Chapter 2

Inventory model for deteriorating items with exponential decreasing demand.

2.0 Introduction

In this chapter an inventory model for deteriorating items in declining market conditions is developed. The inventory model is developed by considering exponentially decreasing demand with time dependent deteriorating items. Here shortages are allowed and completely back-logged. The total cost of the inventory system is minimized.

2.1 Assumptions and Notations

The following assumptions and notations which are used to derive this model

2.1.1 Assumptions

- The inventory system deals with single item.
- The production rate is finite.
- The demand rate $R(t)$ decreases exponentially.
• The shortage are allowed and completely backlogged. The lead time is zero or negligible.

• The units in inventory deteriorate with time. The deterioration of units follows two-parameter Weibull distribution i.e. \( \theta(t) = \alpha \beta t^{\beta-1} \) where \( \alpha \) and \( \beta \) are non-negative. \( \alpha \) Denote the rate of deteriorating of units (scale parameter), denotes increasing, constant, decreasing rate of deteriorating rate of deterioration with time and \( t \) stands for time to deteriorate. The deteriorated units can neither be replaced nor repaired during the cycle time.

### 2.1.2 Notations

- \( I(t) \): inventory level at any instant of time \( t \)
- \( I_m \): maximum inventory level
- \( P \): production rate
- \( R(t) = ae^{-bt} \); demand rate. Where \( a > 0 \) is constant demand and \( b > 0 \) denotes the decreasing rate of the demand. Note that \( P > R(t) \)
- \( t_1 \): time at which maximum inventory level is reached
\( t_2 \)  \( t_3 \)  \( t_4 \)  
\( t_2 \) time at which inventory level is reached to zero  
\( t_3 \) time at which maximum back-ordered level is reached  
\( t_4 \) time at which again production starts  

\( A \) Ordering cost per order  
\( C \) purchase cost per unit of an item  
\( h \) inventory holding cost per unit per time unit  
\( l_b \) maximum back-ordered units  

\( T \) \( = t_1 + t_2 + t_3 + t_4 \): cycle time (decision variable)  
\( \pi \) Shortage cost per unit short  
\( K \) total cost per time unit of an inventory system
2.2 Mathematical Formulation

A typical one time-inventory cycle is shown in the figure 2.2.1

Inventory level $I(t)$

The inventory depletion due to production, demand and deterioration of units during $[0, T]$ can be represented by the following differential equations:

$$\frac{dI(t)}{dt} = P - R(t) - \theta(t)l(t), \quad 0 \leq t \leq t_1$$

(2.2.1)
\[
\frac{dl(t)}{dt} = -R(t) - \theta(t)I(t), \quad t_1 \leq t \leq t_2
\]  
(2.2.2)

\[
\frac{dl(t)}{dt} = -R(t), \quad t_2 \leq t \leq t_3
\]  
(2.2.3)

and

\[
\frac{dl(t)}{dt} = P - R(t), \quad t_3 \leq t \leq T
\]  
(2.2.4)

with the boundary conditions: \(I(0) = 0\), \(I(t_1) = I_m, I(t_2) = 0, I(t_3) = -I_b\) and \(I(T) = 0\).

### 2.3 Theoretical results

Using series expansion of exponential series and under the assumption that \(0 < \alpha < 1\), (neglecting \(\alpha^2\) and its higher powers), the solutions of the above differential equations are

\[
I(t) =
\begin{cases}
\frac{abt^2}{2} - \frac{ab\alpha t^{\beta+2}}{2(\beta+2)} + (P - a)t - \frac{(P-a)abt^{\beta+1}}{(\beta+1)}, & 0 \leq t \leq t_1 \\
\frac{t_2 - t + \frac{b}{2}(t^2 - t_2^2)}{a} - \frac{abt^{\beta+1}}{\beta+1} - \frac{abt^{\beta+2}}{\beta+2} + \frac{abt_{1+2}^2}{2} + \frac{abt_{1+2}^2}{2} + \alpha t_1 t_2^\beta, & t_1 \leq t \leq t_2 \\
\frac{t - t_2 + \frac{b}{2}(t_2^2 - t^2)}{a} - \frac{abt_{2+2}^2}{\beta+1} + \frac{abt_{2+2}^2}{\beta+2} - \alpha t_1 t_2^\beta + \frac{abt_{2+2}^2}{3}, & t_2 \leq t \leq t_3 \\
P(t - T) + a(T - t + \frac{b}{2}(t^2 - T^2)), & t_3 \leq t \leq t_4 
\end{cases}
\]  
(2.3.1)
Using $I(t_1) = I_m$, we have

$$I_m = (P - a)t_1 + \frac{ab t_1^2}{2} - \frac{aba \beta t_1^{\beta+2}}{2(\beta+2)} - \frac{(P-a) \beta t_1^{\beta+1}}{\beta+1}$$  \hspace{1cm} (2.3.2)

And $I(t_3) = -I_b$, gives

$$I_b = a \left[ t_2 - t_3 + \frac{b}{2} (t_3^2 - t_2^2) \right]$$  \hspace{1cm} (2.3.3)

Hence, total purchase quality $Q = I_m + I_b$.

