CHAPTER 3

A Generalized Measure of ‘Useful’ R-norm Information of Degree $\beta$
with Application

3.1 Introduction

Let us consider the set of positive real numbers, not equal to 1 and defined as $R^+ = \{R: R > 0 \neq 1\}$. Let $\Delta_n$ with $n \geq 2$ be the set of all probability distributions

$$P = \left\{ (p_1, p_2, \ldots, p_n), p_i \geq 0 \text{ and } \sum_{i=1}^{n} p_i = 1 \right\}.$$

Boekee and Lubbe (1980) considered R-norm information of distribution $P$ defined for $R \in R^+$ by

$$H^R_P(P) = \frac{R}{R-1} \left[ 1 - \left( \sum_{i=1}^{n} p_i^R \right)^{\frac{1}{R}} \right]. \quad (3.1.1)$$

The measure (3.1.1) can be generalized in so many ways; Hooda and Ram (1998) proposed and characterized the following parametric generalization of (3.1.1):

$$H^\beta_P(P) = \frac{R}{R + \beta - 2} \left[ 1 - \left( \sum_{i=1}^{n} p_i^{\frac{R}{R-R}} \right)^{\frac{2-\beta}{R}} \right], \quad 0 < \beta \leq 1, \quad R(> 0) \neq 1. \quad (3.1.2)$$

The above measure (3.1.2) was called generalized R-norm information measure of degree $\beta$ and it reduces to (3.1.1) when $\beta = 1$.

Belis and Guiasu (1968) characterized a ‘qualitative-quantitative’ measure which was called ‘useful’ information by Longo (1972) of the experiment E and is given as:

$$H \left( P; U \right) = H \left( p_1, p_2, \ldots, p_n; u_1, u_2, \ldots, u_n \right). \quad (3.1.3)$$
Bhaker and Hooda (1993) characterized the following measure of ‘useful’ information:

\[
H(P;U) = -\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i}.
\]  

(3.1.4)

Hooda et al. (2013) characterized the ‘useful’ R-norm information measure given below:

\[
H_R(P;U) = \frac{R}{R - 1} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{R}} \right], \quad R(> 0) \neq 1.
\]  

(3.1.5)

Analogous to (3.1.2) we consider the ‘useful’ R-norm information measure of degree \( \beta \) as given below:

\[
H_R^\beta(P;U) = \frac{R}{R + \beta - 2} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2-\beta}}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{2-\beta}{R}} \right], \quad R(> 0) \neq 1, \quad 0 < \beta \leq 1, \quad R + \beta \neq 2.
\]  

(3.1.6)

where \( U = \{u_1, u_2, \ldots, u_n\} \) is the utility distribution and \( u_i > 0 \) corresponding to probability distribution \( P = \{p_1, p_2, \ldots, p_n\} \). In case \( \beta = 1 \), measure (3.1.6) reduces to (3.1.5) and further \( R \to 1 \) it reduces to (3.1.4).

In the present chapter the ‘useful’ R-norm information measure of degree \( \beta \) is characterized axiomatically in section 3.2. In section 3.3 the properties of ‘useful’ R-norm information measure of degree \( \beta \) are discussed. In section 3.4 give a brief account of mean codeword lengths and their bounds and a new generalized ‘useful’ mean codeword length of
degree $\beta$ is introduced. In section 3.5 the lower and upper bounds of generalized ‘useful’ mean codeword length of degree $\beta$ are derived in terms of ‘useful’ R-norm information measure of degree $\beta$. In section 3.6 the monotone behaviour of $H_{\beta}^\beta(P;U)$ is discussed and conclusion in 3.7.

### 3.2 Characterization of a Generalized Measure of ‘Useful’ R-norm Information of Degree $\beta$

Let $S_n = \Delta_n \times \Delta_n^* \to R^*$, $n = 2,3,\ldots$ and $G_n$ be a sequence of functions of $p_i$’s and $u_i$’s, $i = 1,2,\ldots,n$ defined over $S_n$ satisfying the following axioms:

**Axiom 3.2.1** $G_n(P:U) = a_1 + a_2 \sum_{i=1}^{n} h(p_i,u_i)$, where $a_1$ and $a_2$ are non zero constants, and $p,u \in J = \{(0,1) \times (0,\infty)\} \cup \{(0,y); 0 \leq y \leq 1\} \cup \{(y',\infty); 0 \leq y' \leq \infty\}$.

