CHAPTER - 2

PARAMETRIC DECAY OF A WHISTLER WAVE AT THE DIFFERENCE FREQUENCY OF TWO ELECTROMAGNETIC WAVES IN A PLASMA

2.1 Introduction

Wave mixing was used as a means of increasing the usually very small cross sections for scattering of laser light for plasma diagnostics. When two intense beams of electromagnetic radiation are incident on a plasma, the electrons acquire a velocity \( \mathbf{v} \) component in the direction of the superposed electric fields \( \mathbf{E} \) of the radiation. As a result of this motion, a lorentz force \( \mathbf{F} = \mathbf{v} \times \mathbf{B} \) is produced by the magnetic field \( \mathbf{B} \) of the radiation. It is evident that the force is proportional to \( \mathbf{E}^2 \), so that for two beams of different frequency \( \omega_a \) and \( \omega_b \) and wave vector \( \mathbf{k}_a \) and \( \mathbf{k}_b \), the lorentz force will contain components of frequency \( \Delta \omega = |\omega_a - \omega_b| \) and direction \( \Delta \mathbf{k} = |\mathbf{k}_a - \mathbf{k}_b| \). By arranging \( \Delta \omega \) and \( \Delta \mathbf{k} \) to coincide with the parameters of a normal mode of the plasma, electromagnetic wave can be used for resonant excitation of the mode. The laser plasma beat-wave accelerator is a potential contender that utilizes the large electric field of the high power laser beams in a plasma. In this scheme a large amplitude electrostatic electron-plasma wave is generated at the beat frequency of two collinear laser beams in a plasma.
with phase velocity slightly less than the velocity of light in a vacuum (Darrow et al 1986). Such plasma processes frequently occur in space also.

Naturally occurring atmospheric lightning radiates a broad spectrum of electromagnetic waves (Martelli et al 1974; Shoucri et al 1982). In principle ionospheric discharges and high-frequency acoustic waves can trigger secondary radiation of whistler waves by the mechanism of beating of two high-frequency electromagnetic waves. In ionospheric experiments the part of the radiated whistler wave that is earthbound undergoes damping due to collisions while travelling through the ionospheric D region. Under favourable conditions, the signal may be detected at magnetically conjugate points or by orbiting satellites (Shoucri et al 1982).

Absorption at the level of the beat-excited mode, i.e. conversion into electrostatic waves, is assumed to be the driving mechanism of the parametric instability. In the presence of the interplanetary magnetic field a plasma can support a variety of plasma modes. There has been a great deal of interest in the non-linear propagation of whistler waves in magnetized plasmas (Shukla et al 1986). Since large amplitude whistler waves are employed for supplementary plasma heating as well as in beat-wave particle accelerators (Katsouleas & Dawson 1983), their nonlinear interaction with the background plasma cannot be ignored.
We identify the instability as decay of a whistler wave (excited at the difference frequency of two electromagnetic pump waves) into two plasmons, i.e., a high-frequency lower-hybrid wave and a low-frequency ion-Bernstein wave in a multi-ion-species plasma. Parametric decays are among the most promising heating mechanisms for plasmas (Porkolab & Chang 1977; Tripathi & Sharma 1988). The idea of heating by parametric decay is to pump power into modes at technically available frequencies; the pump modes themselves are not effectively absorbed by the plasmas, but one or both of the decay modes are (Krause et al. 1980).

In this chapter, plasma heating by the decay waves in a multi-species plasma is considered. Particular attention is paid to ion heating by parametrically excited ion-Bernstein mode. The lower-hybrid waves can also energize ions normal to the interplanetary magnetic field (Marsch & Chang 1983). They may occasionally be responsible for the observed temperature anisotropy in high-speed streams and possibly for the acceleration of heavier ions such as oxygen ions and alpha particles. Ion heating using excitation of low-frequency waves in a multi-species plasma has been suggested by Kitsenko & Stepanov (1973) and Kaw & Lee (1973). Parametric instabilities in a multi-species plasma are of interest because the threshold fields in such a plasma can be lower than in a single-species plasma (Baikov 1977).
In the following sections nonlinear wave equations are obtained for the lower hybrid and ion - Bernstein wave, and, after analysing the growth rate, results are discussed, together with their applications.

2.2 Mathematical analysis

We consider the propagation of two collinear high-frequency electromagnetic waves \( (\omega_a, \mathbf{k}_a) \) and \( (\omega_b, \mathbf{k}_b) \) along the same direction in a uniform, collisionless and magnetized plasma:

\[
\mathbf{E}_{a,b} = \mathbf{E}_{a,b} \exp \{-i (\omega_{a,b} t - k_{a,b} x)\}
\]

where \( k_{a,b} = \frac{\omega_{a,b}}{c} \left[ 1 - \frac{\omega_{pe}^2}{\omega_{a,b}^2} \right]^{1/2} \).

