CHAPTER 5

A time dependent Deteriorating Inventory Model for Demand Declining Market associated with discounted cash flow and supplier credit period.

5.0 Introduction

In this chapter we deal with a time dependent deteriorating inventory model for demand declining market associated with discounted cash flow and supplier credit period. We divide this chapter in three different sections.

In Section 5.1 our aim is to develop a mathematical model with discount selling price using Weibull distribution for deteriorate items in inventory in declining market. Using different type of deterioration discount on selling price here we find respective purchase quantity. The computational steps are explored for a retailer to determine the optimal purchase units which maximize the profit per time unit. The numerical examples
are given to support the development of the mathematical model. A sensitivity analysis is carried out to study the effect on decision variable in the optimal solution.

Section 5.2 contains the mathematical model for weibull deterioration of items in inventory in declining market in which demand is decreasing by time when the supplier offers his retailers a progressive credit period to settle the accounts against the dues. With the assumption that if retailer settle account after first credit period then retailer suppose to pay interest at some rate say \( (IC_1) \), if he settles account after second credit period then he is suppose to pay interest on the remaining amount at the rate of say \( (IC_2) \), \( IC_2 > IC_1 \).

Moreover it is assumed that as soon as retailer gets revenue from sale it is deposited in some interest bearing account. The computational steps are explored for a retailer to determine the optimal purchase units which minimizes total inventory cost per time unit. The numerical example is given to demonstrate retailer’s optimal decision. Sensitivity analysis is carried out to study the variations in the optimal solution.
Section 5.3 contains an extension of above derived model using the assumption that declining demand is time and price sensitive. In this research article, an ordering and pricing policy is formulated for a retailer when the supplier offers a delay in payments to settle the accounts against the retailer's due. A decision policy is sketched to determine the optimal selling price and the ordering quantity to maximize the retailer's profit. The numerical examples are given to support the development of the mathematical model. The sensitivity analysis of critical parameter is carried out to observe the changes in the decision variable and profit.

5.1: A deteriorating inventory model with discounted selling price in declining market

Assumptions and Notations which are used to derive model 5.1 is as follows:

5.1.1: Assumptions and Notations

5.1.1.1 Assumption

- The inventory system under consideration deals with single item.
- Replenishment rate is infinite
• Shortages are allowed and lead-time is zero.

• The deterioration rate; \( \theta(t) \) follows three parameter Weibull distribution (say)

\[
\theta(t) = \alpha \beta (t - \gamma)^{\beta - 1}
\]

Where, \( t \) is time measured from arrival of the fresh stock, \( \alpha \) denotes rate of deterioration of units also termed as scale parameter \( 0 < \alpha < 1 \), \( \beta \) denotes shape parameter. \( \beta > 1 \) and \( \gamma \) the time after which deterioration starts.

• The demand; \( R(t) = a(1 - bt) \) is decreasing with respect to time. Where \( a > 0 \) is fixed demand and \( 0 < b < 1 \) denotes rate of change of demand.

• \( f_1 = (1 - d_1)^{-\eta_1}, \eta_1 \in R \) is the effect of pre-deterioration discount on selling price, \( f_2 \) can be determined from the retailer’s past experience. The discount offered on fresh goods is more than discount offered because demand of the product is subject to decline with time. The \( d_1 \) and \( d_2 \) are related by

\[
d_2 = \frac{d_1^2}{1 + \gamma - t_1}, \lambda > 0 \quad (5.1.1.1.1)
\]

here, \( t_1 \) is the length of time after the arrival of fresh stock from which discount \( d_1 \) on selling price for fresh stock is offered. \( t_1 \) is a decision variable when \( t_1 = \gamma \)
and $\lambda = 1$ then $d_1$ and $d_2$ are equal and there is no pre-deterioration discount on fresh stock.

### 5.1.1.2 Notations

- $C$ the unit purchase cost
- $P$ the unit selling price with $(P > C)$
- $h$ the inventory holding cost per unit per year excluding interest charges
- $A$ the ordering cost per order
- $d_1$ Percentage discount offered before deterioration per unit (decision variable).
- $d_2$ Percentage discount offered after deterioration per unit (decision variable).
- $Q(t)$ Inventory level at any instant of time $t$.
- $\theta$ a constant fraction of deterioration per unit.
- $T_1$ Cycle time for pre- and post- deterioration discount on selling price
  
  Equivalently; purchase quantity $Q_1$. (Decision variable)
- $T_2$ Cycle time for only post- deterioration on selling price, Equivalently; purchase
quantity $Q_2$. (decision variable).

$T_3$ Cycle time for no discount on selling price, Equivalently; procurement quantity $Q_3$. (decision variable).

$T_4$ cycle time for post- deterioration discount on selling price, Equivalently; purchase quantity $Q_4$. (decision variable).