This axiom is also called sum property.

**Axiom 3.2.2** For $P \in \Delta_n, U \in \Delta_n^*, P' \in \Delta_m$, and $U' \in \Delta_m^*$, $G_{mn}$ satisfies the following property:

$$G_{mn}(PP':UU') = G_n(P:U) + G_m(P':U') - \frac{1}{a_i} G_n(P:U) G_m(P':U').$$

**Axiom 3.2.3** $h(p,u)$ is a continuous function of its arguments $p$ and $u$.

**Axiom 3.2.4** Let all $p_i$'s and $u_i$'s are equiprobable and of equal utility of events respectively, then

$$G_n\left(\frac{1}{n},\ldots,\frac{1}{n};u_i,\ldots,u_i\right) = \frac{R}{R + \beta - 2} \left[1 - n^{-\beta + 2 - R} \right],$$

where $n = 2,3,\ldots$, and $0 < \beta \leq 1, R(> 0) \neq 1$.

First of all we prove three lemmas to facilitate the proof of the main theorem.
Lemma 3.2.1 From axiom 3.2.1 and 3.2.2, it is very easy to arrive at the following functional equation:

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} h(p_i, p'_j, u_i u'_j) = \left( -\frac{a_2}{a_1} \right) \sum_{i=1}^{n} h(p_i, u_i) \sum_{j=1}^{m} h(p'_j, u'_j),
\]

where \((p_i, u_i), (p'_j, u'_j) \in J\) for \(i = 1, 2, \ldots, n\) and \(j = 1, 2, \ldots, m\).

Lemma 3.2.2 The continuous solution that satisfies (3.2.1) is the continuous solution of the functional equation:

\[
h(p p', u u') = \left( -\frac{a_2}{a_1} \right) h(p, u) h(p', u'), \quad \text{where } a_1 \text{ and } a_2 \text{ are arbitrary constants.}
\]

Proof: This Lemma can be proved on following the lines of Hooda et al. (2013).

Next we obtain the general solution of (3.2.2).

Lemma 3.2.3 One of the general continuous solutions of equation (3.2.2) is given by

\[
h(p, u) = \left( -\frac{a_1}{a_2} \right) \left( \frac{p^\mu u^\nu}{pu} \right)^{\nu/\mu}, \quad \text{where } \mu \neq 0, \nu \neq 0
\]

and \(h(p, u) = 0\).

Proof: Taking \(g(p, u) = \left( -\frac{a_2}{a_1} \right) h(p, u)\) in (3.2.2), we have

\[
g(pp', uu') = g(p, u) g(p', u').
\]

The most general continuous solution of (3.2.5) [Ref. to Aczel (1966)] is given by

\[
g(p, u) = \left( \frac{p^\mu u^\nu}{pu} \right)^{\nu/\mu}, \quad \text{where } \mu \neq 0 \text{ and } \nu \neq 0
\]

and \(g(p, u) = 0\).
On substituting \( g(p,u) = \left( -\frac{a_i}{a_1} \right) h(p,u) \) in (3.2.6) and (3.2.7) we get (3.2.3) and (3.2.4) respectively. This proves the lemma 3.2.3 for all rationals \( p \in ]0,1[ \) and \( u > 0 \), however, by continuity it holds for all reals \( p \in ]0,1[ \) and \( u > 0 \).

Theorem 3.2.1 The measure (3.1.6) can be determined by the axioms 3.2.1 to 3.2.4.

