CHAPTER 2

On ‘Useful’ R-norm Information Measure and its Application

2.1 Introduction

Let us consider the set of positive real numbers, not equal to 1 and defined as \( \mathbb{R}^+ = \{ R : R > 0, R \neq 1 \} \). Let \( \Delta_n \) with \( n \geq 2 \) is the set of all probability distributions

\[
P = \left\{ (p_1, p_2, \ldots, p_n), p_i \geq 0, \text{for each } i \text{ and } \sum_{i=1}^{n} p_i = 1 \right\}.
\]

Boekee and Lubbe (1980) studied R-norm information of the distribution \( p \) defined for \( R \in \mathbb{R}^+ \) by

\[
H_{R}(P) = \frac{R}{R-1} \left[ 1 - \left( \sum_{i=1}^{n} p_i^R \right)^{\frac{1}{R}} \right].
\]  

(2.1.1)

The R-norm information measure (2.1.1) is a real function \( \Delta_n \to \mathbb{R}^+ \) defined on \( \Delta_n \), where \( n \geq 2 \) and \( \mathbb{R}^+ \) is the set of real positive numbers. The measure (2.1.1) is different from entropies of Shannon (1948), Renyi (1961), Havrda and Charvat (1967) and Daroczy (1970). The main property of this measure is that when \( R \to 1 \), (2.1.1) approaches to Shannon’s entropy and when \( R \to \infty \), \( H_{R}(P) \to 1 - \max p_i \), where \( i = 1, 2, \ldots, n \).

In order to distinguish the events \( E_1, E_2, \ldots, E_n \) with respect to a given qualitative characteristic of physical system taken into account, a non negative number \( u(E_i) = u_i (> 0) \) was ascribed to each \( E_i \) and that was directly proportional to its importance \( u_i \) was called the utility or importance of event \( E_i \) whose probability of occurrence was \( p_i \). In general \( u_i \) is independent of \( p_i \) [Refer to Longo (1972)].
Belis and Guiasu (1968) characterized a quantitative-qualitative measure which was called ‘useful’ information by Longo (1972) of the experiment E and is given as

\[ H(P; U) = H(p_1, p_2, \ldots, p_n; u_1, u_2, \ldots, u_n) \]

\[ = -\sum_{i=1}^{n} u_i p_i \log p_i, \quad u_i > 0, \quad 0 < p_i \leq 1, \quad \sum_{i=1}^{n} p_i = 1. \] (2.1.2)

Bhaker and Hooda (1993) characterized the following measure of ‘useful’ information:

\[ H(P; U) = \frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i}. \] (2.1.3)

We consider the following ‘useful’ R-norm information measure corresponding to (2.1.1):

\[ H_R(P; U) = \frac{R}{R-1} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i^R} \right)^{\frac{1}{R}} \right], \quad R > 0 (\neq 1). \] (2.1.4)

Let \( P = \left\{ p_1, p_2, \ldots, p_n, 0 \leq p_i < 1, \sum_{i=1}^{n} p_i = 1 \right\} \) be replaced by \( \beta \) degree probability distribution given by \( P^\beta = \left\{ p_1^\beta, p_2^\beta, \ldots, p_n^\beta, 0 \leq p_i^\beta < 1, \sum_{i=1}^{n} p_i^\beta \leq 1 \right\} \), where \( \beta \geq 1 \). Then (2.1.4) becomes

\[ H_R^\beta(P; U) = \frac{R}{R-1} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{R\beta}} \right)^{\frac{1}{R}} \right], \quad R > 0 (\neq 1), \beta \geq 1, p_i \geq 0 \forall i = 1, 2, \ldots, n, \] (2.1.5)

where \( U = (u_1, u_2, \ldots, u_n) \) is the utility distribution and \( u_i > 0 \) is the utility of an event with probability \( p_i \). It may be noted that if \( R \rightarrow 1 \), (2.1.4) reduces to (2.1.3). Further let \( \Delta_n^* \) be a set of
utility distributions s.t. \( U \in \Delta_n^* \) is utility distribution corresponding to probability distribution \( P \in \Delta_n \). In case \( \beta = 1 \), (2.1.5) reduces to (2.1.4).

In this chapter a non-additive ‘useful’ R-norm information measure is characterized axiomatically in section 2.2. In section 2.3 the properties of the new measure of ‘useful’ R-norm information measure are studied. A new generalized ‘useful’ mean codeword length is introduced in section 2.4 and obtained its lower and upper bounds in terms of generalized ‘useful’ R-norm information measure in section 2.5. In section 2.6 the lower and upper bounds of generalized cost measure of degree \( \beta \) is derived in terms of generalized ‘useful’ R-norm information measure of degree \( \beta \) and conclusion is also given in 2.7.

### 2.2 Characterization of ‘Useful’ R-norm Information Measure

Let \( S_n = \Delta_n \times \Delta_n^* \rightarrow R^+ \), \( n = 2,3, \ldots \) and \( G_n \) be a sequence of functions of \( p, s \) and \( u, s \), \( i = 1,2, \ldots, n \), defined over \( S_n \) satisfying the following axioms:

**Axiom 2.2.1** \( G_n(P : U) = a_1 + a_2 \sum_{i=1}^{n} h(p_i, u_i) \), where \( a_1 \) and \( a_2 \) are non zero constants, and \( p, u \in J = \{(0,1)\times(0,\infty)\} \cup \{(0,y); 0 \leq y \leq 1\} \cup \{(y', \infty); 0 \leq y' \leq \infty\} \). This axiom is also called sum property.

