

Chapter-4

DESIGN OF AN EXTENDED KALMAN FILTER FOR THE ULTRA-TIGHT GPS/INS INTEGRATION

4.1 INTRODUCTION

In Ultra-tight GPS/INS integration, the integration was carried out for I (in-phase) and Q (quadrature) variables from correlator of GPS receiver with velocity, position and attitude from INS and it even refers to very closely linked relationships among the GPS and INS variables. In loosely and tightly integrated system INS aiding of GPS receiver is optional as per Farrell & Barth (1999) [28], whereas in the case of ultra-tight integration it is inherent feature that offers the several benefits for the system. Doppler that is derived from INS navigation information and GPS satellite ephemeris is fed to tracking loop of receiver to diminish the dynamics on GPS signal which enters the tracking loops. In conventional GPS receiver, tracking loops are more sensitive to receiver dynamics like jerk or acceleration that transform as Doppler rate and rate change of Doppler rate on GPS signal. These dynamics on GPS signal must be tracked by tracking the loops continuously for offering consistent navigation measurements. Yet, the tracking loops mislay track of signal whenever the dynamics go above threshold of loops [44, 46]. In order track the dynamics continuously in an unaided or conventional receiver, carrier tracking loop bandwidth is made wider and results in raise in its thermal noise on measurements degrading signal to noise ratio. As a result, in conventional receiver optimising the tracking loop bandwidth in order
to receive the dynamic signals and simultaneously maintaining the accuracy of measurements is very difficult, if not impossible [7]. This inconsistent situation can be addressed by accepting the ultra-tight integration architecture.

INS that is collocated with GPS antenna also measures the vehicle dynamics on which it is mounted. If the navigation data from INS can be converted into Doppler domain (Doppler and Doppler rate) then it can be easily integrated with the GPS tracking loops in order to diminish the Doppler on GPS signal. With Doppler because of vehicle dynamics on GPS signal diminished, the costas tracking loop bandwidth can be substantially reduced and results in a significant threshold improvement [42, 65]. On the other hand, the first question which come up when assessing the ultra-tight system are: how I, Q variables are related with the position and velocity? Its very normal since these variables, so far normally used in context of data demodulation, power calculations in order to switch among the various loops (acquisition, wideband/narrowband tracking) and discriminator corrections. Perhaps, for the initial time these measurements are utilized for estimating the errors in external systems like INS, although such methods were adoyed in stand-alone receiver in implementations of VDLL [3]. In ultra-tight integration, the integration of Kalman filter makes use of these dimensions in order to calculate approximately the inertial sensor errors during the
measurement update process, and later removes those errors from raw inertial measurements.

It's very essential to set up a relationship among the states and measurements in order to carry out the measurement update process as with any other Kalman filter design. Here, in this case the states are inertial sensor errors, INS navigation measurements and receiver clock errors whereas the measurements I and Q are acquired through multiple channels of correlator. In order to set up this relationship a good understanding of composite GPS signal is necessary as every other parameter inside the receiver is estimated from this composite signal. This section, even shows that the I and Q estimates from correlator are related to velocity and position data from INS that is states-measurements relationship via phase and frequency errors. Here, the accent is placed upon the measurement relationship matrix since it is unique when compared with the loosely and tightly integrated systems. In order to prove the validity of theoretical argument and derived mathematical relationships, the simulation experiments were carried out. Filter performance has been evaluated by assessing the error differences among the original and estimated trajectories. Further, the results of covariance analysis were also provided. Hence, the results prove that the theoretical arguments and mathematical assumptions that are considered in the design of Kalman filter are valid.
4.2 ANALYSIS OF ULTRA-TIGHT GPS/INS INTEGRATION

The block diagram of inertial aided carrier tracking loop of ultra-tight integrated system is represented in Fig. 4.1. Doppler signal that is inertially derived is integrated with the carrier NCO in order to eliminate the dynamics on GPS signal caused by receiver movement.

In the conventional GPS receiver, both the code and carrier loops of each channel and produces the corrections in order to align the locally generated signals with incoming signal based on their respective discriminator functions (FLL, CPLL and DLL). All these correction signals are low pass filtered in order to take away any of the high frequency noise and high order terms on them. Generally, the carrier-tracking bandwidth is in the order of 5 to 15 Hz for low dynamic signals and more than 20 Hz for high dynamic signals. Higher bandwidths results in degradation of measurements because of the
increased thermal noise [7]. Ultra-tight tracking loops (where the tracking loops are aided with INS derived Doppler) can circumvent this issue by just tracking the high dynamic signals without raising the bandwidth. The carrier-tracking loop bandwidth can be decreased to even less than 3Hz under high dynamic scenarios.