Proof: Substituting the solution (3.2.3) in axiom 3.2.1 we have

\[
G_n(P;U) = a_1 \left[ 1 - \left( \frac{\sum_{i=1}^{n} p_i u_i^{1/\nu}}{\sum_{i=1}^{n} p_i u_i} \right)^{1/\mu} \right], \quad \mu \nu \neq 0. \tag{3.2.8}
\]

Taking \( p_i = \frac{1}{n} \) and \( u_i = u \) for each \( i \) in (3.2.8), we have

\[
G_n\left( \frac{1}{n}, \ldots, \frac{1}{n}, u, \ldots, u \right) = a_1 \left( 1 - n^{-1/\nu} u^{1/\mu} \right), \quad n = 2,3,\ldots. \tag{3.2.9}
\]

Axiom 3.2.4 together with (3.2.9) gives

\[
a_1 \left( 1 - n^{-1/\nu} u^{1/\mu} \right) = \frac{R}{R + \beta - 2} \left[ 1 - n^{-\frac{\beta + 2 - R/\mu}{\beta}} \right].
\]

It implies

\[
a_1 = \frac{R}{R + \beta - 2}, \quad \mu = \frac{R}{2 - \beta}, \quad \nu = 1.
\]

Putting these values in (3.2.8) we have

\[
G_n(P;U) = \frac{R}{R + \beta - 2} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^{R/2-\beta}}{\sum_{i=1}^{n} u_i p_i} \right)^{2-\beta/\mu} \right] = H^\beta_n(P;U).
\]

Hence this completes the proof of theorem 3.2.1.
3.3 Properties of $H_R^\beta(P;U)$

Property 3.3.1 $H_R^\beta(P;U)$ is a symmetric function of their arguments provided that permutation of $p_i^i$ and $u_i^i$ are taken together.

\[
H_R^\beta(p_1, p_2, \ldots, p_{n-1}, p_n; u_1, u_2, \ldots, u_{n-1}, u_n) = H_R^\beta(p_n, p_1, p_2, \ldots, p_{n-1}; u_n, u_1, u_2, \ldots, u_{n-1}).
\]

Property 3.3.2 For complete probability distribution ‘useful’ R-norm information measure of degree $\beta$, $H_R^\beta(P;U)$ does not satisfy the normality property. However, for incomplete probability distribution the following holds:

\[
H^1_2\left(\frac{1}{4}, \frac{1}{4}; 1, 1\right) = 1.
\]

Proof: \[
H_R^\beta(P;U) = \frac{R}{R + \beta - 2} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^{\frac{2-\beta}{R}}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{R}{2-\beta}} \right], \text{ for } i = 1, 2
\]

It implies

\[
H_R^\beta(P;U) = \frac{R}{R + \beta - 2} \left[ 1 - \left( \frac{u_1 p_1^{\frac{2-\beta}{R}} + u_2 p_2^{\frac{2-\beta}{R}}}{u_1 p_1 + u_2 p_2} \right)^{\frac{R}{2-\beta}} \right]
\]

Setting

\[
p_1 = p_2 = \frac{1}{4}; u_1 = u_2 = 1, \beta = 1 \text{ and } R = 2, \text{ we get}
\]

\[
H^1_2\left(\frac{1}{4}, \frac{1}{4}; 1, 1\right) = 1.
\]

Property 3.3.3 Addition of one event whose probability of occurrence is zero or utility is zero has no effect on ‘useful’ information, i.e.

\[
H_R^\beta(p_1, p_2, \ldots, p_n, 0; u_1, u_2, \ldots, u_{n+1}) = H_R^\beta(p_1, p_2, \ldots, p_n; u_1, u_2, \ldots, u_n)
\]

\[
= H_R^\beta(p_1, p_2, \ldots, p_n, p_{n+1}; u_1, u_2, \ldots, u_n, 0).
\]
Proof: Let us consider

\[ H_R^\beta (p_1, p_2, \ldots, p_n; 0; u_1, u_2, \ldots, u_{n+1}) = \frac{R}{R + \beta - 2} \left[ 1 - \left\{ \frac{u_1 p_1^{\frac{R}{2-\beta}} + u_2 p_2^{\frac{R}{2-\beta}} + \ldots + u_n p_n^{\frac{R}{2-\beta}} + 0}{u_1 + u_2 p_2 + \ldots + u_n p_n + 0} \right\}^{\frac{2-\beta}{R}} \right] = H_R^\beta (P; U). \]