**Axiom 2.2.2** For \( P \in \Delta_n, U \in \Delta_n^+, P' \in \Delta_m^+, \) and \( U' \in \Delta_m^+, G_{mn} \) satisfies the following property:

\[
G_{mn}(PP' : UU') = G_n(P : U) + G_m(P' : U') - \frac{1}{a_1} G_n(P : U) G_m(P' : U').
\]

**Axiom 2.2.3** \( h(p,u) \) is a continuous function of its arguments \( p \) and \( u \).

**Axiom 2.2.4** Let all \( p_i \)'s and \( u_i \)'s are equi-probable and of equal utility of events respectively, then

\[
G_n\left(\frac{1}{n}, \ldots, \frac{1}{n}, u, \ldots, u\right) = \frac{R}{R-1}\left(1 - n^{\frac{1-R}{R}}\right), \text{ where } n = 2,3, \ldots \text{ and } R(>0) \neq 1.
\]

First of all we prove the following three lemmas to facilitate the main theorem:
Lemma 2.2.1 From axiom 2.2.1 and 2.2.2, it is very easy to arrive at the following functional equation:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} h(p_i, p'_{j}, u_i u'_j) = \left( \frac{-a_2}{a_1} \right) \sum_{i=1}^{n} h(p_i, u_i) \sum_{j=1}^{m} h(p'_j, u'_j). \quad (2.2.1)$$

where \((p_i, u_i), (p'_j, u'_j) \in \mathcal{J}\) for \(i = 1, 2, \ldots, n\) and \(j = 1, 2, \ldots, m\).

Lemma 2.2.2 The continuous solution that satisfies (2.2.1) is the continuous solution of the functional equation:

$$h(pp', uu') = \left( \frac{-a_2}{a_1} \right) h(p, u) h(p', u'). \quad (2.2.2)$$

Proof: Let \(a, b, c, d\) and \(a', b', c', d'\) be positive integers such that

\[1 \leq a' \leq a, 1 \leq b' \leq b, 1 \leq c \leq c', \quad \text{and} \quad 1 \leq d \leq d'.\]

Setting \(n = a - a' + 1 = c' - c + 1\) and \(m = b - b' + 1 = d' - d + 1\),

\[p_i = \frac{1}{a}(i = 1, 2, \ldots, a - a'), \quad p_{a-a'+1} = \frac{a'}{a}, \]

\[u_i = \frac{1}{c}(i = 1, 2, \ldots, c' - c), \quad u_{c-c+1} = \frac{c'}{c}, \]

\[p'_j = \frac{1}{b}(j = 1, 2, \ldots, b - b'), \quad p'_{b-b'+1} = \frac{b'}{b}, \]

\[u'_j = \frac{1}{d}(j = 1, 2, \ldots, d' - d), \quad u'_{d-d'+1} = \frac{d'}{d}.\]

From equation (2.2.1) we have

\[(a - a')(b - b')h\left(\frac{1}{ab}, \frac{1}{cd}\right) + (b - b')h\left(\frac{a'}{ab}, \frac{c'}{cd}\right) + (a - a')h\left(\frac{b'}{ab}, \frac{c'}{cd}\right) + h\left(\frac{a'b'}{ab}, \frac{c'd'}{cd}\right) = \left(\frac{-a_2}{a_1}\right) \left[ (a - a')h\left(\frac{1}{a}, \frac{1}{c}\right) + h\left(\frac{a'}{a}, \frac{c'}{c}\right) \right] \left[ (b - b')h\left(\frac{1}{b}, \frac{1}{d}\right) + h\left(\frac{b'}{b}, \frac{d'}{d}\right) \right] \quad (2.2.3)\]

Taking \(a' = b' = c' = d' = 1\) in (2.2.3), we get

$$h\left(\frac{1}{ab}, \frac{1}{cd}\right) = \left(\frac{-a_2}{a_1}\right) h\left(\frac{1}{a}, \frac{1}{c}\right) h\left(\frac{1}{b}, \frac{1}{d}\right) \quad (2.2.4)$$
Taking \( a' = c' = 1 \) in (2.2.3) and using (2.2.4), we have
\[
h \left( \frac{b'}{ab}, \frac{d'}{cd} \right) = \left( -\frac{a_2}{a_1} \right) h \left( \frac{1}{a\cdot c} \right) h \left( \frac{b'}{b\cdot d} \right)
\]
(2.2.5)

Again taking \( b' = d' = 1 \) in (2.2.3) and using (2.2.5), we get
\[
h \left( \frac{a'}{ab}, \frac{c'}{cd} \right) = \left( -\frac{a_2}{a_1} \right) h \left( \frac{1}{b\cdot d} \right) h \left( \frac{a'}{a\cdot c} \right)
\]
(2.2.6)

Now (2.2.3) together with (2.2.4), (2.2.5) and (2.2.6) reduces to
\[
h \left( \frac{a'b'}{ab}, \frac{c'd'}{cd} \right) = \left( -\frac{a_2}{a_1} \right) h \left( \frac{a'}{a\cdot c} \right) h \left( \frac{b'}{b\cdot d} \right)
\]
(2.2.7)

Putting \( \frac{a'}{a} = p, \frac{c'}{c} = u, \frac{b'}{b} = p', \) and \( \frac{d'}{d} = u' \) in (2.2.7), we get the required result (2.2.2).