$Z(t)$ total profit per unit time

5.1.2 Mathematical formulation

5.1.2.1 Model for pre- and post- deterioration discount on selling price.

The system starts with $Q_1$-units in the inventory system. These $Q_1$-units depletes due to only demand up to the time $\gamma$. After time $\gamma$, deterioration starts and inventory level changes due to demand and deterioration units. Inventory level becomes zero at $T_1$. It is assumed that pre-deterioration discount $d_1$ % on unit selling price is given to enhance the demand of the fresh items. This discount is available for the time
Once deterioration starts at time $\gamma$, a $d_2$ % discount on unit selling price is given for the deteriorated units. This discount is continued till the end of the cycle time. Thus, the inventory level at any instant of time $t$ can be described by the following system of linear differential equations

$$\frac{dI(t)}{dt} = \begin{cases} -(a - bt), & 0 \leq t \leq t_1 \\ -f_1(a - bt), & t_1 \leq t \leq \gamma \\ -f_2(a - bt) - \theta I(t), & \gamma \leq t \leq T_1 \end{cases} \tag{5.1.2.1.1}$$

With the boundary condition $I(0) = Q_1$ and $I(T) = 0$.

The solution of differential equation is

$$I(t) = \begin{cases} Q_1 - at + \frac{abt^2}{2} & , 0 \leq t \leq t_1 \\ af_1(\gamma - t) - \frac{abf_1(\gamma^2 - t^2)}{2} + \frac{af_2(e^{\theta(t_1-\gamma)} - 1)}{\theta} + \frac{abf_2(e^{\theta(t_1-\gamma)} - 1)}{\theta^2} + \frac{abf_2\gamma}{\theta} - \frac{abf_2 t}{\theta} - \frac{abf_2 t_1(e^{\theta(t_1-t)} - 1)}{\theta} & , t_1 \leq t \leq \gamma \\ \frac{af_2(e^{\theta(t_1-t)} - 1)}{\theta} + \frac{abf_2(e^{\theta(t_1-t)} - 1)}{\theta^2} + \frac{abf_2 t}{\theta} - \frac{abf_2 t_1(e^{\theta(t_1-t)} - 1)}{\theta} & , \gamma \leq t \leq T_1 \end{cases} \tag{5.1.2.1.2}$$

using continuity of $I(t)$ at $t = t_1$, we get

$$Q_1 = at - \frac{abt_1^2}{2} + af_1(\gamma - t_1) - \frac{abf_1(\gamma^2 - t_1^2)}{2} + \frac{af_2(e^{\theta(t_1-\gamma)} - 1)}{\theta} + \frac{abf_2(e^{\theta(t_1-\gamma)} - 1)}{\theta^2} + \frac{abf_2\gamma}{\theta} - \frac{abf_2 t_1(e^{\theta(t_1-t)} - 1)}{\theta} \tag{5.1.2.1.3}$$
The total profit of a retailer comprises of the following cost components:

Sales revenue per cycle is

\[ SR = P \left( \int_{0}^{r_1} R(t) \, dt + f_1(1 - d_1) \int_{t_1}^{r} R(t - t_1) \, dt + f_2(1 - d_2) \int_{r}^{T_t} R(t) \, dt \right) \]  
(5.1.2.1.4)

Purchase cost per cycle is \( PC = CQ_1 \)  
(5.1.2.1.5)

Inventory holding cost per cycle is

\[ IHC = h \left( \int_{0}^{t_1} L(t) \, dt + \int_{t_1}^{r} L(t) \, dt + \int_{r}^{T_t} L(t) \, dt \right) \]  
(5.1.2.1.6)

Ordering cost \( OC = A \)  
(5.1.2.1.7)

Hence, total profit per unit time is

\[ Z_1(d_1, t_1, T_1) = \frac{1}{T_1} [SR - PC - IHC - OC] \]  
(5.1.2.1.8)

5.1.2.2 Model with only post- deterioration discount on selling price

Here, discount on selling price will be given only when deterioration of units starts. Then \( t_1 = r \) and \( \lambda = 1 \) in (5.1.1.1.1) gives \( d_1 = d_2 \). The purchase quantity \( Q_2 \) in the beginning of the cycle is given by

\[ Q_2 = a \gamma - \frac{ab\gamma^2}{2} + \frac{a_{f_2}(e^{\theta(T_2 - \gamma)} - 1)}{\theta} + \frac{ab_{f_2}(e^{\theta(T_2 - \gamma)} - 1)}{\theta^2} + \frac{ab_{f_2} \gamma}{\theta} - \frac{ab_{f_2} T_2 (e^{\theta(T_2 - \gamma)})}{\theta} \]  
(5.1.2.2.1)

and the profit function per time unit can be given by
\[ Z_2(d_2,T_2) = \frac{1}{T_2} [SR - PC - IHC - OC] \quad (5.1.2.2) \]

The profit function has decision variables post- deterioration discount, \( d_2 \) on selling price and cycle time.

