CHAPTER VI

SOME FIXED POINT THEOREMS ON
CONTRACTIVE TYPE MAPPINGS
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6.1. Pachpatte [39] proved some unique fixed point theorems for a mapping T of a metric space (X, d) into itself satisfying

\[ d(Tx, Ty) \leq q \max \left\{ d(x, y) ; \frac{d(y, Ty)[1+d(x,Tx)]}{1+d(x,y)} \right\} \]

\[ \frac{1}{2} \cdot \frac{d(x,Ty) \left[ 1+d(x,Tx) + d(y,Tx) \right]}{1 + d(x,y)} \]

for all x, y in X, where 0 < q < 1.

Using the technique of Rhoades [48] for a self mapping T of a closed convex subset X of a normed space satisfying contractive condition (6.1.1) of Pachpatte, Yuel and Sharma [66] have shown that if \{x_n\}, the sequence of Mann iterates associated with T converges in X then it converges to a fixed point of T.

The same result has also been extended for two mappings.

The object of the present chapter is to generalize the result of Yuel and Sharma [66] and to obtain two fixed point theorems for self mappings defined on a subset of normed space using G-iterative process.

6.2. RESULTS! We establish the following:

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THEOREM 1. Let $X$ be a closed convex subset of a normed linear space $N$ and let $T$ be a self mapping on $X$ satisfying contractive condition (6.1.1) and ${\{x_n\}}$ be the sequence of $G$-iterates associated with $T$ defined as follows:

Let $x_0, x_1 \in X$ and for $n \geq 0$

$$x_{n+2} = (\lambda_n - \mu_n) x_{n+1} + \lambda_n T x_{n+1} + (1 - \mu_n) T x_n$$

where $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy

(i) $\lambda_0 = \mu_0 = 1$

(ii) $0 < \lambda_n < 1$, $n > 0$ and $\mu_n \geq \lambda_n$

for $n \geq 0$

(iii) $\lim_{n \to \infty} \lambda_n = h > 0$

(iv) $\lim_{n \to \infty} \mu_n = 1$

If ${\{x_n\}}$ converges in $X$, then it converges to a fixed point of $T$.

PROOF. Let $z \in X$ such that

$$\lim_{n \to \infty} x_n = z .$$

We shall show that $z$ is the fixed point of $T$. Now consider

$$\|z - T z\| \leq \|z - x_{n+2}\| + \|x_{n+2} - T z\|$$

$$\leq \|z - x_{n+2}\| + \|(\mu_n - \lambda_n) x - T z\|$$

$$+ \lambda_n \|T x_{n+1} - T z\|$$

$$\leq \|z - x_{n+2}\| + (\mu_n - \lambda_n) \|x - T z\|$$

$$+ \lambda_n \|T x_{n+1} - T z\| + (1 - \mu_n) \|T x_n - T z\|$$

$$\leq \|z - x_{n+2}\| + (\mu_n - \lambda_n) \|x_{n+1} - T z\|$$
\[ \| z - Tz \| \leq \| z - x \| + \| x - Tz \| + \lambda_n q \max \left\{ \| x - Tz \| \left[ 1 + \| x - Tz \| + \frac{1}{\lambda_n} \| x - x \| \right] \right\}, \]

\[ \| x - Tz \| \left[ 1 + \| x - Tz \| + \frac{1}{\lambda_n} \| x - x \| \right] \]

\[ + \lambda_n q \max \left\{ \| x - Tz \| \left[ 1 + \| x - Tz \| + \frac{1}{\lambda_n} \| x - x \| \right] \right\}, \]

\[ l + \| x - z \| \]

\[ + (1-\mu_n) \| Tz - Tz \| \]

We observe that

\[ \| x - Tz \| = \frac{1}{\lambda_n} \| x - x \| + \frac{1-\mu_n}{\lambda_n} \| Tz - x \| \]

and

\[ \| z - Tz \| \leq \| z - x \| + \| x - Tz \| \]

\[ = \| z - x \| + \frac{1}{\lambda_n} \| x - x \| \]

\[ + \frac{1-\mu_n}{\lambda_n} \| Tz - x \| . \]

Therefore the above inequality reduces to

\[ \| z - Tz \| \leq \| z - x \| + (\mu_n - \lambda_n) \| x - Tz \| + \lambda_n q \max \left\{ \| x - Tz \|, \right\}, \]

\[ \| z - Tz \| \left[ 1 + \| x - Tz \| + \frac{1-\mu_n}{\lambda_n} \| Tz - x \| \right] \]

\[ l + \| x - z \| \]
Letting \( n \to \infty \) and using (iii) and (iv), we obtain
\[
\|z - Tz\| \leq (1-h)\|z - Tz\| + hq \max\left\{ 0, \|z - Tz\|, \frac{1}{2} \|z - Tz\| \right\}
\leq (1-h+hq)\|z - Tz\|
\]
a contradiction. Hence \( z = Tz \).
i.e. \( z \) is a fixed point of \( T \).

**REMARK 1.** Taking \( \{ \mu_n \} = \{ 1 \} \) theorem 1 reduces to theorem 1 of Yuel and Sharma [66].

Now we present an example to prove the validity of our theorem 1.