Next we obtain the general solution of (2.2.2).

**Lemma 2.2.3** One of the general continuous solution of equation (2.2.2) is given by
\[
h(p, u) = \left( -\frac{a_2}{a_1} \right) \left( \frac{p^\mu u^\nu}{pu} \right)^{1/\mu}, \text{ where } \mu \neq 0, \nu \neq 0
\]
(2.2.8)

and \( h(p, u) = 0. \)
(2.2.9)

**Proof:** Taking \( g(p, u) = \left( -\frac{a_2}{a_1} \right) h(p, u) \) in (2.2.2), we have
\[
g(pp', uu') = g(p, u)g(p', u').
\]
(2.2.10)

The most general continuous solution of (2.2.10) [Ref. to Aczel (1966)] is given by
\[
g(p, u) = \left( \frac{p^\mu u^\nu}{pu} \right)^{1/\mu}, \mu \neq 0 \text{ and } \nu \neq 0
\]
(2.2.11)

and
\[
g(p, u) = 0.
\]
(2.2.12)
On substituting \( g(p,u) = \left( \frac{-a_i}{a_i} \right) h(p,u) \) in (2.2.11) and (2.2.12) we get (2.2.8) and (2.2.9) respectively. This proves the lemma 2.2.3 for all rationals \( p \in ]0,1[ \) and \( u > 0 \). However, by continuity, it holds for all reals \( p \in ]0,1[ \) and \( u > 0 \).

**Theorem 2.2.1** The measure (2.1.4) can be determined by the axiom 2.2.1 to 2.2.4.

**Proof:** Substituting the solution (2.2.8) in axiom 2.2.1 we have

\[
G_n(P;U) = a_i \left[ 1 - \left( \frac{\sum_{i=1}^{n} p_i^\mu u_i^\nu}{\sum_{i=1}^{n} p_i u_i} \right)^{\frac{1}{\mu}} \right], \quad \mu \nu \neq 0
\]  

(2.2.13)

Taking \( p_i = \frac{1}{n} \) and \( u_i = u \) for each \( i \) in (2.2.13) we have

\[
G_n\left(\frac{1}{n}, \ldots, \frac{1}{n}, u, \ldots, u\right) = a_i \left( 1 - n^{\frac{1-\mu}{\nu}} u^{\frac{1}{\mu}} \right), \quad n = 2, 3, \ldots ,
\]  

(2.2.14)

Axiom 2.2.4 together with (2.2.14) gives

\[
a_i \left( 1 - n^{\frac{1-\mu}{\nu}} u^{\frac{1}{\mu}} \right) = \frac{R}{R-1} \left[ 1 - n^{\frac{1-R}{R}} \right].
\]

It implies

\[
a_1 = \frac{R}{R-1}, \quad \mu = R, \quad \nu = 1.
\]

Putting these values in (2.2.13) we have

\[
G_n(P;U) = \frac{R}{R-1} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{R}} \right] \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{R}} \right] \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{R}} \right] = H_R(P;U).
\]

This completes the proof of theorem 2.2.1.
Particular cases:
(a) When utilities are ignored i.e. \( u_i = 1 \) for each \( i \), (2.1.4) reduces to (2.1.1).
(b) Further \( R \to 1 \), (2.1.1) reduces to Shannon’s entropy (1948).

2.3 Important Properties

The ‘useful’ R-norm information measure \( H_R(P;U) \) satisfies the following properties:

Property 2.3.1 \( H_R(P;U) \) is symmetric function of their arguments provided that the permutation of \( p_i \)'s and \( u_i \)'s are taken together.

\[
H_R(p_1, p_2, \ldots, p_{n-1}, p_n; u_1, u_2, \ldots, u_{n-1}, u_n) = H_R(p_n, p_1, p_2, \ldots, p_{n-1}; u_n, u_1, u_2, \ldots, u_{n-1}).
\]

Property 2.3.2 For complete probability distribution ‘useful’ R-norm information measure \( H_R(P;U) \) does not satisfy the normality property. However, for incomplete probability distribution the following holds:

\[
H_R\left(\frac{1}{4}, \frac{1}{4}; 1, 1\right) = 1.
\]

Proof: \( H_R(P;U) = \frac{R}{R-1} \left[ 1 - \left( \sum_{i=1}^{n} u_i p_i^R \right)^{\frac{1}{R}} \right] \)

For \( i = 1, 2 \)

\[
H_R(P;U) = \frac{R}{R-1} \left[ 1 - \left( \frac{u_1 p_1^R + u_2 p_2^R}{u_1 p_1 + u_2 p_2} \right)^{\frac{1}{R}} \right]
\]

Taking \( p_1 = p_2 = \frac{1}{4} ; u_1 = u_2 = 1 \) and \( R = 2 \)

\[
H_R\left(\frac{1}{4}, \frac{1}{4}; 1, 1\right) = \frac{2}{1} \left[ 1 - \left( \frac{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2}{\frac{1}{4} + \frac{1}{4}} \right)^{\frac{1}{2}} \right] = 1.
\]
**Property 2.3.3** Addition of one event whose probability of occurrence is zero or utility is zero has no effect on useful information, i.e.