Property 3.3.4 \( H_R^\beta (P; U) \) satisfies the non additivity of the following form:

\[ H_R^\beta (P \ast Q; U \ast V) = H_R^\beta (P; U) + H_R^\beta (Q; V) - \frac{R + \beta - 2}{R} H_R^\beta (P; U) H_R^\beta (Q; V), \]

where

\[ P \ast Q = (p_1 q_1, \ldots, p_1 q_m; p_2 q_1, \ldots, p_2 q_m, p_n q_1, \ldots, p_n q_m) \]
\[ U \ast V = (u_1 v_1, \ldots, u_1 v_m, u_2 v_1, \ldots, u_2 v_m, u_n v_1, \ldots, u_n v_m). \]

Proof: R.H.S. = \( H_R^\beta (P; U) + H_R^\beta (Q; V) - \frac{R + \beta - 2}{R} H_R^\beta (P; U) H_R^\beta (Q; V) \)

\[ = \frac{R}{R + \beta - 2} \left[ 1 - \left\{ \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} u_i v_j (p_i q_j)^{\frac{R}{2-\beta}}}{\sum_{i=1}^{n} \sum_{j=1}^{m} u_i v_j p_i q_j} \right\}^{\frac{2-\beta}{R}} \right] = H_R^\beta (P; Q; U; V) = \text{L.H.S.} \]

Property 3.3.5 Let \( A_i, A_j \) be two events having probabilities \( p_i, p_j \) and utilities \( u_i, u_j \) respectively, and then define the utility \( u \) of compound event \( A_i \cap A_j \) as

\[ u(A_i \cap A_j) = \frac{u_i p_i + u_j p_j}{p_i + p_j}. \quad (3.3.1) \]
Theorem 3.3.1 Under the composition law (3.3.1), the following holds:

\[ n_1 H_R^\beta (p_1, p_2, \ldots, p_{n-1}, p', p^*, u_1, u_2, \ldots, u_{n-1}, u', u^*) = n H_R^\beta (P, U) + (p' + p^*)^{R+2-\beta \frac{R}{R}} H_R^\beta \left( \frac{p'}{p' + p^*}, \frac{p^*}{p' + p^*}; u', u^* \right) \]

where \( p_n = p' + p^* \) and \( u_n = \frac{p'u' + p^* u^*}{p' + p^*} \).

**Proof:** L.H.S. = \( n_1 H_R^\beta (p_1, p_2, \ldots, p_{n-1}, p', p^*; u_1, u_2, \ldots, u_{n-1}, u', u^*) \)

\[ = n_1 H_R^\beta (p_1, p_2, \ldots, p_{n-1}; u_1, u_2, \ldots, u_{n-1}) + \frac{R}{R + \beta - 2} \left[ 1 - \left\{ \frac{R}{R} u'p'^{2-\beta} + u'^{2-\beta} \right\} \right] \]

\[ = n H_R^\beta (P; U) + \frac{R}{R + \beta - 2} \left[ \frac{(p' + p^*)^{R-2+\beta \frac{R}{R}} - \left\{ \frac{R}{R} u'p'^{2-\beta} + u'^{2-\beta} \right\} \right] \]

\[ = n H_R^\beta (P; U) + \frac{R}{R + \beta - 2} \left[ (p' + p^*)^{R-2+\beta \frac{R}{R}} - \left\{ \frac{R}{R} u'p'^{2-\beta} + u'^{2-\beta} \right\} \right] \]

\[ = n H_R^\beta (P; U) + (p' + p^*)^{R-2+\beta \frac{R}{R}} H_R^\beta \left( \frac{p'}{p' + p^*}, \frac{p^*}{p' + p^*}; u', u^* \right) \]

This completes the proof of theorem 3.3.1
3.4 A Generalized ‘Useful’ Mean Codeword Length

Let a finite set of \( n \) input symbols \( X = \{x_1, x_2, \ldots, x_n\} \) be encoded using \( D \) size alphabets with Probability distribution \( P = \left\{ (p_1, p_2, \ldots, p_n), p_i \geq 0, \sum_{i=1}^{n} p_i = 1 \right\} \). It was shown by Kraft (1949) that there is a unique decipherable code with codewords lengths \( l_i, i = 1, 2, \ldots, n \) satisfying the following inequality:

\[
\sum_{i=1}^{n} D^{-l_i} \leq 1, \quad (3.4.1)
\]

which is known as Kraft’s inequality.