The basic principle behind the ultra-tight integration is, if INS can compute the same vehicle dynamics as GPS receiver and somehow integrated with tracking loops, then the dynamics on GPS signal as seen by receiver tracking loops are reduced significantly. With the help of almanac or ephemeris data, a Doppler signal is derived from INS navigation solution and then integrated with carrier tracking loops. If multipath is ignored then the ionospheric and satellite oscillator are effect. So, the tracking loops are then need to track the receiver clock dynamics. In ultra-tight integration, both I and Q signals are produced from tracking loops through the measurements in Kalman filter and thus approximate the inertial sensor errors during measurement update process. Since the GPS is referred as the sensor in this architecture, INS error states can be approximated even with less than four satellites.

4.2.1 Mathematics of Phase and Frequency Estimates

The mathematical expression model for L1 C/A signal is (because of its orthogonal nature, the P code can be conveniently removed from the analysis) given as:
\[ y(t) = AC(t-\pi)D(t-\tau)\cos(\omega*(t-\tau)+\phi_d + \eta) \] ... (4.1)

where,

- \( A \) – amplitude of signal
- \( C(t) \) – C/A code sequence
- \( D(t) \) – 50Hz navigation data
- \( \omega \) – L1 angular frequency in radians/sec
- \( \tau \) – propagation delay between the satellite and receiver
- \( \phi_d \) – L1 carrier phase
- \( \eta \) – Gaussian noise

By ignoring the atmospheric and oscillator effects, propagation delay (\( \tau \)) is expanded as:

\[ \tau = \frac{|X_s(t) - X_u(t)|}{c} \] ... (4.2)

where,

- \( X_s(t) \) – GPS satellite position
- \( X_u(t) \) – user receiver position
- \( c \) – velocity of light

By referring both the satellites and receivers motion, in equation (4.2) the numerator is expanded through Taylor’s series into:

\[
|X_s(t) - X_u(t)| \approx |X_s(t_o) - X_u(t_o)| + \frac{d}{dt} |X_s(t_o) - X_u(t_o)| (t-t_o) \\
+ \frac{1}{2} \frac{d^2}{dt^2} |X_s(t_o) - X_u(t_o)| (t-t_o)^2 
\] ... (4.3)
Here, in this case the dynamics are limited up to the order 2 that is acceleration, $t_0$ specify the time at a reference point. Substitute the equations (4.2) and (4.3) into equation (4.1) yield:

$$y(t) = AC(t-\tau) \cos \left( \omega \left( t - \frac{X_S(t_o) - X_u(t_o)}{c} - \frac{v_r}{c} (t - t_o) - \frac{a_r}{c} (t - t_o)^2 \right) + \phi_d(t) \right) - \eta$$

... (4.4)

where,

$$v_r = \frac{d}{dt} |X_S(t_o) - X_u(t_o)|$$

and

$$a_r = \frac{1}{2} \frac{d^2}{dt^2} |X_S(t_o) - X_u(t_o)|$$

Rearranging the Eqn. (4.4) gives:

$$y(t) = AC(t-\tau)D(t-\tau) \cos \left( \omega \left( 1 - \frac{v_r}{c} - \frac{a_r}{c} (t + 2t_o) \right) \right) + \eta$$

... (4.5)

The Eqn. (4.5) can be rewritten as:

$$y(t) = AC(t-\pi)D(t-\pi) \cos(\omega't + \phi') + \eta$$

... (4.6)

where,

$$\omega' = \omega \left( 1 - \frac{v_r}{c} - \frac{a_r}{c} (t + 2t_o) \right)$$

and

$$\phi' = -\frac{\omega}{c} \left( |X_S(t_o) - X_u(t_o)| - v_r t_o + a_r t_o^2 \right) + \phi_d$$

... (4.7)
In Eqn. (4.7), the acceleration term is ignored and leads to the simplified expression as:

$$\omega' = \omega \left(1 - \frac{v_r}{c}\right) \quad \text{and} \quad \phi' = -\frac{\omega}{c} \left(\|X_s(t_o) - X_d(t_o)\| - v_r t_o\right) + \phi_d \quad \text{(4.8)}$$

where,

- $\omega'$ — receiver carrier frequency and
- $\phi'$ — Phase of GPS signal at receiver input

The equation (4.8) represents how the frequency and phase are related with the position and velocity. The main function of tracking loops is to just track these two parameters to extract the navigation information. A frequency lock loop tracks the carrier frequency ($\omega'$) and while the Costas phase lock loop tracks carrier phase ($\phi'$).