\[ H_R(p_1, p_2, \ldots, p_n, 0; u_1, u_2, \ldots, u_{n+1}) = H_R(p_1, p_2, \ldots, p_n; u_1, u_2, \ldots, u_n) = H_n(p_1, p_2, \ldots, p_{n+1}; u_1, u_2, \ldots, u_n, 0). \]

**Proof:** Let us consider

\[ H_R(p_1, p_2, \ldots, p_n, 0; u_1, u_2, \ldots, u_{n+1}) = \frac{R}{R-1} \left[1 - \left\{ \frac{u_1 p_1^R + u_2 p_2^R + \ldots + u_n p_n^R + \ldots + 0^R u_{n+1}}{u_1 p_1 + u_2 p_2 + \ldots + u_n p_n + \ldots + 0 u_{n+1}} \right\}^{\frac{1}{R}} \right] = H_R(P;U). \]

Similarly, we can prove that

\[ H_n(p_1, p_2, \ldots, p_n, p_{n+1}; u_1, u_2, \ldots, u_n, 0) = H_R(P;U). \]

**Property 2.3.4** \( H_R(P;U) \) satisfies the non-additivity of the following form:

\[ H_R(P * Q; U * V) = H_R(P; U) + H_R(Q; V) - \frac{R-1}{R} H_R(P; U) H_R(Q; V), \]

where \( P * Q = (p_1 q_1, \ldots, p_1 q_m, p_2 q_1, \ldots, p_2 q_m, \ldots, p_n q_1, \ldots, p_n q_m) \), and \( U * V = (u_1 v_1, \ldots, u_1 v_m, u_2 v_1, \ldots, u_2 v_m, \ldots, u_n v_1, \ldots, u_n v_m) \).

**Proof:** R.H.S = \( H_R(P; U) + H_R(Q; V) - \frac{R-1}{R} H_R(P; U) H_R(Q; V) \)

\[
= \frac{R}{R-1} \left[1 - \left( \sum_{i=1}^{n} u_ip_i^R \right)^{\frac{R}{R}} \right] + \frac{R}{R-1} \left[1 - \left( \sum_{j=1}^{m} v_jq_j^R \right)^{\frac{R}{R}} \right] - \frac{R}{R-1} \left[1 - \left( \sum_{i=1}^{n} u_ip_i^R \right)^{\frac{R}{R}} \right] \frac{R}{R-1} \left[1 - \left( \sum_{j=1}^{m} v_jq_j^R \right)^{\frac{R}{R}} \right]
\]

\[
= \frac{R}{R-1} \left[1 - \left( \sum_{i=1}^{n} u_ip_i^R \right)^{\frac{R}{R}} \right] + \frac{m}{R-1} \left( \sum_{j=1}^{m} v_jq_j^R \right)^{\frac{R}{R}} - \frac{R}{R-1} \left[1 + \left( \sum_{i=1}^{n} u_ip_i^R \right)^{\frac{R}{R}} \right] - \frac{m}{R-1} \left( \sum_{j=1}^{m} v_jq_j^R \right)^{\frac{R}{R}}
\]

\[
= \frac{R}{R-1} \left[1 - \left( \sum_{i=1}^{n} u_ip_i^R \right)^{\frac{R}{R}} \right] \left( \sum_{j=1}^{m} q_jv_j^R \right)^{\frac{R}{R}} = \frac{R}{R-1} \left[1 - \left( \sum_{i=1}^{n} u_ip_i^R \right)^{\frac{R}{R}} \right] \left( \sum_{j=1}^{m} q_jv_j^R \right)^{\frac{R}{R}}
\]

\[= \frac{R}{R-1} \left[1 - \left( \sum_{i=1}^{n} u_ip_i^R \right)^{\frac{R}{R}} \right] \left( \sum_{j=1}^{m} q_jv_j^R \right)^{\frac{R}{R}} = \frac{R}{R-1} \left[1 - \left( \sum_{i=1}^{n} u_ip_i^R \right)^{\frac{R}{R}} \right] \left( \sum_{j=1}^{m} q_jv_j^R \right)^{\frac{R}{R}}
\]

\[= \frac{R}{R-1} \left[1 - \left( \sum_{i=1}^{n} u_ip_i^R \right)^{\frac{R}{R}} \right] \left( \sum_{j=1}^{m} q_jv_j^R \right)^{\frac{R}{R}} \]
\[ H_R(P \ast Q; U \ast V) = \text{L.H.S.} \]

**Property 2.3.5** Let \( A_i, A_j \) be two events having probabilities \( p_i, p_j \) and utilities \( u_i, u_j \) respectively, then we define the utility \( u \) of the compound event \( A_i \cap A_j \) as

\[ u(A_i \cap A_j) = \frac{u_i p_i + u_j p_j}{p_i + p_j}. \quad (2.3.1) \]

**Theorem 2.3.1** Under the composition law (2.3.1), the following holds:

\[ R_{i=1} H_R(p_1, p_2, \ldots, p_{n-1}, p', p''; u_1, u_2, \ldots, u_{n-1}, u', u'') = \frac{R}{R-1} \left[ 1 - \left( \frac{u' p'^R + u'' p''^R}{u' p' + u'' p''} \right)^{R-1} \right] \]

where \( p_n = p' + p'' \) and \( u_n = \frac{p' u' + p'' u''}{p' + p''} \).