\( \omega_{a,b} \) and \( k_{a,b} \) are the angular frequencies and wave vectors of the incident electromagnetic waves, and \( \omega_{pe} = (4 \pi n_0 e^2/m)^{1/2} \) is the electron plasma frequency. \( e, m, n_0 \) and \( c \) being the electron charge, electron rest mass, equilibrium unperturbed electron density and velocity of light in vacuum. Because of the nonlinear interaction of the incident electromagnetic wave with the plasma, a large-amplitude electromagnetic wave in the whistler mode \( (\omega_c, \mathbf{k}_c; \omega_c = \omega_a - \omega_b, \mathbf{k}_c = \mathbf{k}_a - \mathbf{k}_b) \) is generated at the difference frequency through a parametric process or a resonant excitation mechanism. The
excited wave propagates obliquely to the magnetic field, and
the frequency and wavenumber are related by the cold-plasma
dispersion relation

$$\omega^2 = k_0^2 c^2 + \frac{\omega_{pe}^2}{\omega - \omega_{ce}}$$

which is valid for $\omega - \omega_{ce} \gg k_z c_e$ where $c_e = T_e/m_e$ is the
electron thermal speed, $T_e$ being the electron temperature in
units of the Boltzmann constant, and $\omega_{ce} = eB_0/m_ec$ is the
electron-cyclotron frequency.

We now take this excited electromagnetic whistler wave at
the difference frequency of the two high-frequency electromagnetic
wave to be the pump wave $(\omega_p, k_0)$, which parametrically
decays into a high-frequency lower-hybrid mode $(\omega_1, k_1)$ and a
low-frequency ion-Bernstein mode $(\omega_2, k_2)$. All three waves are
assumed to propagate in the $(x,z)$ plane in a multi-species
(two ion species and one electron species) homogeneous low-$\beta$
plasma ($\beta = 8\pi n_0(T_e + T_i)/B_0^2 < 1$) embedded in a background
magnetic field $B_0^2$. The nonlinearity arises owing to the
ponderomotive force only.

The basic equations governing the dynamics of an electro-
static wave are

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \vec{v}_j) = 0. \quad (2.1)$$
where $n_j$ is the plasma density of species $j = \alpha, e$ ($\alpha = \text{ion A or ion B, e = electron}$), $m_j, e_j, v_j$ and $c_j$ are the mass, charge, velocity and thermal velocity of species $j$, $\Phi$ is the electrostatic potential of the wave and

$$\vec{F}_j = -\left[ m_j (\vec{v}_j, \vec{v}) \vec{v}_j - \frac{e_j}{c} (\vec{v}_j \times \vec{B}) \right]$$

represents the ponderomotive force on species $j$. In terms of the potential, $\vec{F}_j = -e \vec{E}_p j$.

### 2.3 Lower-hybrid wave equation

For the lower-hybrid wave to exist, we must have the condition $\omega_{ce} > \omega_1 > \omega_{c\alpha}$, where $\omega_{c\alpha} = eB_0/m_j c$ is the cyclotron frequency of the charged-particle species and $\alpha = \text{A or B}$ represents the two ion species. We must also have $\omega_1 > k_z c$, and therefore the ions may be taken to be cold and unmagnetized. Now applying the perturbation technique and taking the dependence of all the variable quantities as

$$\exp \left[ i (\omega t - k_x \hat{x} - k_z \hat{z}) \right],$$

we obtain the perturbed densities for the lower-hybrid wave from equations (2.1) and (2.2) as
\[ n_{e1} = \frac{(-e/m_e)n_{eo} \left( \phi_1 + \phi_{pe,1} \right) k_x^2/\Delta + k_z^2 \phi_1 + \phi_{pe,1}}{\Delta_e} \frac{1}{\omega_1^2} \]

\[ n_{k1} = \frac{e(n_{OA} + n_{OB})}{m_A + m_B} \left[ \frac{k_x^2 \phi_1 + \phi_{pe,1}}{\omega_1^2} + \frac{k_z^2 \phi_1 + \phi_{pe,1}}{\omega_1^2} \right] \]

where

\[ \Delta_e = 1 - \frac{k_x c_e^2}{\Delta} - \frac{k_z c_e^2}{\omega_1^2} \]

\[ \Delta = \omega_1^2 - \omega_{ce}^2 \]