### 4.2.2 Mathematics of I and Q Estimates

Further to formulate the measurement relationship, a relation is defined among $I$ and $Q$ signals with phase and frequency errors. At antenna the composite GPS signal is down converted and digitized by RF down converter. In order to demodulate 50 bps navigation information from pseudorandom noise (PRN) code and carrier, an $I$ component is produced by multiplying the digitized IF by NCO output and $Q$ component by multiplying 90° phase shifted NCO output. Because of the presence of Doppler on signal, NCO output is shifted in frequency and phase in order to drive the output of multiplier to zero frequency. Both the frequency and phase corrections ($\omega_c$ and $\phi_c$) to NCO is acquired from discriminator algorithms that in turn identifies
the magnitude and direction of correction for aligning the frequency $(\omega')$ and phase $(\phi')$ of incoming carrier signal. Code is removed by multiplying the each component of $I$ and $Q$ signals through three replicas of C/A code where those are separated by about 0.5 chip. These three replicas are generally termed as early, prompt and late. Multiplier outputs from $I_E$, $I_P$, $I_L$ from I-arm and $Q_E$, $Q_P$, $Q_L$ from Q-arm are utilized by code discriminator algorithms to produce the correction ($\sigma_e$) in order to align the incoming code signal. Multiplied signals are thereafter integrated over the pre-detection integration interval $T$. for regular operation, the prompt signals must be precisely aligned such a way that the output power $\sqrt{I_P^2 + Q_P^2}$ is at the peak of cross-correlation function. At this state, the magnitudes of early and late outputs are smaller. Both the carrier and code stripping are carried out in conjunction with the each other that is those are done in parallel.

Fig.4.2: GPS receiver tracking loop showing the Early, Prompt and Late $I$ and $Q$ signals to the discriminator algorithms
Mathematically relating the I, Q signals with \( \omega', \phi' \) are explained here (in order to make the analysis simpler and code signals are not considered). Let us consider

\[
\hat{\omega} - \text{Local frequency} \\
\hat{\phi} - \text{phase estimates of receiver} \\
k - \text{measurement epoch} \\
T - \text{integration interval}
\]

Thus, I and Q signals are produced by multiplying the local estimates with incoming signal and integrating across pre-detection interval given the expression as:

\[
I = \frac{\int_{kT}^{(k+1)T} \sin(\hat{\omega} t + \hat{\phi}) [A \cos(\omega' t + \phi') + \eta] dt}{kT} 
\quad \ldots \ (4.9)
\]
\[
Q = \frac{\int_{kT}^{(k+1)T} \cos(\hat{\omega} t + \hat{\phi}) [A \cos(\omega' t + \phi') + \eta] dt}{kT} 
\quad \ldots \ (4.10)
\]

In both the Eqns. (4.9) and (4.10), the first expression match with the locally generated signals on the other hand the second expression represented within the square brackets are related to the incoming carrier signal (Note that, only carrier signals are considered for analysis as the impact of dynamics on code signals is less). These equations are further expanded and represented as:

\[
I = \frac{\int_{kT}^{(k+1)T} \left[ \frac{A}{2} \left( \sin(\hat{\omega} + \omega') t + \hat{\phi} + \phi') + \sin(\hat{\omega} - \omega') t + \hat{\phi} - \phi') \right] + \eta_t \right] dt}{kT} 
\quad (4.11)
\]
\[
Q = \int_{kT}^{(k+1)T} \left( \frac{A}{2} \left[ \cos \left( \omega + \omega' t + \phi + \phi' \right) + \cos \left( \omega - \omega' t + \phi - \phi' \right) \right] + \eta_Q \right) \, dt \quad (4.12)
\]

where, \( \eta_I \) and \( \eta_Q \) are the quadrature noise components.