**5.1.2.3 Model with no discount on unit selling price**

Here, neither pre- deterioration nor post- deterioration discount on unit selling price is available. Substituting \( t_1 = \gamma \) and \( d_2 = 0 \) in (5.1.2.1.3) and (5.1.2.1.8) gives purchase quantity

\[ Q_3 = a\gamma - \frac{ab\gamma^2}{2} + \frac{a(e^{\theta(T_3-\gamma)} - 1)}{\theta} + \frac{ab(e^{\theta(T_3-\gamma)} - 1)}{\theta^2} + \frac{aby}{\theta} - \frac{abT_3(e^{\theta(T_3-\gamma)})}{\theta} \quad (5.1.2.3.1) \]

and the profit function per time unit time

\[ Z_3(T_3) = \frac{1}{T_3} [SR - PC - IHC - OC] \quad (5.1.2.3.2) \]

**5.1.2.4 Model with instant deterioration**

Here, the unit starts deteriorating as soon as they arrive in the system. So, there is no pre- deterioration. One only gets post- deterioration discount. Here, \( t_1 = \gamma = 0 \) and \( \lambda = 1 \) and hence \( d_1 = d_2 \). The lot- size is given by

\[ Q_4 = \frac{af_2(e^{\theta T_4} - 1)}{\theta} + \frac{abf_2(e^{\theta T_4} - 1)}{\theta^2} - \frac{abf_2 T_4(e^{\theta T_4})}{\theta} \quad (5.1.2.4.1) \]
and the profit $Z_4(d_2, T_4)$ per unit time is function of post deterioration discount on selling price and cycle time.

### 5.1.2.5 Model with instant deterioration and no discount

Here, $d_2 = 0$. Hence eq. (5.1.2.4.1) gives procurement quantity as

$$Q_5 = \frac{a(e^{\theta T_5} - 1)}{\theta} + \frac{ab(e^{\theta T_5} - 1)}{\theta^2} - \frac{abT_5(e^{\theta T_5})}{\theta}$$

(4.1.2.5.1)

and $Z_5(T_5)$ is the profit per unit time only a function of cycle time.

### 5.1.3 Numerical example and observation

Consider the following parametric values in proper units:

$[a, b, C, P, h, A, n_1, n_2, \lambda, \gamma, \theta] = [500, 0.13, 5, 20, 0.8, 300, 5, 2, 1.2, 1.2, 0.05]

For the model with both pre and post- deterioration discounts, a starts at $t_1 = 0.58$ units and it is 31.78%. Then deteriorated units are sold at 15.62%. Here pre-deterioration discount on selling price is higher than that of the post- deterioration because the demand of the product is declining. The profit of $7754.21$ is obtained during cycle time of 6.97 units. The corresponding optimal purchase quantity is 4248-units. The optimal profit and procurement quantity for the scenario of only post-deterioration discount are $7554.20$ and 2363 units during cycle time of 6.75- units. The
profit decreases by $200. The purchase quantity decreases by 44.37% and cycle time by 4%. The post-deterioration discount is 9.23% which is very less than what is offered in the case of both types of discount. The profit lowers by $667.44 when no discount offered. It agrees with the prevailing situations in the market. For instant deterioration, 3.87% discount on selling price is offered which results in the profit of $5476.31 which is 14% more than the profit obtained when no discount offered. All this optimal values are exhibited in following Table 5.1.3.1
Table 5.1.3.1

Optimal values of decision variables and profit per unit time of the models.

<table>
<thead>
<tr>
<th>Model</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$t_1$</th>
<th>$T_i$</th>
<th>$Q_i$</th>
<th>$\pi_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both type of discounts with all discounts</td>
<td>0.3178</td>
<td>0.1562</td>
<td>0.58</td>
<td>6.97</td>
<td>4248</td>
<td>7754.21</td>
</tr>
<tr>
<td>With Post deterioration discount</td>
<td>0.0</td>
<td>0.0923</td>
<td>0.0</td>
<td>6.75</td>
<td>2363</td>
<td>7554.20</td>
</tr>
<tr>
<td>No discount</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>6.53</td>
<td>2025</td>
<td>7086.77</td>
</tr>
<tr>
<td>Instant deterioration Post deterioration</td>
<td>---</td>
<td>0.0387</td>
<td>---</td>
<td>4.76</td>
<td>946</td>
<td>5476.31</td>
</tr>
<tr>
<td>Instant deterioration</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No discount</td>
<td>---</td>
<td>0.0</td>
<td>---</td>
<td>5.03</td>
<td>937</td>
<td>4709.63</td>
</tr>
</tbody>
</table>

In the next section, we change the policy by applying trade credit period as a promotional tool to the retailer.
5.2: Optimal inventory policies for Weibull deterioration under trade credit in declining market

Additional Assumptions and Notations which are used to derive model 5.1 is as follows:

5.2.1: Assumptions and Notations

5.2.1.1 Assumption

- The inventory system under consideration deals with the single item.

- The planning horizon is infinite.

- The demand of the product is decreasing function of the time.

- Shortages are allowed and lead-time is zero.

- The units in inventory deteriorate with respect to time. The deteriorated units can neither be repaired nor replaced during the cycle time.

- The retailer can deposit generated sales revenue in an interest bearing account during the allowable credit period. At the end of this period, the retailer settles the account for all the units sold keeping the difference for day-to-day expenses,
and starts paying the interest charges on the unsold items in the inventory system.