Using \( n_{e1} \) and \( n_{k1} \) in Poisson's equation (2.3) and neglecting the ponderomotive force on ions as being weak, we obtain the nonlinear dispersion relation for a lower-hybrid wave as

\[ \epsilon_1 \phi_1 = \mu_1 \phi_0 \phi_2 \tag{2.4} \]

where

\[ \mu_1 = \frac{-e}{4m_e \omega_0 \omega_2} \frac{x_{e1}}{k_0 k_2} \tag{2.4'} \]

is the coupling coefficient and

\[ \epsilon_1 = 1 - x_{e1} \tag{2.5} \]

gives the linear dispersion relation for the lower-hybrid wave; here,

\[ x_{k1} = \frac{\omega_{pe}^2}{k_1^2 \Delta_e} \left( \frac{k_x^2 + k_z^2}{\omega_1^2} \right) \quad (\alpha = A, B), \]

\[ x_{e1} = \frac{\omega_{pe}^2}{k_1^2 \Delta_e} \left[ \frac{k_x^2}{\Delta} + \frac{k_z^2}{\omega_1^2} \right] \]

are the dielectric susceptibilities.
2.4 Ion-Bernstein wave equation

The low-frequency ion-Bernstein mode is an electrostatic plasma mode that occurs in the parameter regime.

\[ \omega_{ce} > \omega_2 ; \omega_2 > k_{2e} \cdot c_\text{e} ; k_{2e} \cdot k_\text{e} < 1 ; \]

i.e. in the long-wavelength case, where we can use the fluid equations (2.1) and (2.2). We shall consider oscillations near the ion-cyclotron frequency \( \omega_{ci} \).

Again applying the perturbation technique and Fourier analysing, we obtain the perturbed electron and ion densities as

\[
\begin{align*}
n_{e2} &= \frac{-e}{m_e} n_{eo} \left[ \frac{k_{2e}^2 (\phi_2 + \phi_{pe,2})}{\Delta} + \frac{k_{2e}^2 (\phi_2 + \phi_{pe,2})}{\omega_2^2} \right] \\
n_{i2} &= \frac{e (n_{oA} + n_{oB})}{(m_A + m_B) \Delta_i} \left[ \frac{k_{2i}^2 \phi_i}{\Delta} + \frac{k_{2i}^2 \phi_i}{\omega_2^2} \right]
\end{align*}
\]

The low-frequency ponderomotive force \( \vec{F}_{pe} = -e \nabla \phi_{pe} \) originates from the beating of the pump wave and the side band, and enhances the density perturbation. The low-frequency ion response is assumed to be linear.

Using Poisson's equation, we obtain the nonlinear wave equation for the ion-Bernstein mode as

\[
\epsilon_2 \phi_2 = \frac{1}{2} \phi_c \phi_1
\]

(2.6)
where

$$
\mu_2 = \frac{-e^x e_2 k_0 \cdot k_1}{4m_e \omega_0 \omega_1}
$$

(2.6')

is the coupling coefficient and

$$
\varepsilon_2 = 1 - X e(2) - X \alpha(2) - X \alpha(2')
$$

(2.7)

is the linear dispersion relation for the electrostatic ion-Bernstein mode; here

$$
X e(2) = \frac{\omega^2_{Be}}{\omega_2^2} \frac{k_2^2}{k_{2x}^2}
$$

$$
X \alpha(2) = \frac{\omega^2_{p\alpha}}{\Delta}
$$

$$
X \alpha(2') = \frac{k_{2x}^2 \omega^2_{p\alpha}}{\Delta^2}
$$

are the dielectric susceptibilities, since for the ion-Bernstein mode $\omega - \omega c_{\alpha} > k_2 c_{\alpha}$.

2.5 The dispersion relation for the decay instability

On combining (2.4) and (2.6), we obtain the coupled nonlinear dispersion relation as

$$
\varepsilon_1 (\omega_1, \vec{k}_1) \varepsilon_2 (\omega_2, \vec{k}_2) = \mu_1 \mu_2 |E_{o2}|^2
$$

(2.8)

where $\varepsilon_1 (\omega_1, \vec{k}_1)$ and $\varepsilon_2 (\omega_2, \vec{k}_2)$ are given by (2.5) and (2.7) respectively. Thus, in the absence of the pump wave, we obtain the following dispersion relations:
(i) the high-frequency decay mode as the lower-hybrid wave described by the dispersion relation (Ott 1975) $\epsilon_1 = 0$ in a cold-plasma system,

$$k_1^2 - k_1^2 = -\frac{\omega_{pe}^2}{\omega_1^2} + -\frac{\omega_{pe}^2}{\omega_{ce}^2} k_{1x}^2 - -\frac{\omega_{pe}^2}{\omega_1^2} k_{1z}^2 = 0$$