Let \( L = \sum_{i=1}^{n} p_i l_i \) \( (3.4.2) \) be the mean codeword length associated with input symbols \( \{x_1, x_2, \ldots, x_n\} \) then under the Kraft’s inequality given by (3.4.1) Shannon’s (1948) proved the following result for a noiseless channel:

\[
H(P) \leq L < H(P) + 1, \quad D \geq 2. \quad (3.4.3)
\]

With equality if and only if \( l_i = -\log_D p_i \).

Guiasu and Picard (1971) considered the problem of encoding by means of a single letter prefix code whose code words \( w_1, w_2, \ldots, w_n \) are of lengths \( l_1, l_2, \ldots, l_n \) respectively and satisfy the Kraft’s inequality (3.4.1)

They introduced the following ‘useful’ mean length of code:

\[
L = \frac{\sum_{i=1}^{n} u_i p_i l_i}{\sum_{i=1}^{n} u_i p_i}, \quad (3.4.4)
\]

Later on Longo (1972) interpreted as the average transmission cost of letters \( x_i \) and obtained the following bounds for the cost measure \( L \) given by (3.4.4):

\[
H(P;U) \leq L < H(P;U) + 1. \quad (3.4.5)
\]
Hooda et al. (2013) considered the following generalization of (3.4.4):

\[
L_R^\beta (P; U) = \frac{R}{R - 1} \left[ 1 - \frac{\sum_{i=1}^{n} u_i p_i D_{-i}^{\left( \frac{R-1}{R} \right)}}{\sum_{i=1}^{n} u_i p_i} \right],
\]

(3.4.6)

and proved

\[
H_R (P; U) \leq L_R (P; U) < H_R (P; U) + 1,
\]

(3.4.7)

under the condition

\[
\sum_{i=1}^{n} u_i D^{-i} \leq \sum_{i=1}^{n} u_i p_i.
\]

(3.4.8)

It may be noted that (3.4.8) reduces to Kraft’s inequality if utilities are ignored i.e. \( u_i = 1 \) for all \( i \).

Here we give the following new generalized ‘useful’ mean code word length of degree \( \beta \) as given below:

\[
L_R^\beta (P; U) = \frac{R}{R + \beta - 2} \left[ 1 - \frac{\sum_{i=1}^{n} u_i p_i D_{-i}^{\left( \frac{R+\beta-2}{R} \right)}}{\sum_{i=1}^{n} u_i p_i} \right].
\]

(3.4.9)

In case \( \beta = 1 \), then (3.4.9) reduces to (3.4.6). Further if utilities are ignored for all \( i = 1, 2, \ldots, n \) it reduces to

\[
L_R (P) = \frac{R}{R - 1} \left[ 1 - \sum_{i=1}^{n} p_i D_{-i}^{\left( \frac{R-1}{R} \right)} \right],
\]

(3.4.10)

which is average codeword length due to Boekee and Lubbe (1980).

### 3.5 Coding Theorem on Bounds of \( H_R^\beta (P; U) \)

In this section we study the lower and upper bounds of \( L_R^\beta (P; U) \) in terms of ‘useful’ \( R \)-norm information measure of degree \( \beta \) given by (3.1.6).
Theorem 3.5.1 If \( l_i, i = 1, 2, \ldots, n \) is length of codeword \( x_i \) satisfying (3.4.1), then

\[
H^p_R(P; U) \leq L^p_R(P; U),
\]

(3.5.1)

under the condition

\[
\sum_{i=1}^{n} u_i D^{-l_i} \leq \sum_{i=1}^{n} u_i p_i.
\]

(3.5.2)

Proof: By Holder’s inequality we have

\[
\left( \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}} \leq \sum_{i=1}^{n} x_i y_i,
\]

(3.5.3)

where \( p^{-1} + q^{-1} = 1, p \neq 0 < 1, q < 0 \) or \( q \neq 0 < 1, p < 0, x_i, y_i \geq 0 \) for each \( i \).