5.2.1.2 Notations

\( R(t) \) \( a(1 - bt) \): the annual demand as a decreasing function of time where \( a > 0 \) is constant demand and \( b(0 < b < 1) \) denotes the rate of change of demand with respect to time

\( C \) the unit purchase cost

\( P \) the unit selling price with \( (P > C) \)

\( h \) the inventory holding cost per unit per year excluding interest charges

\( A \) the ordering cost per order

\( M \) the permissible credit period offered by the supplier to the retailer for settling the account

\( I_c \) the interest charged per monetary unit in stock per annum by the supplier

\( I_e \) the interest earned per monetary unit per year

Note: \( I_c > I_e \)

\( Q \) the order quantity (a decision variable)
θ(t) deterioration with respect to time

\[ θ(t) = α \beta t^{\beta-1} \]

Where, \( t \) is time of deterioration. \( α \) denotes rate of deterioration of units also termed as scale parameter. \( 0 < α < 1 \) and \( β \) denotes shape parameter. \( β > 1 \)

\( I(t) \) the inventory level at any instant of time \( t, 0 ≤ t ≤ T \)

\( T \) the replenishment cycle time (a decision variable)

\( K(T) \) the total cost per time unit of an inventory system

### 5.2.2 Mathematical formulation

The inventory level; \( I(t) \) depletes to meet the demand and deterioration of units.

The rate of change of inventory level can be described by the following differential equation:

\[
\frac{dI(t)}{dt} + \theta(t)I(t) = -R(t), \quad 0 ≤ t ≤ T \tag{5.2.2.1}
\]

With the initial condition \( I(0) = Q \) and boundary condition and \( I(T) = 0 \).

Consequently, the solution of (5.2.2.1) is given by

\[
I(t) = -\left[ \int_0^T a(1 - bt)e^{at^\beta} dt \right] e^{-at^\beta}, \quad 0 ≤ t ≤ T \tag{5.2.2.2}
\]
The solution is obtained using series expansion of exponential and neglecting \( \alpha^2 \) and its higher powers because \( 0 < \alpha < 1 \). Using \( I(0) = Q \), the order quantity is

\[
Q = I(0) = \frac{a}{8(1+\beta)(2+\beta)(1+2\beta)} \left[ 16T + 72aT^{1+\beta} + 32a^2T^{1+2\beta} - 28T^2\beta^3 - 28bT^2\beta - 8b\alpha^2T^{2+2\beta} - 8b\alpha T^{2+\beta} + 56T\beta + 56T\beta^2 + 16T^3 - 8bT^2 - 10b\alpha^2T^{2+2\beta} - 16aT^{2+\beta}\beta^2 - 24b\alpha T^{2+\beta} \right]
\]

(5.2.2.3)

The total cost of inventory system per time unit consists of the following cost components:

Ordering cost; \( OC = A/T \)  \hfill (5.2.2.4)

Cost due to deterioration; \( DC \) per time unit;

\[
DC = \frac{C}{T} \left[ Q - \int_0^T R(t) \, dt \right] = \frac{C}{T} \left[ \frac{a}{8(1+\beta)(2+\beta)(1+2\beta)} \left[ 16T + 72aT^{1+\beta} + 32a^2T^{1+2\beta} - 28T^2\beta^3 - 28bT^2\beta - 8b\alpha^2T^{2+2\beta} - 8b\alpha T^{2+\beta} + 56T\beta + 56T\beta^2 + 16T^3 - 8bT^2 - 10b\alpha^2T^{2+2\beta} - 16aT^{2+\beta}\beta^2 - 24b\alpha T^{2+\beta} \right] + \frac{abT^2}{2} - aT \right]
\]

(5.2.2.5)

Inventory holding cost; IHC per time unit;

\[
IHC = \frac{h}{T} \int_0^T I(t) \, dt \hfill (5.2.2.6)
\]
5.2.3 Theoretical results

Regarding interest charges and earned, two cases may arise based on the lengths of the allowable credit period \( M \) and cycle time \( T \).

**Case1: \( M \leq T \)**

The retailer sells \( R(M) \) units by the end of the permissible tread credit \( M \) and has \( CR(M) \) to pay the supplier. For the unsold items in the stock, the supplier charges an interest rate \( I_c \) from time \( M \) onwards. Hence, the interest charged, \( IC_1 \) per time unit is

\[
IC_1 = \frac{cI_c}{T} \int_M^T I(t) \, dt
\]  
(5.2.3.1)

During \([0,M]\) the retailer sells the product and deposits the revenue into an interest earning account at the rate \( I_e \) per monetary unit per year. Therefore, the interest earned, \( IE_1 \) per time unit is

\[
IE_1 = \frac{pI_e}{T} \int_0^M R(t) \, dt = \frac{pI_e}{T} \left[ \frac{aM^2}{2} - \frac{bM^3}{3} \right]
\]  
(5.2.3.2)
**Case 2: T ≤ M**

