}{5} - \frac{1}{3}} + \alpha_4 \frac{x}{4} - \frac{3}{4} + \alpha_5 \frac{1-x}{5} + \frac{x}{6} \frac{3}{4} - \frac{1}{3} \cdot \]

(1) When \( \frac{x}{5} \geq \frac{1}{3} \) then taking \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \), \( \alpha_4 = \frac{4}{5} \), \( \alpha_5 = 5 \), \( \alpha_6 = \frac{4}{5} \) we have
\[
\frac{x}{5} - \frac{1}{3} \leq \frac{9x}{20} - \frac{4}{15}.
\]

(11) When \( \frac{x}{5} \leq \frac{1}{3} \) then putting \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \), \( \alpha_5 = 1 \), \( \alpha_6 = \frac{9}{10} \), we obtain
\[
\frac{1}{3} - \frac{x}{5} \leq \frac{x}{20} - \frac{3}{8}.
\]

The above relations show that condition (4.2.1) is satisfied in each case. When \( x = 0 \), \( y = 0 \) then also condition (4.2.1) is satisfied trivially. Thus all the assumptions of theorem 2 are satisfied except condition (4.1.4) at \( x = 0 \) and clearly \( T_1 \), \( T_2 \) and \( T_3 \) have no common fixed point in \( X \).

The following example illustrates the validity of our theorem 2.

**Example 2.** Let \( X = [0, 1] \) with the usual metric \( d \) and define
\[
T_1 x = \frac{x}{4}, \quad T_2 x = \frac{x}{2}, \quad T_3 x = \frac{x}{5}
\]
for all \( x \) in \( X \).

Then as shown in example 1, all the assumptions of theorem 2 are satisfied and clearly \( x = 0 \) is the unique common fixed point of \( T_1 \), \( T_2 \) and \( T_3 \).

Now we extend the result of theorem 2 for five mappings and obtain the following:
THEOREM 3. Let $T_i, i = 1, 2, \ldots, 5$ be five self-maps of a complete metric space $(X, d)$ satisfying the following conditions:

\[
\alpha d(T_i T_j y, T_j y) \left[ 1 + d(T_i T_j x, T_j x) \right] \\
\frac{1}{1 + d(T_i T_j x, T_j y)} \\
\alpha_1 d(T_i T_j x, T_j x) \left[ 1 + d(T_i T_j y, T_j x) \right] \\
\frac{1}{1 + d(T_i T_j x, T_j y)} \\
\alpha_2 d(T_i T_j x, T_j y) \left[ 1 + d(T_i T_j y, T_j x) \right] \\
\frac{1}{1 + d(T_i T_j x, T_j y)} \\
\alpha_3 d(T_i T_j y, T_j y) \left[ 1 + d(T_i T_j x, T_j y) \right] \\
\frac{1}{1 + d(T_i T_j x, T_j y)} \\
\alpha_4 d(T_i T_j x, T_j y) + \alpha_5 d(T_i T_j x, T_j y) \\
\frac{1}{1 + d(T_i T_j x, T_j y)} \\
\alpha_6 d(T_i T_j y, T_j y) \\
\frac{1}{1 + d(T_i T_j x, T_j y)}
\]

for all $x, y \in X$, where $0 \leq \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 < 1$

(4.2.4) $T_i T_j = T_j T_i, T_i T_j = T_j T_i, T_i T_j = T_j T_i, T_i T_j = T_j T_i$

(4.2.5) $\{T_i, T_j\}$ is a compatible pair.

(4.2.6) $T_1, T_2$ are continuous

(4.2.7) $d(x, T_i T_j x) \leq d(x, T_i T_j x)$ for all $x$ in $X$

(4.2.8) There exists an asymptotically $T_i$-regular sequence w.r.t. both $T_i T_j$ and $T_i T_j$. 
Then $T_i$, $i = 1, 2, \ldots, 5$ have a unique common fixed point.