Where $\omega_1$ is the lower-hybrid wave frequency;

(ii) the low-frequency decay product satisfying the dispersion relation for the ion-Bernstein wave (Kumar & Sharma 1989) i.e. $\epsilon_2 = 0$,

$$1 - -\frac{\omega_{pe}^2}{\omega_2^2} k_{2z}^2 - -\frac{\omega_{pe}^2}{\omega_{ce}^2} - -k_{2x}^2 - \frac{\omega_{pe}^2}{\omega_2^2} c_{e\alpha}^2 - \frac{\omega_{pe}^2}{\omega_{ce}^2} \left( \frac{\omega_2^2 - \omega_{c\alpha}^2}{\omega_{ce}^2} \right)^2 = 0.$$ 

In the cold-plasma approximation, i.e. $\omega_2^2 - \omega_{ce}^2 k_{2x}^2 c_{e\alpha}^2$, and in the high density region $\omega_{pe}^2 >> \omega_2^2 - \omega_{c\alpha}^2$ we obtain the usual cold-plasma ion-Bernstein wave dispersion relation for $\omega_2 \leq \omega_{c\alpha}$ (Kumar & Sharma 1989) as

$$\omega_2 \approx \omega_{c\alpha} \frac{k_{2z}^2}{k_{2x}} \left[ \frac{m_\alpha}{m_e} \right]^{1/2} \left[ 1 + \frac{m_\alpha}{m_e} \frac{k_{2z}}{k_{2x}} \right]^{1/2}$$

Since $\epsilon_1 = 0$ and $\epsilon_2 = 0$, for the above decay process we must have

$$\eta^2 = \frac{-\mu}{(\partial \epsilon_1 / \partial \omega_1) (\partial \epsilon_2 / \partial \omega_2)}$$
where \( \gamma \) is the growth rate. Expanding, we have

\[
\omega_{1,2} = \omega_{1,2}(r) + i\gamma_{1,2}
\]

\[
\epsilon_1 = i(\gamma_1 + \gamma_{L1}) \frac{\partial \epsilon_1}{\partial \omega_1}
\]

\[
\epsilon_2 = i(\gamma_2 + \gamma_{L2}) \frac{\partial \epsilon_2}{\partial \omega_2}
\]

where \( \gamma_{L1} \) and \( \gamma_{L2} \) are the linear damping constants of the corresponding mode. In our case the damping constants are zero, the system being collisionless. Using expansions of the coupled nonlinear dispersion relation (2.8), the growth rate of the three-wave interaction process can be obtained as

\[
\gamma^2 = \frac{\mu_1 \mu_2}{RS}
\]

where

\[
R = \frac{\partial \epsilon_1}{\partial \omega_1} = \frac{2 \omega^2}{\omega_1^3} \left[ 1 + \frac{n_{0B} m_A}{n_{0A} m_B} \right] \frac{k_{1x}^2 + k_{1z}^2}{k_1^2}
\]
\[ s = \frac{2 \varepsilon_2}{\omega_2} = \frac{2 \omega_p^2 k^2}{\omega^3 k^2} + \frac{2 \omega_2^2}{\omega^2 p_A} \left[ 1 + \frac{n_{0B}}{n_{0A}} \right] \]

\[ \left[ \frac{\omega^2 - \omega_A}{c_A} \left[ 1 + \frac{m_A^2}{m_B^2} \right] \right] \]

and the coupling constants are given by (2.4') and (2.6'). Since for the decay modes \( k_x \ll k_x \) and \( k^2 \rho^2 \ll 1 \) (long-wavelength case), for \( k_{0x} \), \( k_{1x} \) and \( k_{2x} \) of the same order, the growth rate in the frequency regime

\[ \omega_{c} < \omega_1 < \omega_{ce}, \quad \omega_{c\alpha} < \omega_2 < 2 \omega_{c\alpha} \]

is given roughly by
\[
\gamma' = \frac{e E_0 z}{8 m_e \omega_o} \frac{\omega_1 \omega_2}{\omega_p A} \left( \frac{k z}{k^2 \omega_2} \right)^{1/2} \left( \frac{x e_1 x e_2}{x \frac{m_A}{m_B}} \right)^{1/2} \left[ 1 + \frac{n_{oB}}{n_{oA}} \frac{m_A}{m_B} \right]^{-1/2}
\]

\[
x \left[ \omega_p e^{-\frac{k^2}{k^2 2x \omega_p}} + \omega_p A \left[ 1 + \frac{n_{oB}}{n_{oA}} \frac{m_A}{m_B} \right]^{-1/2} \right]
\]

\[
x \left[ \frac{2k^2}{2x \omega_2} \frac{T_e}{m_A} \left[ 1 + \frac{m_A}{m_B} \right] \right]^{-1/2}
\]

where we have assumed \( T_e > T_A = T_B \).