In this scenario, the retailer sells \( R(T)T \) a unit in all by the end of the cycle time and has \( C R(T)T \) to pay the supplier in full by the end of the credit period \( M \). Hence, interest charges

\[
IC_2 = 0 \tag{5.2.3.3}
\]

and the interest earned per time unit is

\[
IE_2 = \frac{P_I}{T} \left[ \int_0^T R(t) t \, dt + R(T)T(M - T) \right]
= \frac{P_I}{T} \left[ \frac{aT^2}{2} - \frac{abT^3}{3} + aTM - aT^2 - abT^2M + abT^3 \right] \tag{5.2.3.4}
\]

As a result, the total cost of an inventory system per time unit in both the cases is given by

\[
K_i(T) = OC + DC + IHC + IC_i - IE_i \quad \text{where, } i = 1,2 \tag{5.2.3.5}
\]

Hence, the total cost; \( K(T) \) of an inventory system per time unit is

\[
K(T) = \begin{cases} 
K_1(T), & M \leq T \\
K_2(T), & M \geq T 
\end{cases} \tag{5.2.3.6}
\]

For \( T = M \), we have \( K_1(M) = K_2(M) \) \tag{5.2.3.7}

For \( M \leq T \), the value of \( T \) can be obtained by solving \( \frac{\partial K_1(T)}{\partial T} = 0 \) \tag{5.2.3.8}
The obtained $T = T_1$ (say) minimizes the total cost provided

$$\frac{\partial^2 K_1(T)}{\partial T^2} > 0 \text{ for } T = T_1 \tag{5.2.3.9}$$

For $M \geq T$, the value of $T = T_2$ (say) can be obtained by solving $\frac{\partial K_2(T)}{\partial T} = 0 \tag{5.2.3.10}$

The obtained $T = T_2$ (say) minimizes the total cost provided

$$\frac{\partial^2 K_2(T)}{\partial T^2} > 0 \text{ for } T = T_2 \tag{5.2.3.11}$$

5.2.4 Computational algorithm

To obtain optimal solution, decision maker is advised to observe the following steps.

**Step1**: Initialize all parametric values.

**Step2**: Compute $T_1$ from equation (5.2.3.8),

If $M < T_1$ then $K_1(T_1)$ from equation (5.2.3.9) gives minimum cost; else go to step3.

**Step3**: Compute $T_2$ from equation (5.2.3.10),

If $M > T_2$ then $K_2(T_2)$ from equation (5.2.3.11) gives minimum cost for decision maker else.

**Step4**: $K_1(M) = K_2(M)$ from equation (5.2.3.7) is the minimum cost.

**Step5**: stop.
5.2.5 Numerical example and observation

Here we derive two examples relative to above discussed cases:

**Example 1:** Consider the following parametric values in proper units:

\[ [a, b, C, P, h, A, L_c, L_e, M, \alpha, \beta] = [300, 0.2, 20, 40, 1, 200, 0.12, 0.09, 30/365, 0.3, 3.5] \]

Then \( T_1 = 0.5270 \) years which is less than \( M \). So \( K_1(T_1) = 612.86 \) is minimum cost (see Fig. 5.2.5.1) procuring 150.79 units. Also \( \frac{\partial^2 K_1(T)}{\partial T^2} = 3878.12 > 0 \) proves convexity of the total cost.

**Fig 5.2.5.1:** Cost convexity when \( M \leq T \)
Example 2: for parametric values:

\[ [a, b, C, P, h, A, l_e, M, \alpha, \beta] = [2500, 0.1, 28, 40, 1, 250, 0.02, \frac{60}{365}, 0.3, 1.8] \]

Then \( T_2 = 0.1591 \) years which is greater than \( M \). So, \( K_2(T_2) = 974.61 \) is minimum cost (see Fig.5.2.5.2) for purchase of 396.17 units. Also \( \frac{\partial^2 K_2(T)}{\partial T^2} = 137787 > 0 \) shows that the total cost per time unit of an inventory system is minimum.

![Graph](image)

**Fig 5.2.5.2:** cost convexity when \( M > T \)

Using parametric values defined in example 1, sensitivity analysis is performed by changing values as -40%, -20%, 20%, 40% on decision variable and the objective function.
## Table 5.2.5.1: sensitivity analysis

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<th>Parameter</th>
<th>% changes</th>
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<td>T</td>
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</tbody>
</table>
It is observed that increases in deterioration rate decreases cycle time because retailer will have to put order frequently. The more deterioration rate of units increases total cost of an inventory system. To reduce deterioration rate of units he replenish order of smaller size. Increase in shape parameter of deterioration with respect to time increases cycle time and procurement quantity significantly. The total cost is of an inventory system is very sensitive to shape parameter. The permissible delay period decrease cycle time and total cost. Decrease in total cost is due to the fact that the retailer can earn interest on generated sales revenue for a longer period. Increase in the declining demand rate directs retailer to put order of smaller size after a long time and decrease in total cost.

In section 5.3, we expand above derive model by applying time and price sensitive demand function.

5.3 Retailer’s pricing and ordering strategy for Weibull distribution deterioration under trade credit in declining market

Additional Assumptions and Notations which are used to derive model 5.1 is as follows:
5.3.1: Assumptions and Notations

5.3.1.1 Assumption

- The inventory system under consideration deals with the single item only.

- The planning horizon is infinite.

- The demand of the product is decreasing function of the time and the sale price.