**Proof.** From condition (4.2.3), we have for any positive integers $m, n$.

$$d(T x, T x) \leq \frac{\alpha_1 d(T T x, T x) [1 + d(T T x, T x)]}{1 + d(T x, T x)} + \frac{\alpha_2 d(T x, T x) [1 + d(T T x, T x)]}{1 + d(T x, T x)} + \frac{\alpha_3 d(T x, T x) [1 + d(T T x, T x)]}{1 + d(T x, T x)} + \frac{\alpha_4 d(T T x, T T x) + \alpha_5 d(T T x, T x)}{1 + d(T x, T x)} + \frac{\alpha_6 d(T T x, T x)}{1 + d(T x, T x)}$$
\[ + \alpha \left\{ d(T_T^{2m}x, T_T^m x) + d(T^m x, T_T^x) + d(T_T^x, T_T^{34m} x) \right\} \]

\[ + \alpha d(T_T^x, T_T^x) + \alpha d(T_T^x, T_T^x) \]

\[ = \frac{5}{12} n, \frac{5}{n} \quad \frac{6}{34} m, \frac{5}{m} \]

Taking the limit as \( m, n \to \infty \) and using (4.2.2), we get

\[ \lim \left( \lim_{m,n \to \infty} d(T^m x, T_T^x) \right) = 0 \]

which implies that \( \{ T_T^x \}_{5} \) is a Cauchy sequence and hence by completeness of \( X \), there is a \( z \) in \( X \) such that \( T_T^x \to z \).

Since

\[ d(T_T^x, z) \leq d(T_T^x, T_T^x) + d(T_T^x, z) \]

\[ \to 0 \text{ as } n \to \infty \]

So

\[ T_T^{12} x_n \to z \]

and similarly \( T_T^{34} x_n \to z \).

Applying the conditions (4.2.5), (4.2.6) and the lemma of Jungck, we have

\[ (T_T^{12})^2(x_n) \to T_T^{12}z \text{ and } T_T^{12}(T_T^{34}x_n) \to T_T^{12}z \]

Further condition (4.2.3) gives

\[ d(T(T_T^{34}x_n), T_T^{12}x_n) \leq \frac{\alpha d(T_T^{12}x_n, T_T^{34}x_n) \left[ 1 + d(T_T^{12}x_n, T_T^{34}x_n) \right]}{1 + d(T_T^{12}x_n, T_T^{34}x_n)} \]
\[
\begin{align*}
&\alpha_2 d((T_{12}^2 x, T_{512}^n x_n) \left[ 1 + d(T_{512}^n x, T_{512}^n x) \right] \\
&+ \frac{1 + d(T_{512}^n x, T_{512}^n x)}{1 + d(T_{512}^n x, T_{512}^n x)} \\
&+ \frac{1 + d(T_{512}^n x, T_{512}^n x)}{1 + d(T_{512}^n x, T_{512}^n x)} \\
&+ \frac{1 + d(T_{512}^n x, T_{512}^n x)}{1 + d(T_{512}^n x, T_{512}^n x)} \\
&\alpha_3 d(T_{34}^n x, T_{5}^n x) \left[ 1 + d((T_{12}^2 x, T_{5}^n x) \right] \\
&+ \frac{1 + d(T_{512}^n x, T_{512}^n x)}{1 + d(T_{512}^n x, T_{512}^n x)} \\
&+ \frac{1 + d(T_{512}^n x, T_{512}^n x)}{1 + d(T_{512}^n x, T_{512}^n x)} \\
&+ \frac{1 + d(T_{512}^n x, T_{512}^n x)}{1 + d(T_{512}^n x, T_{512}^n x)} \\
&\alpha_4 d((T_{12}^2 x, T_{5}^n x) + \alpha_5 d((T_{512}^n x, T_{512}^n x) \\
&+ \alpha_6 d(T_{512}^n x, T_{512}^n x)
\end{align*}
\]