The multiplication yields \( \omega + \omega' \) and \( \omega - \omega' \), where these are sum and difference of terms. Because of its low pass nature, the predetection integrator pull out the sum term. Since, \( \omega + \omega' \approx 2\omega \) and \( \phi + \phi' \approx 2\phi \) in both the Eqns. (4.11) and (4.12) and are much above the cut-off frequency of loop filters and the equations are categorized into:

\[
I = \int_{kT}^{(k+1)T} \left( \frac{A}{2} \left[ \sin(\omega_c t + \phi_c) \right] + \eta_I \right) \, dt \quad \ldots \quad (4.13)
\]

\[
Q = \int_{kT}^{(k+1)T} \left( \frac{A}{2} \left[ \cos(\omega_c t + \phi_c) \right] + \eta_Q \right) \, dt \quad \ldots \quad (4.14)
\]

where, \( \omega_c = \omega - \omega' \) and \( \phi_c = \phi - \phi' \) are the frequency and phase errors. A rough knowledge of the user position help converge these errors faster to steady state values. Rate of convergence even depends on the loop bandwidth settings. Still, even without any prior knowledge, loops correct themselves to steady state condition eventually. Signals in both the Eqns. (4.13) and (4.14) are integrated over the predetection interval \( T \), where this corresponds to one complete C/A code period, under the zero Doppler conditions is 1 ms. the full correlation is attained only when its locally produced signals are exactly produced with incoming signal. In the first signal search, because of the
presence of noise the integration of above Eqns. (4.13) and (4.14) becomes a time averaged process. Hence, only the predictable value can be evaluated. Integrating and taking the expectation of Eqns. (4.13) and (4.14) yields:

\[ E[I] = -\frac{A}{2\omega_e} \left[ \cos(\omega_e (k + 1)T + \theta_e) - \cos(\omega_e kT + \phi_e) \right] \]  
(4.15)

\[ E[Q] = -\frac{A}{2\omega_e} \left[ \sin(\omega_e (k + 1)T + \theta_e) - \sin(\omega_e kT + \phi_e) \right] \]  
(4.16)

From Eqns. (4.15) and (4.16) it can be seen that the expectation of \( I \) and \( Q \), represented by \( E[I] \) and \( E[Q] \) depends on the phase \( (\phi_e) \) and frequency \( (\omega_e) \) errors. All these errors are considered as constant during the integration interval \( T \) and are described in terms of position and velocity as

\[ \omega_e = \frac{\omega}{c} \left| \hat{u}_u - \hat{v}_u \right| = \frac{\omega}{c} v_e \]  
(4.17)

\[ \phi_e = -\frac{\omega}{c} \left[ \hat{x}_e - \hat{x}_u \left| \hat{u}_u - \hat{v}_u \right| t \right] = \frac{\omega}{c} \left[ x_e - v_e t \right] \]  
(4.18)

where, \( x_u \) and \( v_u \) are measured position and velocity of receiver \( x_u \) and \( \hat{v}_u \) are the receiver approximate of position and velocity and \( x_e \) and \( v_e \) are the errors in position and velocity among the measured and approximated values. These two variables \( \omega_e \) and \( \phi_e \) behaves as the link among \( I \) and \( Q \) and position (\( P \)) and velocity (\( V \)). In the steady state tracking condition, that is when \( \omega_e, \phi_e \leq \) threshold, the
magnitude of I raises to a maximum while magnitude of Q decreases to minimum.

The functions of I and Q signals in conventional GPS receiver are:

1) To produce the discriminator corrections, $\sigma_e$, $\omega_e$, $\phi_e$ that align their respective NCO’s to incoming signal.

2) To calculate the power $\sqrt{I^2 + Q^2}$ that facilitates a switching among the loops (FLL, PLL and Wideband/Narrowband DLL).

3) 50 Hz navigation information demodulation.

In this study, it can also be seen that an additional function is attributed to these quadrature signals that is the estimation of inertial sensor errors. A superior understanding on I and Q signals and their relation with the phase and frequency errors is thus necessary, especially whenever they are dealing with the ultra-tight integration systems as these form the measurements in integrated Kalman filter.

4.2.3 Derivation of States (P, V) and Measurements (I, Q) relationship

The previous section given a relevant relationship, now it is a straightforward to relate I and Q with P and V through the following expressions:

$$\text{d}E[I] = \frac{1}{2} \left[ \frac{\partial E[I]}{\partial \phi_e} \frac{\partial \phi_e}{\partial x} + \frac{\partial E[I]}{\partial \omega_e} \frac{\partial \omega_e}{\partial x} \right] \text{d}x \quad \text{... (4.19)}$$
\[
\frac{dE}{dQ} = \frac{1}{2} \left[ \frac{\partial E[Q]}{\partial \phi_e} \frac{\partial \phi_e}{\partial x} + \frac{\partial E[Q]}{\partial \omega_e} \frac{\partial \omega_e}{\partial x} \right] dx \quad \text{... (4.20)}
\]