- Shortages are allowed and lead-time is zero.

- The units in inventory are subject to deteriorate with respect to time. The deterioration rate follows Wiebull distribution. The deteriorated units can neither be repaired nor replaced during the cycle time.

- The retailer generates revenue by selling the product. The generated revenue is deposited in an interest earning account during the allowable credit period. At the end of this period, the retailer settles the account for all the units sold keeping the difference for day-to-day expenses, and starts paying the interest charges on the unsold items in the inventory.
5.3.1.2 Notations

\[ R(t, P) = a(1 - bt)P^\eta; \]
Where \( a > 0 \) is fixed demand, \( b \) is the rate of change of demand, \( 0 \leq b < 1 \) and \( \eta > 0 \) is mark up parameter.

- \( C \) the unit purchase cost
- \( P \) the unit selling price with \( (P > C) \)
- \( h \) the inventory holding cost per unit per year excluding interest charges
- \( A \) the ordering cost per order
- \( M \) the permissible credit period offered by the supplier to the retailer for settling the account
- \( I_c \) the interest charged per monetary unit in stock per annum by the supplier
- \( I_e \) the interest earned per monetary unit per year
  Note : \( I_c > I_e \)
- \( Q \) the order quantity (a decision variable)
- \( \theta(t) \) deterioration with respect to time

\[ \theta(t) = \alpha \beta^t \] Where, \( t \) is time of deterioration. \( \alpha \) denotes rate of deterioration of units also termed as scale parameter. \( 0 < \alpha < 1 \) and \( \beta \)
denotes shape parameter. $\beta > 1$

$I(t)$ the inventory level at any instant of time $t$, $0 \leq t \leq T$

$T$ the replenishment cycle time (a decision variable)

$Z(P, T)$ The total profit per time unit.

5.3.2 Mathematical formulation

The retailer’s inventory level gradually decrease due to the time dependent and
sale price and deterioration of units in inventory. The instantaneous state of inventory
level at any instant of time $t$ during the cycle period $[0, T]$ can be described by the
differential equation:

$$\frac{dI(t)}{dt} + \theta(t)I(t) = -R(t, P), \quad 0 \leq t \leq T \quad (5.3.2.1)$$

With the initial condition $I(0) = Q$ and boundary condition and $I(T) = 0$.

Consequently, the solution of (5.3.2.1) is given by

$$I(t) = [Q - \int_0^t R(t, P)e^{a\tau} d\tau] e^{-at\beta}, \quad 0 \leq t \leq T \quad (5.3.2.2)$$

Under the assumption that $\alpha(0 < \alpha < 1)$ is very small, expanding exponential
series by neglecting $\alpha^2$ and its higher powers, the solution (5.3.2.2) can be written as
\[ I(t) = \left[ Q - \int_0^T a(1 - bt)P^{-\eta}(1 + \alpha t^\beta)dt \right] (1 - \alpha t^\beta) \]

\[ Q = I(0) = \frac{a^p - \eta}{8(1+\beta)(2+\beta)(1+2\beta)} \begin{pmatrix} -8bT^2\beta^3 & -28T^2\beta^2 & 56T\beta^4 & -4b\alpha^2T^2+2\beta \vspace{1em} 2 & 4b\alpha^2T^2+2\beta^2 \vspace{1em} + 12\alpha^2T^4+2\beta \vspace{1em} -8b\alpha T^4+2\beta + 16T \vspace{1em} -16\alpha T^4+2\beta + 40\alpha T^4+2\beta + 28T^2 \vspace{1em} 8\alpha T^4+2\beta + 16T \vspace{1em} -16\alpha T^4+2\beta + 16\alpha T^4+2\beta \vspace{1em} 2 \end{pmatrix} \]

(5.3.2.3)

Next we compute different cost involved in the total profit per time unit.

Sales revenue; per time unit is \( SR = \frac{PQ}{T} \) (5.3.2.4)

Purchase cost of procuring \( Q \)- units per time unit is \( PC = \frac{CQ}{T} \) (5.3.2.5)

Inventory holding cost; \( IHC \) per time unit excluding interest charges is ;

\[ IHC = \frac{h}{T} \int_0^T I(t) \, dt \] (5.3.2.6)

Ordering cost; \( OC \) per order is \( OC = \frac{A}{T} \) (5.3.2.7)

**5.3.3 Theoretical Results**

Regarding interest charges and earned, two cases may arise based on the length of \( T \) and \( M \); Viz \( M \leq T \) and \( M > T \).