Taking limit as \( n \to \infty \), we have

\[
\begin{align*}
d(T_{12}^n z, z) &\leq \alpha_4 d(T_{12}^n z, z) \\
\text{a contradiction. Hence } T_{12}^n z &= z
\end{align*}
\tag{4.2.9}
\]

From (4.2.7), we obtain

\[
\begin{align*}
d(z, T_{34}^n z) &\leq d(z, T_{12}^n z) = 0 \\
i.e. & T_{34}^n z = z
\end{align*}
\tag{4.2.10}
\]

Again by (4.2.3), (4.2.9) and (4.2.10), we have

\[
\begin{align*}
&\alpha_1 d(T_{34}^n z, T_{5}^n z) \left[ 1 + d((T_{12}^2 x, T_{5}^n x) \right] \\
&d(T_{512}^n x, T_{512}^n x) \leq \frac{1 + d((T_{12}^2 x, T_{34}^n z) \\
&\alpha_2 d((T_{12}^2 x, T_{512}^n x) \left[ 1 + d(T_{34}^n z, T_{512}^n x) \right] \\
&+ \frac{1 + d(T_{512}^n x, T_{512}^n x)}{1 + d(T_{512}^n x, T_{512}^n x)} \\
&\alpha_4 d((T_{12}^2 x, T_{512}^n x) + \alpha_6 d(T_{512}^n x, T_{512}^n x)
\end{align*}
\]
\[ \alpha_d(T_2 z, T z) = d(T T z, T z) = d(T T z, T z) \]
\[ \leq \frac{\alpha_d(T T z, T z) [1 + d(T T z, T T z)]}{1 + d(T T z, T T z)} \]
\[ \leq \frac{\alpha_d(T T z, T z) [1 + d(T T z, T T z)]}{1 + d(T T z, T T z)} \]
\[ \leq \frac{\alpha_d(T T z, T z) [1 + d(T T z, T T z)]}{1 + d(T T z, T T z)} \]
\[ \leq \frac{\alpha_d(T T z, T z) [1 + d(T T z, T T z)]}{1 + d(T T z, T T z)} \]
\[
\sum_{d(4, \overline{4})} \sum_{d(3, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right] \\
+ \frac{1 + d(4, \overline{4})}{5} \sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

This contradicts. Hence \( T_2 z = z \) and then from (4.2.11) we get
\( T_1 z = z \).

Now if possible let \( z \neq T_4 z \), then using (4.2.11),
(4.2.3) and (4.2.4), we obtain
\[
d(4, \overline{4}) = d(4, \overline{4}) = d(4, \overline{4}) = d(4, \overline{4})
\]
\[
\sum_{d(4, \overline{4})} \sum_{d(3, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right] \\
+ \frac{1 + d(4, \overline{4})}{5} \sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

\[
\sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

\[
\sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

\[
\sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

\[
\sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

\[
\sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

\[
\sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

\[
\sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

\[
\sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

\[
\sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]

\[
\sum_{d(5, \overline{4})} \sum_{d(6, \overline{4})} \left[ \frac{1 + d(4, \overline{4})}{5} \right]
\]
a contradiction. Therefore $T_4z = z$ and then (4.2.11) implies that $T_3z = z$.

Combining the above results, we have

$$T_1z = T_2z = T_3z = T_4z = z.$$ 

The uniqueness follows directly from (4.2.3).

**REMARK 3.** Putting $T_1 = T_2 = T_3 = T_4 = I$ (Identity map), $a_2 = a_3 = a_5 = a_6 = 0$, $a_1 = a$, $a_4 = b$ we get the theorem of Sharma and Yuel [58].

**REMARK 4.** Taking $T_2 = T_4 = I$ (Identity map)

$$T_5 = T_3 = T_2, \quad a_2 = a_3 = a_5 = a_6 = 0, \quad a_1 = a, \quad a_4 = b$$

we obtain theorem 1 of Sharma and Sahu [56].