**Proof:**

\[
= \frac{R}{R-1} \left[ 1 - \left( \frac{u' p'^R + u'' p''^R}{u' p' + u'' p''} \right)^{R-1} \right] - \frac{R}{R-1} \left( 1 - \frac{u_n p_n^{R-1}}{u_n} \right)
\]

\[
= \frac{R}{R-1} \left[ (p' + p'')^{R-1} - \left\{ \frac{u' p'^R + u'' p''^R}{u' p' + u'' p''} \right\}^{R-1} \right]
\]

\[
= \frac{R}{R-1} \left[ (p' + p'')^{R-1} - \left\{ \frac{u' p'^R + u'' p''^R}{u' p' + u'' p''} \right\} \left( \frac{p' + p''}{(p' + p'')^R} \right)^{R-1} \right]
\]

\[
= \frac{R}{R-1} \left( \frac{p' + p''}{(p' + p'')^R} \right) \left[ 1 - \left\{ \frac{u' p'^R + u'' p''^R}{u' p' + u'' p''} \right\} \left( \frac{p' + p''}{(p' + p'')^R} \right)^{1-R} \right]
\]

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\[= n \, H_R(P;U) + \frac{R}{R-1} \left( p' + p'' \right)^{\frac{R-1}{R}} \left[ 1 - \left( \frac{p'}{p' + p''} \right)^R + \left( \frac{p''}{p' + p''} \right)^R \right] \]

\[= n \, H_R(P;U) + \left( p' + p'' \right)^{\frac{R-1}{R}} \frac{R}{R} \left( \frac{p'}{p' + p''} \right)^R H_R \left( \frac{p'}{p' + p''} ; u', u'' \right) \]

This completes the proof of theorem 2.3.1

2.4 Generalized ‘Useful’ Mean Codeword Lengths

Let a finite set of \( n \) source symbols \( X = \{x_1, x_2, \ldots, x_n\} \) with probabilities \( P = \{p_1, p_2, \ldots, p_n\} \) be encoded using \( D \geq 2 \) code alphabets, then there is a uniquely decipherable instantaneous code with lengths \( l_1, l_2, \ldots, l_n \) if and only if

\[ \sum_{i=1}^{n} D^{-l_i} \leq 1, \quad (2.4.1) \]

which is known as Kraft’s inequality given by Kraft (1949)

Let \( L = \sum_{i=1}^{n} p_i l_i \quad (2.4.2) \)

be the average codeword length associated with input symbols \( X = \{x_1, x_2, \ldots, x_n\} \) under the Kraft’s inequality given by (2.4.1) Shannon’s (1948) proved the following result for a noiseless channel:

\[ H(P) \leq L < H(P) + 1 \), where \( D \geq 2 \quad (2.4.3) \]

with equality if and only if \( l_i = -\log_{10} p_i \) for \( i = 1, 2, \ldots, n \).

Guiasu and Picard (1971) considered the problem of encoding by means of a single letter prefix code whose code words \( w_1, w_2, \ldots, w_n \) are of lengths \( l_1, l_2, \ldots, l_n \) respectively and satisfy the Kraft’s inequality (2.4.1). They introduced the following ‘useful’ mean length of code:
Later on Longo (1976) interpreted (2.4.4) as the average transmission cost of the letters \( x_i \)'s and obtained the following bounds for the cost measure (2.4.4):

\[
H(P;U) \leq L(P;U) < H(P;U) + 1,
\]

where \( H(P;U) = \frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{j=1}^{n} u_j p_j} \).

It may be noted that the mean codeword length (2.4.4) had been generalized parametrically by many authors and their bounds had been studied in terms of generalized measures of entropies. Here we define the two non-additive following new generalizations of (2.4.4) and study their bounds in terms of ‘useful’ R-norm information measures given by (2.1.4) and (2.1.5) respectively:

\[
L_R(P;U) = \frac{R}{R-1} \left( 1 - \frac{\sum_{i=1}^{n} u_i p_i D^{-i(R-1)/R}}{\sum_{i=1}^{n} u_i p_i} \right),
\]

and

\[
L_R^\beta(P;U) = \frac{R}{R-1} \left[ 1 - \frac{\sum_{i=1}^{n} u_i p_i^\beta D^{-i(R-1)/R}}{\sum_{i=1}^{n} u_i p_i^\beta} \right].
\]

If \( u_i = 1 \) for all \( i = 1, 2, \ldots, n \), (2.4.6) reduces to

\[
L_R(P) = \frac{R}{R-1} \left( 1 - \sum_{i=1}^{n} p_i D^{-i(R-1)/R} \right).
\]
which is average codeword length due to Boekee and Lubbe (1980). In case $R \to 1$, then (2.4.6) reduces to (2.4.4) and after ignoring utilities it reduces to (2.4.2).

2.5 Bounds on ‘Useful’ Mean Codeword Length

In this section we study the lower and upper bounds of $L_R(P;U)$ in terms of ‘useful’ $R$-norm information measure $H_R(P;U)$.