**Case1:** \( M \leq T \)
Using the assumption the retailer sells $R(M)M$ - units by the end of the permissible tread credit $M$ and has $CR(M)M$ to pay the supplier. For the unsold items in an inventory system, the supplier charges an interest rate $I_c$ during the period $[M, T]$. Hence, the interest charged, $IC_1$ per time unit is

$$IC_1 = \frac{Cl_c}{T} \int_M^T I(t) \, dt$$

(5.3.3.1)

During $[0, M]$ the retailer sells the product and deposits the revenue into an interest earning account at the rate $I_c$ per monetary unit per year. Therefore, the interest earned, $IE_1$ per time unit is

$$IE_1 = \frac{Pl_e}{T} \int_0^M R(t, P) \, dt = \frac{aP^{-\eta+1}I_c}{T} \left[ \frac{M^2}{2} - \frac{bM^3}{3} \right]$$

(5.3.3.2)

Hence retailer’s profit in inventory system per time unit is given by

$$Z_1(T, P) = SR - OC - PC - IH\bar{C} - IC_1 + IE_1$$

The optimum value of $(T, P) = (T_1, P_1)$ (say) can be obtained by solving

$$\frac{\partial Z_1(T, P)}{\partial P} = 0, \quad \frac{\partial Z_1(T, P)}{\partial T} = 0$$

(5.3.3.3)

The obtained $(T, P) = (T_1, P_1)$ (say) maximizes the profit $Z_1(T, P)$ provided

$$\frac{\partial^2 Z_1(T, P)}{\partial P^2} < 0, \quad \frac{\partial^2 Z_1(T, P)}{\partial T} < 0 \quad \text{and} \quad \frac{\partial^2 Z_1(T, P)}{\partial P^2} \frac{\partial^2 Z_1(T, P)}{\partial T^2} - \left( \frac{\partial^2 Z_1(T, P)}{\partial P \partial T} \right)^2 > 0$$

(5.3.3.4)
Here, the retailer sells $R(T)T$ a unit in all by the end of the cycle time and has $C R(T) T$ to pay the supplier in full by the end of the credit period $M$. Hence, interest charges

$$IC_2 = 0$$  \hspace{1cm} (5.3.3.5)$$

and the interest earned per time unit is

$$IE_2 = \frac{P_{le}}{T} \left[ \int_0^T R(t, P) t \, dt + R(T, P)T(M - T) \right]$$

$$= \frac{aP^{-\eta+1}l_e}{T} \left[ \frac{T^2}{2} - \frac{bT^3}{3} + TM - T^2 - bT^2M + bT^3 \right]$$ \hspace{1cm} (5.3.3.6)$$

As a result retailer’s profit in inventory system per time unit is given by

$$Z_2(T, P) = SR - OC - PC - IH - IC_2 + IE_2$$

The optimum value of $(T, P) = (T_2, P_2)$ (say) can be obtained by solving

$$\frac{\partial Z_2(T, P)}{\partial P} = 0, \quad \frac{\partial Z_2(T, P)}{\partial T} = 0$$ \hspace{1cm} (5.3.3.7)$$

The obtained $(T, P) = (T_2, P_2)$ (say) maximizes the profit $Z_2(T, P)$ provided

$$\frac{\partial^2 Z_2(T, P)}{\partial P^2} < 0, \quad \frac{\partial^2 Z_2(T, P)}{\partial T^2} < 0 \quad \text{and} \quad \frac{\partial^2 Z_2(T, P)}{\partial P^2} \frac{\partial^2 Z_2(T, P)}{\partial T^2} - \left( \frac{\partial^2 Z_2(T, P)}{\partial P \partial T} \right)^2 > 0$$ \hspace{1cm} (5.3.3.8)$$

For, $T = M$, $Z_1(M, P) = Z_2(M, P)$ \hspace{1cm} (5.3.3.9)
The complexity of the expression suggests that it is not easy to get good closed form for the necessary and sufficient conditions. One can solve it by using mathematical software.

5.3.4 Computational Algorithm

The decision maker can use following steps to maximize his profit.

**Step1** Take parametric values in proper units.

**Step2** Calculate \((p_1, T_1)\) using equation (5.3.3.3) and (5.3.3.4). If \(M < T_1\) then case 1 is best policy to have maximum profit else go to step 3.

**Step3** Compute \((p_2, T_2)\) using equation (5.3.3.7) and (5.3.3.8). If \(M > T\) then case2 is best policy to have maximum profit else go to step 4.

**Step4** Compute \(P\) from (5.3.3.3) and (5.3.3.7). Here, \(Z_1(M, P) = Z_2(M, P)\) is maximum profit.

**Step 5** Stop

5.3.5 Numerical example and observation

Following two examples regarding to both cases is:
**Example1:** For $a = 200000, b = 0.2, \eta = 2.0, h = $1.00 / unit / year, $C = $20.00 per unit, $A = $250 / order, $=I_c = 0.12 / $ / year, $I_e = 0.09 / $ / year, $\alpha = 10%, \beta = 1,$ $M = 30/365$ year, the optimal selling price $P_1 = $42.59 per unit and cycle time $T_1 = 0.9163$ year. The maximum profit per time unit is $Z_1(P_1,T_1) = $1952.5 and optimum purchase quantity is 95 units. For, $P_1 = $42.59 / unit and $T_1 = 0.9163$ years

\[
\frac{\partial^2 Z_1(T,P)}{\partial P^2} = -2.454, \quad \frac{\partial^2 Z_1(T,P)}{\partial T^2} = -624.51 \quad \text{and} \quad \frac{\partial^2 Z_1(T,P)}{\partial P \partial T} = \left(\frac{\partial^2 Z_1(T,P)}{\partial P \partial T}\right)^2 = 1475.5 > 0
\]
guarantees maximum profit. The 3D-plot drawn in the range $[35,55]$ for $P$ and $[0.5,1.5]$ for $T$ exhibits that $Z_1 = (42.59, 0.9163) = $1952.52 is the maximum profit.