2.6 Discussion

In this paper ion heating in connection with parametric decay at the beat frequency has been discussed and explained analytically as heating by parametrically excited low-frequency ion-Bernstein wave (Tripathi & Sharma 1988) in a multi-ion-species plasma (Dash, Sharmma & Buti 1984; Sharma, Yashvir & Bhatnagar 1986; Milic & Krstic 1987). It has been found that the growth rate can be controlled by several parameters, such as the mass, density and temperature of two ion species. It is also clear that the growth rate depends upon coupling coefficients, decay wave frequencies and magnetic field. It increases with
increasing magnetic field and concentration ratio of two ions, and decreases with increasing ion temperature, ion mass ratio and pump-wave frequency. At $\omega_1 = \omega_{ce}$ the growth rate is zero. For small pump fields the decay into lower-hybrid and ion-Bernstein waves clearly dominates owing to its dependence on $|E_0|$ rather than $|E_0|^2$.

Multi-species plasmas with low $B$ exist in the earth's ionosphere, magnetosphere and upper ionosphere, and in type III solar bursts. The ion composition ratio changes with height. In such regions amplified ULF hisses of longer wavelengths occur during magnetic storms. The result of our investigation suggests that the whistler wave excited at the beat frequency should stimulate finite-amplitude ion-Bernstein and lower-hybrid waves.

The relevance of our work to the ionospheric plasma is now considered. In spite of the low system efficiency, ionospheric experiments detect signals at comparable levels. The beating of two high-frequency electromagnetic waves gives rise to the radiation of a whistler at a frequency $6.28 \times 10^5$ rad s$^{-1}$ (Shoucri et al. 1982). Taking parameters typical of the auroral ionosphere,

- $B = 0.33$ G; $n_{oe} = 10^5$ cm$^{-3}$,
- $n_{OA} = (1-3) \times 10^4$ cm$^{-3}$; $n_{OB} = 5 \times 10^5$ cm$^{-3}$,
- $k = 10^{-3}$ cm$^{-1}$; $\omega_1 \approx 10^4$ rad s$^{-1}$,
- $\omega_2 = 17.1$ rad s$^{-1}$; $c_e = 2.56 \times 10^4$ cm s$^{-1}$,
the growth rate for this decay process is
\[ \gamma = 8.4 \text{ s}^{-1} \]

Figures (2.1) and (2.2) show the variation of growth rate with the concentration ratio of two ion species and the magnetic field respectively. Thus it can be seen that by varying the ratio \( n_{0A}/n_{0B} \) and the magnetic field \( B_0 \) the growth rate can be controlled. The wave is thus expected to modify the dynamics of the space plasmas, and the ion-Bernstein wave as the low-frequency decay mode is assumed to be excited to a sufficiently high level to explain the observed ion heating.

The decay instability of an excited whistler wave at the difference frequency of two electromagnetic pump waves may lead to difficulties in accelerating particles uniformly in plasma beat-wave accelerators (Katsouleas & Dawson 1983; Shukla, Yu & Stenflo 1986).

For parameters realizable in a large laboratory plasma device, a whistler wave excites at a frequency 1.884 MHz. Taking typical parameters for an accelerator plasma (Shoucri et al. 1982),

\[ n = 10^{17} - 10^{20} \text{ cm}^{-3}, \quad T_\alpha = 13 \text{ keV}, \]
\[ B_0 = (90 - 600) \text{ kG}, \quad k = 4.33, \]

the growth rate for this decay process is
\[ \gamma = 2.82 \times 10^5 \text{ s}^{-1} \]
Figure 2.1 Variation of growth rate $\gamma$ with concentration ratio $\frac{n_{0A}}{n_{0B}}$. 
Figure 2.2 Variation of growth rate $\gamma$ with magnetic field $B_0$. 
Thus the process is relevant to experiments. The present paper suggests an interesting hypothesis for the origin of low-frequency ion-mode fluctuations observed in the presence of large amplitude whistler waves in both a laboratory and ionospheric plasma study. These decay instabilities however are expected to play an important role in the saturation of the parametrically excited whistlers.
REFERENCES