However, I and Q are the functions of both the position and velocity. In the analysis, I is related to only the position and Q to only velocity for the sake of simplicity. Equations (4.19) and (4.20) set up a fundamental relationship among the ultra-tight integration a system that is relating to I, Q to P, V. Expanding the terms within the square brackets in Eqn. (4.19) gives:

\[
\frac{\partial E[I]}{\partial \phi_e} = \frac{A}{2\omega_e} \left[ \sin(\omega_e(k+1)T+\phi_e) + \sin(\omega_e kT + \phi_e) \right] \quad \text{... (4.21)}
\]

\[
\frac{\partial \phi_e}{\partial x} = \frac{-\omega_e}{c R_e} x_e \quad \text{... (4.22)}
\]

\[
\frac{\partial E[I]}{\partial \phi_e} \approx \frac{AkT}{2\omega_e} \left[ - \sin(\omega_e(k+1)T + \phi_e) + \sin(\omega_e kT + \phi_e) - \cos(\omega_e(k+1)T + \phi_e) + \cos(\omega_e kT + \phi_e) \right]
\]

\[
\frac{\partial \omega_e}{\partial x} = 0 \quad \text{... (4.23)}
\]

In Eqn. (4.22),

\[
x_e = x^u - x_u
\]

\[
R_e = \sqrt{x_e^2 - y_e^2 + z_e^2}
\]

Now, expanding the terms in square brackets in Eqn. (4.20) yields:
\[ \frac{\partial E[Q]}{\partial \phi_e} = \frac{-A}{2\omega_e} \left[ \cos(\omega_e(k+1)T+\phi_e) - \cos(\omega_e kT + \phi_e) \right] \]  \quad \ldots (4.25)

\[ \frac{\partial \omega_e}{\partial \dot{x}} = \frac{\omega T}{c} \left[ \frac{v_e}{R_e} \right] \]  \quad \ldots (4.26)

\[ \frac{\partial E[Q]}{\partial \omega_e} = -\frac{AkT}{2\omega_e^2} \begin{bmatrix} \sin(\omega_e (k + 1)T + \phi_e) - \sin(\omega_e kT + \phi_e) + \cos(\omega_e (k + 1)T + \phi_e) \\ -\cos(\omega_e kT + \phi_e) \end{bmatrix} \]  \quad \ldots (4.27)

\[ \frac{\partial \omega_e}{\partial \dot{x}} = \frac{\omega}{c} \left[ \frac{v_e}{R_e} \right] \]  \quad \ldots (4.28)

where, \( v_e = \hat{v}_u - \hat{\hat{v}}_u \). The equations from (4.21) to (4.28) are substituted into the Eqns. (4.19) and (4.20) in order to obtain the relationship among the states and measurements of Kalman filter. Thus, the equations illustrate the different relationships which are used in ultra-tight integration. The next section illustrates how they are used in Kalman filter.

### 4.3 DESIGN OF AN ULTRA-TIGHT INTEGRATION KALMAN FILTER

Kalman filter is referred as the ideal choice invariably for integrating the two navigation systems. This is recursive algorithms that can makeup the filter and it can be easily realized in modern computers [25]. Kalman filter is popular because of its recursive property. Generally, any linear time-invariant systems are expressed in various forms such as transfer function, s-domain, z-domain, state-space and frequency domain. This filter works with the state-space form where states are unknowns that needs to be estimated, and
system dynamics are symbolized for state transitions, i.e., transition of system state is from ‘t’ to ‘t+1’ [48].

The initial step in setting up the ultra-tight integration model is to linearise the system. Basically, any system dynamics (in continuous domain) can be represented in linear form:

Process Model

\[
\dot{x} = Ax + \varepsilon \quad \text{...(4.29)}
\]

Measurement Model

\[
z = Hx + \nu \quad \text{...(4.3)}
\]

where,

- \(x\) – State matrix
- \(A\) – system matrix
- \(H\) – measurement matrix
- \(Z\) – measurements
- \(c\), \(\nu\) – process and measurement noises

Some of the assumptions that are being considered in deriving the filter equations are:

\[
E[\varepsilon(t)] = E[\nu(t)] = 0 \quad \text{...(4.31)}
\]

\[
E[\varepsilon(t)\varepsilon^T(t)] = F\delta[t-t]; \quad E[\nu(t)\nu^T(t)] = R\delta[t-t]; \quad E[\varepsilon(t)\nu^T(t)] = 0 \quad \text{...(4.32)}
\]

Note that, \(\tau\) in Eqn. (4.32) must not be confused with \(\tau\) in Eqn. (4.1) in order to sustain consistency with existing literatures, and the same notation is followed in this equation. Both the process and
measurement noises are Gaussian with covariance matrices \( F \) and \( R \) respectively, and these are statistically independent. One of the assumptions made while designing the Kalman filter is that both the states and measurements are Gaussian in nature. Though even they are not white, the non-whiteness can be augmented in state vector in order to absorb the correlations. In order to further design the Kalman filter for ultra-tight integration, the states and measurements must be defined first.