**Theorem 2.5.1** If $l_i, i = 1, 2, 3, \ldots, n$ are the lengths of codeword’s $x_i$ satisfying (2.4.1), then

$$H_R(P;U) \leq L_R(P;U), \quad (2.5.1)$$

under the condition

$$\sum_{i=1}^{n} u_i D^{-l_i} \leq \sum_{i=1}^{n} u_i p_i, \quad (2.5.2)$$

**Proof:** By Holder’s inequality we have

$$\left( \sum_{i=1}^{n} x_i^p \right)^{1/p} \left( \sum_{i=1}^{n} y_i^q \right)^{1/q} \leq \sum_{i=1}^{n} x_i y_i, \quad (2.5.3)$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p( \neq 0) < 1$, $q < 0$ or $q( \neq 0) < 1$, $p < 0$ and $x_i, y_i \geq 0$ for each $i$.

Putting $x_i = \left( \frac{u_i p_i}{\sum_{i=1}^{n} u_i p_i} \right)^{1/(R-1)} D^{-l_i}$, $y_i = \left( \frac{u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)^{1/(1-R)}$

we have

$$\left[ \sum_{i=1}^{n} u_i p_i D^{-l_i} \left( \frac{R-1}{R} \right) \right]^{1/(R-1)} \left[ \sum_{i=1}^{n} u_i p_i \right]^{1/(1-R)} \leq \sum_{i=1}^{n} u_i \frac{1}{D^{-l_i}} \leq 1.$$
It implies
\[
\left[\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{1}{1-R}} \leq \left[\frac{\sum_{i=1}^{n} u_i p_i D^{-(\frac{R-1}{R})}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{R}{1-R}}.
\]  
(2.5.4)

Case 1 Let \(0 < R < 1\). Raising power \(\frac{1-R}{R} > 0\) both sides of (2.5.4) we have
\[
\left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} \leq \frac{\sum_{i=1}^{n} u_i p_i D^{-(\frac{R-1}{R})}}{\sum_{i=1}^{n} u_i p_i}.
\]
Subtracting both sides from 1, we get
\[
1 - \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} \geq 1 - \frac{\sum_{i=1}^{n} u_i p_i D^{-(\frac{R-1}{R})}}{\sum_{i=1}^{n} u_i p_i}.
\]  
(2.5.5)

Multiplying (2.5.5) by \(\frac{R}{R-1} < 0\) throughout we get
\[
\left[\frac{R}{R-1}\right] \left[1 - \left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}}\right] \leq \left[\frac{R}{R-1}\right] \left[1 - \frac{\sum_{i=1}^{n} u_i p_i D^{-(\frac{R-1}{R})}}{\sum_{i=1}^{n} u_i p_i}\right]
\]

\[
H_R(P; U) \leq L_R(P; U).
\]  
(2.5.6)

Case 2 Let \(R > 1\). Raising power \(\frac{1-R}{R} < 0\) both sides of (2.5.4), we have
\[
\left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} \geq \frac{\sum_{i=1}^{n} u_i p_i D^{-(\frac{R-1}{R})}}{\sum_{i=1}^{n} u_i p_i}
\]
or
\[
1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right) \leq 1 - \frac{\sum_{i=1}^{n} u_i p_i D^{-\lambda(R-1)}}{\sum_{i=1}^{n} u_i p_i}. \tag{2.5.7}
\]
Multiplying (2.5.7) by \( \frac{R}{R-1} > 0 \) throughout we get
\[
\frac{R}{R-1} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right) \right] \leq \frac{R}{R-1} \left[ 1 - \frac{\sum_{i=1}^{n} u_i p_i D^{-\lambda(R-1)}}{\sum_{i=1}^{n} u_i p_i} \right].
\]

\( H_R(P;U) \leq L_R(P;U). \tag{2.5.8} \)

Hence theorem 2.5.1 is proved in both cases.

In (2.5.1) equality holds if and only if
\[
D^{-\lambda} = \frac{p_i^R}{\sum_{i=1}^{n} u_i p_i^R / \sum_{i=1}^{n} u_i p_i}, \quad R > 0(\neq 1)
\]
or
\[
l_i = -\log_D p_i^R + \log_D \left( \sum_{i=1}^{n} u_i p_i^R \right).
\]

or
\[
\log_D p_i^{-R} \left( \sum_{i=1}^{n} u_i p_i^R \right) \leq l_i < \log_D p_i^{-R} \left( \sum_{i=1}^{n} u_i p_i^R \right) + 1. \tag{2.5.9}
\]
It implies
\[
p_i^{-R} \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right) \leq D^l_i < D p_i^{-R} \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right).
\]

(2.5.10)

In the next theorem, we obtain an upper bound on \( L_R(P;U) \) in term of \( H_R(P;U) \).