The cost for deterioration due to cycle is

$$CD = C \left[ \int_0^{t_1} I(t) dt + \int_{t_2}^{t_3} I(t) dt \right]$$  \hspace{1cm} (2.3.4)

The inventory holding cost incurred during the cycle is

$$IHC = h \left[ \int_0^{t_1} I(t) dt + \int_{t_2}^{t_3} I(t) dt \right]$$  \hspace{1cm} (2.3.5)

and the shortage cost is

$$SC = \pi \left[ -\int_{t_2}^{t_3} I(t) dt + \int_{t_3}^{t_4} I(t) dt \right]$$  \hspace{1cm} (2.3.6)

The ordering cost per cycle is

$$OC = A$$  \hspace{1cm} (2.3.7)

Hence the total cost of the per time unit of an inventory system is given by
\[ K = \frac{1}{T} (CD + IHC + SC + OC) \]  

(2.3.8)

The total cost; \( K \) is function of \( t_1, t_2, t_3 \) and \( t_4 \). The optimum values of decision variables \( t_1, t_2, t_3 \) and \( t_4 \) which minimizes total cost per time unit of an inventory system are the solutions of the equations

\[
\frac{\partial K}{\partial t_1} = 0, \frac{\partial K}{\partial t_2} = 0, \frac{\partial K}{\partial t_3} = 0 \quad \text{and} \quad \frac{\partial K}{\partial t_4} = 0
\]  

(2.3.9)

provided that these values satisfy the sufficiency condition

\[
\begin{vmatrix}
\frac{\partial^2 K}{\partial t_1^2} & \frac{\partial^2 K}{\partial t_1 \partial t_2} & \frac{\partial^2 K}{\partial t_1 \partial t_3} & \frac{\partial^2 K}{\partial t_1 \partial t_4} \\
\frac{\partial^2 K}{\partial t_2 \partial t_1} & \frac{\partial^2 K}{\partial t_2^2} & \frac{\partial^2 K}{\partial t_2 \partial t_3} & \frac{\partial^2 K}{\partial t_2 \partial t_4} \\
\frac{\partial^2 K}{\partial t_3 \partial t_1} & \frac{\partial^2 K}{\partial t_3 \partial t_2} & \frac{\partial^2 K}{\partial t_3^2} & \frac{\partial^2 K}{\partial t_3 \partial t_4} \\
\frac{\partial^2 K}{\partial t_4 \partial t_1} & \frac{\partial^2 K}{\partial t_4 \partial t_2} & \frac{\partial^2 K}{\partial t_4 \partial t_3} & \frac{\partial^2 K}{\partial t_4^2}
\end{vmatrix} > 0
\]  

(2.3.10)

The simultaneous non-linear equations in (2.3.9) can be solved by mathematical software. Hence, the optimum values of \( I_m, I_b \) and \( K \) can be obtained from equations (2.3.2), (2.3.3) and (2.3.8) respectively.

2.4 Numerical example and observation

To verify our theoretical result here we use the following parameter values in proper units in above derived model:
\[ [a, b, \alpha, \beta, P, C, A, h, \pi] = [200, 0.30, 0.05, 1.01, 300, 30, 250, 3, 50] \]

The optimum values of \( t_1, t_2, t_3 \) and \( t_4 \) along with minimum total average cost per time unit for different values of parameters are exhibited in Table-2.4.1

<table>
<thead>
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<th>Table-2.4.1 sensitivity analysis</th>
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The production inventory model for time dependent deterioration of units with a decreasing market demand is studied here. It is meant for the decision maker to plan the production and stocking the inventory. It is observed that the model is very sensitive to deterioration of units and rate of change of demand, production rate and shortage cost of unit short. It also helps decision maker to reduce the deterioration of units by deploying advanced techniques for stocking.

### 2.5 Conclusion

Note: IFS stands for infeasible solution

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