**States**

Here in this integration system, state vector includes the inertial error states and receiver clock states. For this analysis, a total of 20 states are considered.

\[
x(t) = \{dx, dy, dz, \dot{x}, \dot{y}, \dot{z}, \psi_x, \psi_y, \psi_z, a_x, a_y, a_z, g_x, g_y, g_z, l_x, l_y, l_z, c_b, c_d\}^T \quad \ldots (4.33)
\]

These 20 states includes 3 inertial error states where each in position, velocity, attitude, accelerometer bias, gyro bias, lever-arm and 1 state each for the receiver clock bias and drift.

**Measurements**

GPS measurements, both I (in-phase) and Q (quadrature) are acquired from correlator of receiver. In complementary configuration, both GPS and INS predicted measurements are differenced out before filtering. Measurements which are given as input to the filter are:

\[
z = \{\text{INS predicted measurements}\} - \{\text{GPS measurements}\}
\]
\[ z = \{I+\delta I, Q+\delta Q\}_t - \{-\eta_I, \eta_Q\}_t \quad \cdots (4.34) \]

\[ = \{dI+\eta_I, dQ+\eta_Q\}_t \]

Here, dI, dQ represents the derivations in INS predicted I and Q measurements caused due to the inertial sensor errors; \( \eta_I, \eta_Q \) are the quadrature noise components in GPS I and Q measurements. In Eqn. (4.34), suffix \( T' \) represents number of channels tracked.

### 4.3.1 Filter Algorithm

The complimentary Kalman filter architecture is used in ultra-tight integration and represented in Fig. 4.3. I and Q predictor block identifies I and Q signals from INS data based on the Eqns. (4.15) to (4.18).

**Fig. 4.3: Complementary Kalman filter for ultra-tight integration**

All these calculate signals are then subtracted from I, Q signals from GPS receiver in order to produce the residuals represented in
Eqn. (4.34). These residual signals are then processed through the
filters for producing the error estimates given in Eqn. (4.33). In order
to start the recursive filtering process, the initialization of state
estimate and its covariance matrix are necessary. Since the exact
values of systems are not known at the start of operation, basically
\( E[x_{t0}] = 0 \) and \( E[x_{t0}x_{t0}^T] = P^- \) or \( P(0) \) are assumed, where \( P^- \) is the
diagonal matrix correspond to error state variances.

The recursive equations for filter algorithm are represented in
Eqn. (4.35) to (4.39). Both the matrices \( P(0) \) and \( x(0) \) are initialiased
before the commencement of filter. The initial state \( x(0) \) at time \( t_0 \) is
considered to be zero mean Gaussian with a covariance \( P(0) \). Through
each measurement update of filter, Kalman gain \( K_k \) is computed and
the error states and its covariance matrix are updated based on the
Eqns. (4.35) to (4.37).

**Measurement Update**

Kalman gain

\[
K_k = P^- H_k^T [H_k P^- H_k^T + R_k]^{-1}
\]  \hspace{2cm} \text{... (4.35)}

Kalman state

\[
x_k = x^- + K_k (z_k - H_k x^-)
\]  \hspace{2cm} \text{... (4.36)}

Kalman covariance

\[
P_k = (I - K_k H_k) P^-_k
\]  \hspace{2cm} \text{... (4.37)}


**Time Update**

Kalman covariance update

\[ P_{k+1} = \varphi_k P_k \varphi_k^T + F \]  

... (4.38)

Kalman state update

\[ x_{k+1} = \varphi_k x_k \]  

... (4.39)

**4.3.1.1 Process Model**

The process model for the physical system can be explained in vector form as the set of first order differential equations. In Eqn. (4.29), the coefficients block in dynamic matrix \( A(t) \) for INS error states is formed with the help of terrestrial INS psi-angle error model [66]:

**Position vector estimate**

\[ d\mathbf{r} = \rho * d\mathbf{r} + d\mathbf{v} \]  