**Fig: 5.3.5.1** Convexity when $M \leq T
Example2: For $a = 200000$, $b = 0.1$, $\eta = 1.0$, $h = \$ 1.00$ per unit/year, $C = \$ 21.00$ per unit, $A = \$ 54$ per unit, $I_e = 0.01$ per unit/year, $a = 0.2$, $\beta = 1.1$, $M = 60/365$ year, $\ell = \$ 1.00$ per unit/year, $g_{1829} = \$ 21.00$ per unit, $g_{1827} = \$ 54$ per order, $g_{2009} = 0.2$, $g_{2010} = 1.1$, $g_{1839} = 60/365$ year, the optimal selling price $P_2 = \$ 55.78$ per unit and cycle time $T_2 = 0.1427$ year. The maximum profit per time unit is $Z_2(P_2, T_2) = \$ 112379$ and optimum purchase quantity $Q = 95$ units. For, $P_2 = \$ 55.78$ per unit and $T_2 = 0.1427$ years

$$\frac{\partial^2 Z_2(T,P)}{\partial P^2} = -34.99, \frac{\partial^2 Z_2(T,P)}{\partial T^2} = -94141.82$$

and $\frac{\partial^2 Z_2(T,P) \partial^2 Z_2(T,P)}{\partial P \partial T} - \left(\frac{\partial^2 Z_2(T,P)}{\partial P \partial T}\right)^2 = 3266920.9 > 0$

guarantees maximum profit. The 3D-plot drawn in the range $[45, 90]$ for $P$ and $[0.05, 1.5]$ for $T$ explores that obtained profit $Z_2 = (55.78, 0.1427) = \$ 1123379$ is the maximum profit.
Using the data of example 1, the sensitivity analysis is carried out by changing values of $M, \alpha, \beta, b$ from -40%, -20%, 20%, 40%. The variation in cycle time, selling price, purchase units and total profit per time unit are exhibited in table 5.3.5.1.
### Table 5.3.5.1: Sensitivity Analysis

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% Changes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$T$   $P$     $Q$  $K_1$</td>
</tr>
<tr>
<td>$M$</td>
<td>-40 0.38 0.37 0.34 -0.46</td>
</tr>
<tr>
<td></td>
<td>-20 0.19 0.18 0.56 -0.20</td>
</tr>
<tr>
<td></td>
<td>20 -0.19 -0.21 0.96 0.20</td>
</tr>
<tr>
<td></td>
<td>40 -0.40 -0.39 0.10 0.46</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>-40 -6.54 -0.42 -6.18 -1.80</td>
</tr>
<tr>
<td></td>
<td>-20 -3.45 -0.23 -2.90 -0.97</td>
</tr>
<tr>
<td></td>
<td>20  3.93  0.23  4.91  1.02</td>
</tr>
<tr>
<td></td>
<td>40  8.44  0.51  9.65  2.04</td>
</tr>
<tr>
<td>$\beta$</td>
<td>-40 -3.93 -0.37 -1.05  1.69</td>
</tr>
<tr>
<td></td>
<td>-20 -1.73 -0.16  0.00  0.72</td>
</tr>
<tr>
<td></td>
<td>20  1.43  0.11  1.05 -0.12</td>
</tr>
<tr>
<td></td>
<td>40  2.55  0.21  1.05 -0.22</td>
</tr>
<tr>
<td>$b$</td>
<td>-40 23.85 -0.63 25.26  5.02</td>
</tr>
<tr>
<td></td>
<td>-20  9.93  0.70 10.52  2.40</td>
</tr>
<tr>
<td></td>
<td>20 -7.57 -0.56 -7.36 -2.20</td>
</tr>
<tr>
<td></td>
<td>40 -13.61 -0.98 -13.68 -4.25</td>
</tr>
</tbody>
</table>
It is observed that increases in delay period decreases cycle time and selling price. While increases in procurement quantity not significantly and profit significantly. Increase in deterioration rate $\alpha$ increases selling price and profit. The variation in shape parameter $\beta$ significantly increases cycle time and decreases profit. The decrease in profit is due to more deterioration unit with respect to time. The increase in declining demand rate decrease cycle time, purchase quantity and profit significantly.

5.4 Conclusion

A mathematical model is developed when demand of the product is declining and units in inventory are subject to constant deterioration. The idea of pre and post deterioration discount on unit selling price is explore to maximize the profit. Different scenarios are studies by relaxing conditions. The model is very sensitive to rate of change of demand, selling price and purchase price of the unit.

Then by generalize the result for time dependent deterioration of units in inventory when demand of the product is declining in the market and supplier offers the
retailer a credit period to settle the account. It is established that the retailer should replenish smaller order more frequently to avail of sales promotional tool as trade credit.

The study is interesting by allowing partial backlogging, inflation, stochastic demand etc.