**Theorem 2.5.2** Let \( l_1, l_2, \ldots, l_n \) be the code words lengths satisfying (2.5.10). Then following inequality holds:
\[
L_R(P;U) < D^{\frac{1-R}{R}} H_R(P;U) + \frac{R}{R-1} \left( 1 - D^{\frac{1-R}{R}} \right).
\]

(2.5.11)

**Proof:** From the R.H.S. of (2.5.10), we have
\[
D^l_i < D p_i^{-R} \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right).
\]

(2.5.12)

Here two cases arise:

**Case 1** Let \( 0 < R < 1 \). Raising both sides of (2.5.12) to the power \( \frac{1-R}{R} > 0 \), we get
\[
D^{l_i \left( \frac{R-1}{R} \right)} < D^{\frac{1-R}{R}} p_i^{R-1} \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right).
\]

(2.5.13)

Multiplying both sides of (2.5.13) by \( \frac{u_i p_i}{\sum_{i=1}^{n} u_i p_i} \) and summing over \( i \), we have
\[
\sum_{i=1}^{n} u_i p_i D^{l_i \left( \frac{R-1}{R} \right)} < D^{\frac{1-R}{R}} \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right) \left( \frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1-R}{R}}.
\]
or
\[
\sum_{i=1}^{n} u_i p_i D_i^{\frac{R-1}{R}} < D^{\frac{1-R}{R}} \left( \sum_{i=1}^{n} \frac{u_i p_i^R}{\sum_{i=1}^{n} u_i p_i} \right)^{\frac{1}{R}}.
\]

Subtracting both sides from 1 and multiplying by \( \frac{R}{R-1} < 0 \), we have
\[
L_R(P;U) < D^{\frac{1-R}{R}} H_R(P;U) + \frac{R}{R-1} \left( 1 - D^{\frac{1-R}{R}} \right).
\]

(2.5.14)

Similarly, we can prove that (2.5.11) holds when \( R > 1 \). Hence theorem 2.5.2 is proved.

Thus from (2.5.1) and (2.5.11)
\[
H_R(P;U) \leq L_R(P;U) < D^{\frac{1-R}{R}} H_R(P;U) + \frac{R}{R-1} \left( 1 - D^{\frac{1-R}{R}} \right),
\]

(2.5.15)

which is Coding theorem for noiseless channel on ‘useful’ R-norm information measure.

### 2.6 Bounds on ‘Useful’ Mean Codeword Length of Degree \( \beta \)

In this section we study the lower and upper bounds for \( L_R^\beta(P;U) \) in terms of the generalized ‘useful’ R-norm information measure of degree \( \beta \) denoted by \( H_R^\beta(P;U) \).

In the next theorem we obtain the lower bound for \( L_R^\beta(P;U) \) in terms of ‘useful’ R-norm information measure of degree \( \beta \).

**Theorem 2.6.1** Let \( \{u_i\}_{i=1}^{n}, \{p_i\}_{i=1}^{n}, \{l_i\}_{i=1}^{n} \), satisfy the following inequality:
\[
\sum_{i=1}^{n} u_i D_i^{-l_i} \leq \sum_{i=1}^{n} u_i p_i^\beta,
\]

(2.6.1)

then
\[
L_R^\beta(P;U) \geq H_R^\beta(P;U), \quad R > 0 \neq 1, \quad \beta \geq 1,
\]

(2.6.2)

where \( H_R^\beta(P;U) \) and \( L_R^\beta(P;U) \) are given by (2.1.5) and (2.4.7).

**Proof:** By Holder’s inequality we have
\[
\left( \sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left( \sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}} \leq \sum_{i=1}^{n} x_i y_i,
\]

(2.6.3)
where \( x_i, y_i \geq 0 \) for each \( i \) and \( \frac{1}{p} + \frac{1}{q} = 1 \); \( p(\neq 0) < 1, q < 0 \) or \( q(\neq 0) < 1, p < 0 \).

Putting \( x_i = \left( \frac{u_i p_i^\beta}{\sum_{i=1}^{n} u_i p_i^\beta} \right)^{\frac{R}{R-1}} D^{-i}, \ y_i = \left( \frac{u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{R\beta}} \right)^{\frac{1}{R-1}} \),

\[ p = \frac{R - 1}{R} \quad \text{and} \quad q = 1 - R \quad \text{in (2.6.3)} \]

we have

\[
\left[ \frac{\sum_{i=1}^{n} u_i p_i^\beta D^{-(\frac{R-1}{R})}}{\sum_{i=1}^{n} u_i p_i^\beta} \right]^{\frac{R}{R-1}} \leq \left[ \frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{R\beta}} \right]^{\frac{1}{R-1}} \leq 1.
\]

It implies

\[
\left[ \frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^\beta} \right]^{\frac{1}{R-1}} \leq \left[ \frac{\sum_{i=1}^{n} u_i p_i^{R\beta} D^{-i(\frac{R-1}{R})}}{\sum_{i=1}^{n} u_i p_i^{R\beta}} \right]^{\frac{R}{R-1}}.
\]  \tag{2.6.4}

**Case 1** Let \( 0 < R < 1 \). Raising power \( \frac{1 - R}{R} > 0 \) both sides of (2.6.4) we have

\[
\left( \frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^\beta} \right)^{\frac{1}{R-1}} \leq \left( \frac{\sum_{i=1}^{n} u_i p_i^{R\beta} D^{-i(\frac{R-1}{R})}}{\sum_{i=1}^{n} u_i p_i^{R\beta}} \right).
\]