...(4.40)

**Velocity error vector estimate**

\[ d\mathbf{v} = -(\Omega^* \omega) * d\mathbf{v} + \nabla - \psi * f \]  

...(4.41)

**Attitude error vector estimate**

\[ d\psi = - \omega * \psi * \varepsilon \]  

... (4.42)

where,

- \( d\mathbf{r} \) – position error vector
- \( d\mathbf{v} \) – velocity error vector
- \( d\psi \) – attitude error vector
- \( \rho \) – true frame rate with respect to Earth
- \( \Omega \) – earth rate vector
ω – true co-ordinate system angular rate with respect to inertial frame
∇ – accelerometer error vector
f – specific force vector
ε – gyro drift rate vector

For further detailed illustrations on this mode, refer Wong (1988); Da (1996) [67,68].

4.3.1.2 Measurement Model

The main intention behind introducing this mode is to relate the states and measurements. Understanding these relationships in earlier sections, now it is straight forward to defined the measurement matrix H in (4.30). The measurement matrix, H is defined as 2i*20 matrix where ‘i’ represents the number of satellites tracked, which is given as:

\[
H = \begin{bmatrix}
h_{x1} & h_{y1} & h_{z1} & 0 & 0 & 0 & 0 & \ldots & \ldots & 1 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
h_{xn} & h_{yn} & h_{zn} & 0 & 0 & 0 & 0 & \ldots & \ldots & 1 & 0 \\
0 & 0 & 0 & \ddot{h}_{x1} & \ddot{h}_{y1} & \ddot{h}_{z1} & 0 & \ldots & \ldots & 0 & 1 \\
0 & 0 & 0 & \ddot{h}_{xn} & \ddot{h}_{yn} & \ddot{h}_{zn} & 0 & \ldots & \ldots & 0 & 1 \\
\end{bmatrix}
\]

… (4.43)

where,

\[ h_{x1} = \left[ \frac{\partial E[I]}{\partial \phi_e} \frac{\partial \phi_e}{\partial x} + \frac{\partial E[I]}{\partial \omega_e} \frac{\partial \omega_e}{\partial x} \right] \]

and

\[ \dot{h}_{x1} = \left[ \frac{\partial E[Q]}{\partial \phi_e} \frac{\partial \phi_e}{\partial x} + \frac{\partial E[Q]}{\partial \omega_e} \frac{\partial \omega_e}{\partial x} \right] \]

… (4.44)
where, $h_{x1}$ is vector which relates $dx$ with I measurement tracked in channel 1 and $\dot{h}_{x1}$ and $d\dot{x}$ to Q measurement. In Eqn. (4.44), the phase and frequency errors can be extracted from the phase and frequency lock loops of correlator respectively. To initialize the R matrix, understand the noise statistics I and Q measurements are necessary. The noise samples both $\eta_I$ and $\eta_Q$ in Eqn. (4.34) are considered as zero mean Gaussian random variables with unity variance that is,

The noise samples in I signal

$$\eta_I = N(0,1) \quad \ldots (4.45a)$$

The noise samples in Q signal

$$\eta_Q = N(0,1) \quad \ldots (4.45b)$$

where, $N$ is the normal random variable.

All these noise samples are statistically independent, that is $E[\eta_I \eta_Q]=0$ [25] and Gaussian implying that off diagonal elements in R matrix are zero as represented in equation (4.46). Matrix R has the dimensions $(2i*2i)$, where ‘i’ is the number of satellites tracked and given as:

$$R = \begin{bmatrix}
\left(\sigma_{I(1,1)}\right)^2 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \left(\sigma_{Q(I,1)}\right)^2 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & \left(\sigma_{Q(2,1)}\right)^2 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & 0 & \ldots & \ldots & \ldots & \left(\sigma_{Q(2i,2i)}\right)^2 
\end{bmatrix} \quad \ldots (4.46)$$
4.4 EXPERIMENTAL RESULTS

The results obtained by Kalman filter in ultra tight configuration for position, velocity and attitude are discussed in this section.

4.4.1 Simulations and Discussions

In order to estimate the Kalman filter performance in ultra-tight configuration, a trajectory with the known dynamics that is acceleration of 0.01 g for 100 secs in north direction was produced through the INS toolbox from GPSsoft™. Both In and Q measurements for the INS systems were simulated through this reference trajectory. INS was modeled with the gyro bias of 1 deg/hr and an accelerometer bias of 1mg, whereas the GPS measurements were added with the zero mean Gaussian noise and clock errors (bias and drift).