Setting \( x_i = \left( \frac{u_i p_i}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{R}{R+\beta-2}}, y_i = \left( \frac{u_i p_i^{2-\beta}}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{2-\beta}{2-\beta-R}} \)

\( p = \frac{R+\beta-2}{R}, \quad q = \frac{2-\beta-R}{2-\beta} \)

and put all these values in (3.5.3), we have

\[
\left( \sum_{i=1}^{n} u_i p_i D^{-l_i} \right)^{\frac{R}{R+\beta-2}} \leq \left( \sum_{i=1}^{n} u_i p_i \right)^{\frac{2-\beta}{2-\beta-R}} \left( \sum_{i=1}^{n} u_i p_i \right)^{\frac{R}{2-\beta-R}} \leq \frac{n}{\sum_{i=1}^{n} u_i p_i}
\]

(3.5.4)

It implies

\[
\left( \sum_{i=1}^{n} u_i p_i \right)^{\frac{2-\beta}{2-\beta-R}} \leq \left( \sum_{i=1}^{n} u_i p_i \right)^{\frac{R}{2-\beta-R}} \left( \sum_{i=1}^{n} u_i p_i \right)^{\frac{R}{R+\beta-2}} \left( \sum_{i=1}^{n} u_i p_i \right)^{\frac{R}{2-\beta-R}} \leq \frac{n}{\sum_{i=1}^{n} u_i p_i}
\]

(3.5.4)
Case 1 Let $0 < R + \beta < 2$. Raising Power $\frac{2 - \beta - R}{R} > 0$ to both sides of (3.5.4), we have

$$
\left( \sum_{i=1}^{n} u_i p_i^{2-\beta} \right)^{\frac{2-\beta}{R}} \leq \frac{\sum_{i=1}^{n} u_i p_i D^{-\frac{R+\beta-2}{R}}}{\sum_{i=1}^{n} u_i p_i}.
$$

Subtracting both sides from 1, we have

$$
1 - \left( \sum_{i=1}^{n} u_i p_i^{2-\beta} \right)^{\frac{2-\beta}{R}} \geq 1 - \frac{\sum_{i=1}^{n} u_i p_i D^{-\frac{R+\beta-2}{R}}}{\sum_{i=1}^{n} u_i p_i}.
$$

(3.5.5)

Multiplying (3.5.5) by $\frac{R}{R + \beta - 2} < 0$ throughout we have

$$
\frac{R}{R + \beta - 2} \left[ 1 - \left( \sum_{i=1}^{n} u_i p_i^{2-\beta} \right)^{\frac{2-\beta}{R}} \right] \leq \frac{R}{R + \beta - 2} \left[ 1 - \frac{\sum_{i=1}^{n} u_i p_i D^{-\frac{R+\beta-2}{R}}}{\sum_{i=1}^{n} u_i p_i} \right].
$$

Hence $H_{k}^{\beta} (P; U) \leq L_{k}^{\beta} (P; U)$. (3.5.6)

Case 2 Let $R + \beta > 2$. Raising power $\frac{2 - \beta - R}{R} < 0$ to both sides of (3.5.4)

$$
\left( \sum_{i=1}^{n} u_i p_i^{2-\beta} \right)^{\frac{2-\beta}{R}} \geq \frac{\sum_{i=1}^{n} u_i p_i D^{-\frac{R+\beta-2}{R}}}{\sum_{i=1}^{n} u_i p_i}.
$$

(3.5.7)

Subtracting both sides from 1, we have

$$
1 - \left( \sum_{i=1}^{n} u_i p_i^{2-\beta} \right)^{\frac{2-\beta}{R}} \leq 1 - \frac{\sum_{i=1}^{n} u_i p_i D^{-\frac{R+\beta-2}{R}}}{\sum_{i=1}^{n} u_i p_i}.
$$
Multiplying (3.5.7) by $\frac{R}{R + \beta - 2} > 0$ throughout we have

$$
\frac{R}{R + \beta - 2} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2-\beta}}}{\sum_{i=1}^{n} u_i p_i} \right)^{2-\beta} R \right] \leq \frac{R}{R + \beta - 2} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i D^{-i} \left( \frac{R+\beta-2}{R} \right)}{\sum_{i=1}^{n} u_i p_i} \right) \right]
$$

Hence $H^\beta_R (P; U) \leq L^\beta_R (P; U)$.

(3.5.8)

Thus theorem 3.5.1 is proved in both cases.