**EXAMPLE 1.** Let \( N = R^4 \), where \( R^4 \) is the set of all 4-tuples \( x = (x_1, x_2, x_3, x_4) \) of real numbers and the norm \( \|x\| \) is defined by
\[
\|x\| = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}, x \in R^4
\]
Further, let \( X = \{ x : \|x - 0\| \leq 1, 0, x \in R^4 \} \) and \( T \) be the mapping from \( X \) into itself such that for any arbitrary \( x = (x_1, x_2, x_3, x_4) \in X \)
\[
Tx = \left( \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{4}, \frac{x_4}{5} \right)
\]
Suppose \( \{x_n\} \) be the sequence of elements of \( X \) such that for \( n \geq 0 \)
\[
x_{n+2} = \left( \frac{\mu_n}{n} \right) x_{n+1} + \lambda_n T x_{n+1} + \left( 1 - \frac{\mu_n}{n} \right) T x_n
\]
where
\[
\{ \mu_n \} = \begin{cases} \frac{2n+3}{2n+4} & , \ n > 0 \text{ and } \mu_0 = 1 \\ \{ \frac{n+1}{2n+1} \} , & , \ n \geq 0
\end{cases}
\]

Consider \( x_0 = (\frac{3}{4}, 0, 0, 0) \), \( x_1 = (\frac{1}{2}, 0, 0, 0) \) then it can be easily seen that \( x_2 = (\frac{1}{4}, 0, 0, 0) \), \( x_3 = (\frac{1}{6}, 0, 0, 0) \), \( x_4 = (\frac{107}{960}, 0, 0, 0) \) etc.

Now it is easy to see that all the conditions of theorem 1 are satisfied, for instance taking \( y = x_2 \), we obtain
\[
||T x - T y|| = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2} ||x-y||
\]
Setting \( \frac{1}{2} < q < 1 \), condition (6.1.1) is satisfied. Clearly \( x = (0, 0, 0, 0) \) i.e. \( 0 \) is the unique fixed point of \( T \).

**REMARK 2.** If the set \( X \) is not closed or convex then mapping \( T \) may not have fixed point in \( X \).

The following example illustrates this fact.

**EXAMPLE 2.** Let \( N = \mathbb{R} \), the set of all real numbers regarded as a normed space. Let \( X = [-1, 0) \cup (0,1] \) and define mapping \( T \) of \( X \) into itself such that
\[
T x = \frac{-x}{3} \text{ for all } x \in X.
\]

Let \( \{x_n\} \) be the sequence of elements of \( X \) as defined in
example 1. Let \( x_0 = -1, x_1 = -\frac{1}{2} \) then \( x_2 = \frac{1}{6}, x_3 = \frac{1}{54} \) etc.

Now it is easy to see that all the conditions of theorem 1 are satisfied except that \( X \) is neither closed nor convex.

Clearly \( T \) has no fixed point in \( X \).

We extend the result of theorem 1 for a pair of mappings in the form of following:

**Theorem 2.** Let \( X \) be a closed, convex subset of a normed linear space \( N \) and let \( T_1, T_2 \) be two self mappings on \( X \) satisfying

\[
\|T_1x - T_2y\| \leq q \max \left\{ \|x - y\|, \frac{\|y - T_2y\| \left[ 1 + \|x - T_1x\| \right]}{1 + \|x - y\|}, \|x - T_2y\| \left[ 1 + \|x - T_1x\| + \|y - T_1x\| \right] \right\}
\]

for all \( x, y \in X \), where \( 0 < q < 1 \)

and \( \{x_n\} \) be the sequence of \( G \)-iterates associated with \( T_1 \) and \( T_2 \) defined as follows:

Let \( x_0, x_1 \in X \) and for \( n \geq 0 \)

\[
x_{2n+2} = (\mu - \lambda) x_{2n+1} + \lambda \frac{T_1 x_{2n+1} + (1 - \mu) T_2 x_{2n+1}}{n} + (1 - \mu) T_2 x_{2n+1}
\]

\[
x_{2n+3} = (\mu - \lambda) x_{2n+2} + \lambda \frac{T_1 x_{2n+2} + (1 - \mu) T_2 x_{2n+2}}{n} + (1 - \mu) T_2 x_{2n+1}
\]

where \( \{\mu\} \) and \( \{\lambda\} \) satisfy (i), (ii), (iii) and (iv).