For GPS I and Q measurements, composite signals for 7 satellites with PRN codes 4, 6, 7, 10, 18, 19, 22 were simulated. All these signals are processed through software receiver’s carrier and code tracking loops with the pre-detection interval of 1msec. Update rate of this filter is 10 Hz, that is at every 100 msecs those measurements were processed. When each of the measurement was updated, the measurement matrix $H$ defined in equation (4.43) was formed. The phase and frequency errors, difference in phase and frequency between the incoming and the composite signals, were used to form the measurement matrix $H$. The matrices, $P$, $F$, $R$, and $x$ are initialized with the prior data. A U-D covariance factorization
algorithm is used in order to make sure that covariance matrix $P$ is always positive definite. Additionally, in order to progress the computational efficiency further, the scalar measurement update method was utilized and detailed algorithm for this is given [69].

**Fig. 4.4**: Measurements (difference among GPS and INS quadrature signals) to Kalman filter for channel (PRN 6)

Input for the Kalman filter is acquired by deducting the quadrature signals of GPS from INS predicted quadrature signals. Fig.4.4 represents the GPS and inertial sensor errors of one channel (PRN 6) which remain after the predicted and measured quadrature signals, I and Q cancel each other.
From correlator, I signal includes the 50 Hz navigation data. Thus, in order to eliminate these data bits effectively, INS predicted I signal is also added with this data. A preceding knowledge of these information bits are assumed for this analysis.

**Fig.4.5: Original and estimated trajectory position errors**

**Fig.4.6: Velocity errors for original and estimated trajectory**
After its initialisation, simulation experiment was carried out for 100 epochs. Figs. 4.5 to 4.7, explains the navigation performance of ultra-tight GPS/INS integration, acquired from the comparison among the reference trajectory and integrated system output.

The mean of all the position components includes -0.8 m (north), 2.3 m (east) and 0.4 m (down). Conversely, the overall results represents that the position differences are within the few meters. Even the velocity and attitude figures also prove that the filter approximated are stable and are within the bounds.

**Fig.4.7: Attitude errors for original and estimated trajectory**
Fig. 4.8: RMS value changes in position components

Fig. 4.9: RMS value changes in velocity components
From Figs. 4.8 to 4.10, it represents the RMS (Root Mean Square) errors in position, velocity and attitude components specifying the convergence of filter. All these values are acquired from diagonal components of covariance matrix. From Fig. 4.8, it can be seen that the position accuracy is within 0.5m and velocity accuracy is within 0.1 m/s. But, it can also be noted that the accuracy of attitude components is within 0.01 degrees.

The model for lever-arm among the INS body frame and GPS antenna phase center is given as [70]:

\[ R_{\text{INS}}(t) = R_{\text{GPS}}(t) - C_n^b(t) \times 1_x \]

where,
\( C^n_b \) – rotation matrix among the body frame and navigation frame

\( L_x \) – vector offset in \( x \) direction

![Lever arm estimates](image)

**Fig. 4-11: Lever-arm errors between the reference trajectory and the filter estimates**

*(Offset Vector North = 2m, East = 2m, Down = 2m)*

In order to further assess the lever-arm estimates of filter, a 2m offset in each direction was set among the INS body and GPS antenna coordinates. Filter calculates approximately for lever-arm in north, east and down direction which are represented in Fig. 4.11. From the figure, it can be seen that the vertical component is slightly biased when compare to horizontal components. Same results were observed in other studies [70].

### 4.5 CONCLUSIONS

Ultra-tight integration of GPS/INS is progressively becoming more important because of its multi-faceted advantages like immunity
to jamming and interference signals, raise in dynamics ranges, improved signal to noise ratio because of low tracking loop bandwidths, etc. This section was stressed more on the design of mathematical equations which lead to the states-measurements relationship of integration Kalman filter. Initially from the composite GPS signal, pertinent mathematical equations, particularly the derivation of I and Q signals, which are important to the design of ultra-tightly integrated system, are offered. The complementary Kalman filter, which is the crux of integrated system, is talked by focusing on the measurement model that relates the measurements and states of filter. This section even proves that the phase and frequency errors of tracking loops are intermediate variables which establish the relationship among GPS measurements and INS states. In order to validate this theory, simulation experiments have been conducted and those results were summarized. Kalman filter performance specifies that the various mathematical relationships, derivations and assumptions are valid. Moreover, covariance analysis point out the convergence of navigation states.