In (3.5.1) equality holds if and only if

$$D^{-i} = \frac{\frac{R}{p_i^{2-\beta}}}{\sum_{i=1}^{n} \frac{R}{u_i p_i^{2-\beta}}} \sum_{i=1}^{n} \frac{u_i p_i}{u_i p_i} \sum_{i=1}^{n} u_i p_i$$

or

$$l_i = -\log \frac{R}{p_i^{2-\beta}} + \log \frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2-\beta}}}{\sum_{i=1}^{n} u_i p_i}$$

Choosing $l_i$ such that

$$\log \frac{R}{p_i^{2-\beta}} \left[ \sum_{i=1}^{n} \frac{u_i p_i^{\frac{R}{2-\beta}}}{\sum_{i=1}^{n} u_i p_i} \right] \leq l_i < \log \frac{R}{p_i^{2-\beta}} \left[ \sum_{i=1}^{n} \frac{u_i p_i^{\frac{R}{2-\beta}}}{\sum_{i=1}^{n} u_i p_i} \right] + 1.$$  

(3.5.9)

It implies

$$\frac{R}{p_i^{2-\beta}} \left[ \sum_{i=1}^{n} \frac{u_i p_i^{\frac{R}{2-\beta}}}{\sum_{i=1}^{n} u_i p_i} \right] \leq D^i \leq D \frac{R}{p_i^{2-\beta}} \left[ \sum_{i=1}^{n} \frac{u_i p_i^{\frac{R}{2-\beta}}}{\sum_{i=1}^{n} u_i p_i} \right].$$  

(3.5.10)
In the next theorem, we obtain an upper bound on $I^{-h_{R\beta}(P;U)}$ in terms of $H_{R\beta}^{-h_{R\beta}(P;U)}$.

**Theorem 3.5.2** Let $l_1, l_2, \ldots, l_n$ be the codeword lengths satisfying (3.5.10). Then following inequality holds:

$$I^{-h_{R\beta}(P;U)} \leq D \frac{2^{\beta - R}}{R} H_{R\beta}^{-h_{R\beta}(P;U)} + \frac{R}{R + \beta - 2} \left( 1 - D \frac{2^{\beta - R}}{R} \right).$$

(3.5.11)

**Proof:** From the R.H.S. of (3.5.10), we have

$$D^i < Dp_i \left( \sum_{i=1}^{n} \frac{R}{u_i} \sum_{i=1}^{n} u_i p_i \right).$$

(3.5.12)

Here two cases arise:

**Case 1** Let $0 < R + \beta < 2$. Raising both sides of (3.5.12) to the power $\frac{2 - \beta - R}{R} > 0$,

we get

$$D^{-i} \left( \frac{R + \beta - 2}{R} \right) < D \frac{2^{-\beta - R}}{R} \frac{R}{p_i} \left( \sum_{i=1}^{n} \frac{R}{u_i} \sum_{i=1}^{n} u_i p_i \right).$$

(3.5.13)

Multiplying both sides of (3.5.13) by $\frac{u_i p_i}{\sum_{i=1}^{n} u_i p_i}$ and summing over $i$, we have

$$\sum_{i=1}^{n} u_i p_i D^{-i} \left( \frac{R + \beta - 2}{R} \right) < D \frac{2^{-\beta - R}}{R} \frac{R}{p_i} \left( \sum_{i=1}^{n} \frac{R}{u_i} \sum_{i=1}^{n} u_i p_i \right).$$

(3.5.14)
or

\[
\sum_{i=1}^{n} u_i p_i D \sum_{i=1}^{n} u_i p_i^{2-\beta R} < D \left( \sum_{i=1}^{n} u_i p_i^{\frac{R}{R}} \right) \left( \sum_{i=1}^{n} u_i p_i^{2-\beta R} \right)^{\frac{R}{2-\beta R}}.
\]

Subtracting both sides from 1 and multiplying by \( R \) \( \frac{R}{R + \beta - 2} < 0 \), we have

\[
L_r^\beta (P;U) < D \frac{2-\beta R}{R} H_r^\beta (P;U) + \frac{R}{R + \beta - 2} \left[ 1 - D \frac{2-\beta R}{R} \right].
\]

(3.5.14)

Similarly, we can prove that (3.5.14) holds when \( R + \beta > 2 \). Hence theorem is proved

Thus (3.5.1) and (3.5.11) together give

\[
H_r^\beta (P;U) \leq L_r^\beta (P;U) < D \frac{2-\beta R}{R} H_r^\beta (P;U) + \frac{R}{R + \beta - 2} \left[ 1 - D \frac{2-\beta R}{R} \right],
\]

which is coding theorem for noiseless channel on ‘useful’ R-norm information of degree \( \beta \).