**REMARK 5.** If we put $T_4 = T_2 = I$ (Identity map),

$$T_5 = T_3, \quad T_3 = T_2$$

in theorem 3 then it reduces to our theorem 2 for three mappings.

The following example illustrates the validity of our theorem 3.

**EXAMPLE 3.** Let $X = [0, 1]$ be the subset of reals with the usual metric $d$. Let $T_i : X \to X$, $i = 1, 2, 3, \ldots, 5$ be mappings defined as follows:

$$T_1x = \frac{2x}{3}, \quad T_2x = \frac{x}{4}, \quad T_3x = \frac{x}{2}, \quad T_4x = \frac{x}{2}, \quad T_5x = \frac{x}{10}$$

for all $x \in [0, 1]$.

Clearly for all $x, y \in [0, 1]$, we have
(4.2.4) \[ T_{T} x = \frac{x}{5} = T_{T} x, \quad T_{T} x = \frac{x}{20} = T_{T} x \]

\[ T_{T} x = \frac{x}{12}, \quad T_{T} x, \quad T_{T} x = \frac{x}{4} = T_{T} x \]

\[ (4.2.5) \quad T_{T} (T_{T} x) = \frac{x}{52} = T_{T} (T_{T} x) \]

\[ : \{T_{5}, T_{12}\} \text{ is a commuting pair and hence compatible.} \]

(4.2.6) \[ T_{1}, T_{2} \text{ are continuous} \]

(4.2.7) \[ d(x, T_{T} x) = \frac{3x}{3} \leq \frac{5x}{6} = d(x, T_{T} x) \]

(4.2.8) \[ \text{By choosing a sequence} \{x_{n}\}, \quad x_{n} \neq 0 \text{ for any positive integer } n, \text{ converging to zero, we deduce that} \]

\[ \lim_{n \to \infty} d(T_{n} x, T_{T} x) = \lim_{n \to \infty} | \frac{x_{n}}{10} - \frac{x_{n}}{6} | = \lim_{n \to \infty} \frac{x_{n}}{15} = 0 \]

\[ \lim_{n \to \infty} d(T_{n} x, T_{T} x) = \lim_{n \to \infty} | \frac{x_{n}}{10} - \frac{x_{n}}{4} | = \lim_{n \to \infty} \frac{3x_{n}}{20} = 0 \]

(4.2.3) becomes

\[ \frac{x}{2} \leq \frac{y}{10} \left[ 1 + \frac{x}{6} - \frac{x}{10} \right] + \frac{y}{10} \left[ 1 + \frac{y}{4} - \frac{y}{10} \right] \]

\[ \frac{y}{2} \leq \frac{x}{2} \left[ 1 + \frac{y}{6} - \frac{x}{10} \right] + \frac{x}{10} \left[ 1 + \frac{y}{4} - \frac{x}{10} \right] \]
Case I. When \( x > y \) then taking \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \)
\[ \alpha_3 > \frac{3}{2}, \ 0 \leq \alpha_6 < 1, \] we get
\[
\frac{x - y}{10} \leq \frac{x}{15} \cdot \frac{3}{2} + \frac{3y}{20} \cdot 0
= \frac{x}{10}
\]

Case II. When \( y > x \) then taking \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \)
\[ \alpha_5 > 0, \ \frac{2}{3} \leq \alpha_6 < 1, \] we obtain
\[
\frac{y - x}{10} \leq \frac{x}{15} \cdot 0 + \frac{3y}{20} \cdot \frac{2}{3}
= \frac{y}{10}
\]

which implies that condition (4.2.3) is also satisfied in this case. Thus all the assumptions of theorem 3 hold and hence \( T_i, i = 1, 2, \ldots, 5 \) have a unique common fixed point namely 0 in \([0, 1]\).