If \( \{x_n\} \) converges in \( X \), then it converges to the common fixed point of \( T_1 \) and \( T_2 \).
PROOF. Let \( z \in X \) such that \( \lim_{n \to \infty} x_n = z \). We shall show that \( z \) is the common fixed point of \( T_1 \) and \( T_2 \). Now consider

\[
\|z - T_1 z\| = \|z - x\|_{2n+3} + \|x - T_1 z\|_{2n+3}
\]

\[
\leq \|z - x\|_{2n+3} + \|\frac{\lambda - \mu}{n} x\|_{2n+2} + \frac{\lambda}{n} \|x - T_1 z\|_{2n+2}
\]

\[
+ (1 - \frac{\lambda}{n}) \|T_x - T_1 z\|_{2n+3}
\]

\[
\leq \|z - x\|_{2n+3} + \|\frac{\lambda - \mu}{n} x\|_{2n+2} + \frac{\lambda}{n} \|x - T_1 z\|_{2n+2}
\]

\[
+ (1 - \frac{\lambda}{n}) \|T_x - T_1 z\|_{2n+3}
\]

\[
\leq \|z - x\|_{2n+3} + \|\frac{\lambda - \mu}{n} x\|_{2n+2}
\]

\[
\leq \max_{n} \{ \|z - x\|_{2n+2}, \frac{\|x - T_1 z\|_{2n+2}}{2} \}
\]

\[
\frac{1}{2} \left( \frac{\|z - T_2 x\|_{2n+2}}{2} + \frac{\|x - T_1 z\|_{2n+2}}{2} \right)
\]

\[
+(1 - \frac{\lambda}{n}) \|T_x - T_1 z\|_{2n+3}
\]

We observe that

\[
\|x - T_2 x\|_{2n+2} = \frac{1}{\lambda_n} \|x - x\|_{2n+2} + \frac{1 - \mu}{\lambda_n} \|T_x - T_1 z\|_{2n+2}
\]

and
Using these conclusions in the above inequality, we obtain

\[
\|z - T_1 z\| \leq \|z - x\| + (\mu - \lambda_n) \|x - T_1 z\| + \lambda q \max\left\{\|z - x\|, \|T_1 z - x\| \right\} \left[1 + \|z - T_1 z\|\right]
\]

\[
\frac{1 - \mu}{\lambda_n} \|z - T_1 z\| + \frac{1 - \mu}{\lambda_n} \|T_1 x - x\| \leq \frac{1 - \mu}{\lambda_n} \|z - T_1 z\| + \frac{1 - \mu}{\lambda_n} \|T_1 x - x\| \leq \frac{1 - \mu}{\lambda_n} \|x - T_1 z\|
\]

\[
\left[\frac{1}{\lambda_n} \|z - x\| + \frac{1}{\lambda_n} \|T_1 x - x\| + \frac{1 - \mu}{\lambda_n} \|T_1 x - x\|\right] \left[1 + \|z - T_1 z\|\right]
\]

Taking limit as \( n \to \infty \) and using (iii), (iv) we have

\[
\|z - T_1 z\| \leq (1 - h) \|z - T_1 z\| + h q \max\{0, 0, 0\}
\]

\[
\Rightarrow \|z - T_1 z\| \leq (1 - h) \|z - T_1 z\|
\]

a contradiction. Therefore \( z = T_1 z \) i.e. \( z \) is a fixed point of \( T_1 \).
Similarly we can show that

$$||z-T_2z|| \leq (1-h)||z-T_2z||.$$

Hence $z$ is a common fixed point of $T_1$ and $T_2$. Uniqueness follows from condition (6.2.1).

**REMARK 3.** When we put $\{\mu_n\} = \{1\}$ then theorem 2 reduces to theorem 2 of Yuel and Sharma [66].

Finally, we furnish an example to discuss the validity of theorem 2.

**EXAMPLE 3.** Let $N = 4$ and

$$X = \{x : ||x-0|| < 1, \ 0, x \in R^4\}$$

as defined in example 1. Suppose that $T_1, T_2$ be two self mappings on $X$ defined by

$$T_1x = \left( \frac{x_1}{3}, \frac{x_2}{2}, \frac{x_3}{4}, \frac{x}{5} \right)$$

and

$$T_2x = (x_1, x_2, x_3, x_4)$$

for any $x = (x_1, x_2, x_3, x_4) \notin X$.

Let $\{x_n\}$ be the sequence of point of $X$ such that for $n \geq 0$

$$x_{2n+2} = (\mu - \lambda)x_n + \lambda T_1x_n + (1-\mu)T_2x_n$$

and

$$x_{2n+3} = (\mu - \lambda)x_n + \lambda T_1x_n + (1-\mu)T_2x_n$$

where
\[
\{ \lambda_n \} = \left\{ \frac{n+1}{2n+1} \right\}, \quad n \geq 0
\]

\[
\mu_0 = 1 \quad \text{and} \quad \left\{ \mu_n \right\} = \left\{ \frac{2n+3}{2n+4} \right\}, \quad n > 0.
\]

Let \( x_0 = \left( \frac{3}{4}, 0, 0, 0 \right), \quad x_1 = \left( \frac{1}{2}, 0, 0, 0 \right) \) \( \in X \)

then it is easy to see that

\[
x_2 = \left( \frac{1}{6}, 0, 0, 0 \right), \quad x_3 = \left( \frac{1}{6}, 0, 0, 0 \right), \quad x_4 = \left( \frac{5}{54}, 0, 0, 0 \right), \text{etc.}
\]

Now, it is easy to see that all the conditions of theorem 2 are satisfied and clearly \( x = (0, 0, 0, 0) \) i.e. \( 0 \) is the unique common fixed point of \( T_1 \) and \( T_2 \).