3.6 Monotone Behaviour of \( H_r^\beta (P;U) \)

In this section we discuss the monotone behaviour of a generalized ‘useful’ R-norm measure of information degree \( \beta \).

Let us assume \( R = 2, 3 \) and 4 corresponding to different values of \( \beta \). Using (3.1.6) we have constructed table 3.1 listed below. Looking at table 3.1, it is clear that the generalized measure of ‘useful’ R-norm information of degree \( \beta \) given by (3.1.6) is monotonically decreasing function of \( \beta \). The figure 3.1 of \( H_r^\beta (P;U) \) with respect to \( \beta \) is plotted based on the following computed table 3.1:
Table 3.1: Monotone Behaviour of $H^B_{R}(P;U)$ with respect to $\beta$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$H^B_{2}(P;U)$</th>
<th>$H^B_{3}(P;U)$</th>
<th>$H^B_{4}(P;U)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.6759</td>
<td>1.27157</td>
<td>1.11107</td>
</tr>
<tr>
<td>0.2</td>
<td>1.5826</td>
<td>1.23609</td>
<td>1.08719</td>
</tr>
<tr>
<td>0.3</td>
<td>1.5337</td>
<td>1.20162</td>
<td>1.06352</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4672</td>
<td>1.16805</td>
<td>1.03996</td>
</tr>
<tr>
<td>0.5</td>
<td>1.4050</td>
<td>1.13252</td>
<td>1.01640</td>
</tr>
</tbody>
</table>
From figure 3.1, it is clear that the function is monotonic decreasing with respect to $\beta$ for given values of $R = 2, 3, 4$. 
Let us assume $\beta=0.1, 0.2, \text{ and } 0.3$, corresponding to different values of $R$. The figure 3.2 of $H_R^{\beta}(P;U)$ with respect to $R$ is plotted based on the following table 3.2:

**Table 3.2: Monotone Behaviour of $H_R^{\beta}(P;U)$ with respect to $R$**

<table>
<thead>
<tr>
<th>$R$</th>
<th>$H_{R}^{0.1}(P;U)$</th>
<th>$H_{R}^{0.2}(P;U)$</th>
<th>$H_{R}^{0.3}(P;U)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.6759</td>
<td>1.2715</td>
<td>1.5329</td>
</tr>
<tr>
<td>3</td>
<td>1.2715</td>
<td>1.2360</td>
<td>1.2016</td>
</tr>
<tr>
<td>4</td>
<td>1.1110</td>
<td>1.0871</td>
<td>1.0635</td>
</tr>
<tr>
<td>5</td>
<td>1.0211</td>
<td>1.0022</td>
<td>0.9831</td>
</tr>
<tr>
<td>6</td>
<td>0.9606</td>
<td>0.9442</td>
<td>0.9276</td>
</tr>
</tbody>
</table>
Fig. 3.2: Graph between $H_R^\beta (P; U)$ and $R$

From figure 3.2, it is clear that the function is monotonic decreasing with respect to $R$ for given values of $\beta = 0.1, 0.2, 0.3$. 
3.7 Conclusion

In the present chapter, we have studied the parametric generalization of R-norm measure of information given by Hooda and Ram (1998) by attaching the utility and characterized it axiomatically.

Various authors have defined generalized mean codeword lengths, which are additive and studied their bounds in terms of generalized ‘useful’ measures of information. However, we have introduced a new generalized ‘useful’ mean codeword length of degree $\beta$, which is non-additive in nature and have studied application of new generalized ‘useful’ R-norm measure of information in source coding.

Further, we have also studied the monotonic behaviour of $H^\beta_R(P;U)$ with respect to respective parameters. This study can be extended to more non-additive generalized mean codeword lengths.