Subtracting both sides from 1, we get

\[
1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^\beta} \right)^{\frac{1}{R-1}} \geq 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^{R\beta} D^{-i(\frac{R-1}{R})}}{\sum_{i=1}^{n} u_i p_i^{R\beta}} \right).
\]  \tag{2.6.5}
Multiplying (2.6.5) by $\frac{R}{R-1} < 0$ throughout we get

$$
\frac{R}{R-1} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^{R\beta} R}{\sum_{i=1}^{n} u_i p_i^{\beta}} \right)^{\frac{1}{R}} \right] \leq \frac{R}{R-1} \left[ 1 - \frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-\ell_{i} \left( \frac{R-1}{R} \right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \right]
$$

$$
H_R^\beta (P;U) \leq L_R^\beta (P;U).
\tag{2.6.6}
$$

**Case 2** Let $R > 1$. Raising power $\frac{1-R}{R} < 0$ both sides of (2.6.4) we have

$$
\left( \frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \right)^{\frac{1}{R}} \geq \frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-\ell_{i} \left( \frac{R-1}{R} \right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}}.
$$

Subtracting both sides from 1, we get

$$
1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \right)^{\frac{1}{R}} \leq 1 - \frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-\ell_{i} \left( \frac{R-1}{R} \right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}}.
\tag{2.6.7}
$$

Multiplying (2.6.9) $\frac{R}{R-1} > 0$, throughout we get

$$
\frac{R}{R-1} \left[ 1 - \left( \frac{\sum_{i=1}^{n} u_i p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \right)^{\frac{1}{R}} \right] \leq \frac{R}{R-1} \left[ 1 - \frac{\sum_{i=1}^{n} u_i p_i^{\beta} D^{-\ell_{i} \left( \frac{R-1}{R} \right)}}{\sum_{i=1}^{n} u_i p_i^{\beta}} \right]
$$

$$
H_R (P;U) \leq L_R (P;U).
\tag{2.6.8}
$$

Hence theorem 2.6.1 is proved in both cases.

In (2.6.2) equality holds if and only if

$$
D^{-\ell_{i}} = \frac{p_i^{R\beta}}{\sum_{i=1}^{n} u_i p_i^{R\beta} / \sum_{i=1}^{n} u_i p_i^{\beta}} , R > 0 (\neq 1) , \beta > 0
$$
or

\[ l_i = -\log_D p_i^{R_\beta} + \log_D \left( \frac{\sum_{i=1}^{n} u_i p_i^{R_\beta}}{\sum_{i=1}^{n} u_i p_i^\beta} \right) \]

or

\[ \log_D p_i^{-R_\beta} \left( \frac{\sum_{i=1}^{n} u_i p_i^{R_\beta}}{\sum_{i=1}^{n} u_i p_i^\beta} \right) \leq l_i < \log_D p_i^{-R_\beta} \left( \frac{\sum_{i=1}^{n} u_i p_i^{R_\beta}}{\sum_{i=1}^{n} u_i p_i^\beta} \right) + 1. \]  

(2.6.9)

It implies

\[ p_i^{-R_\beta} \left( \frac{\sum_{i=1}^{n} u_i p_i^{R_\beta}}{\sum_{i=1}^{n} u_i p_i^\beta} \right) \leq D_i < D p_i^{-R_\beta} \left( \frac{\sum_{i=1}^{n} u_i p_i^{R_\beta}}{\sum_{i=1}^{n} u_i p_i^\beta} \right) . \]  

(2.6.10)

In the next theorem, we obtain an upper bound on \( L_R^\beta (P; U) \) in term of \( H_R^\beta (P; U) \).

**Theorem 2.6.2** Let \( l_1, l_2, \ldots, l_n \) be the code words lengths satisfying (2.6.10). Then following inequality holds:

\[ L_R^\beta (P; U) < D p_i^{-R_\beta} \left( \frac{\sum_{i=1}^{n} u_i p_i^{R_\beta}}{\sum_{i=1}^{n} u_i p_i^\beta} \right) + \frac{R - 1}{R - 1} \left( 1 - D p_i^{-R_\beta} \right) . \]  

(2.6.11)

**Proof:** From the R.H.S. of (2.6.10), we have

\[ D_i < D p_i^{-R_\beta} \left( \frac{\sum_{i=1}^{n} u_i p_i^{R_\beta}}{\sum_{i=1}^{n} u_i p_i^\beta} \right) . \]  

(2.6.12)
Here two cases arise:

**Case 1** Let $0 < R < 1$. Raising both sides of (2.6.12) to the power $\frac{1 - R}{R} > 0$, we get

$$D^{-\frac{R-1}{R}} \leq D^{-\frac{R-1}{R}} p_i^{R-1} \left( \sum_{i=1}^{n} u_i p_i^{R} \right)^{\frac{1-R}{R}}. \tag{2.6.13}$$

Multiplying by $\frac{u_i p_i^R}{\sum_{i=1}^{n} u_i p_i^R}$ and summing over $i$, we have

$$\left( \sum_{i=1}^{n} u_i p_i^R \right) D^{-\frac{R-1}{R}} \left( \sum_{i=1}^{n} u_i p_i^R \right) < \left( \sum_{i=1}^{n} u_i p_i^R \right)^{\frac{1-R}{R}}. \tag{2.6.14}$$

Subtracting both sides from 1 and multiplying by $\frac{R}{R - 1} < 0$, we have

$$L_R^\beta (P; U) < D^{\frac{1-R}{R}} H_R^\beta (P; U) + \frac{R}{R - 1} \left( 1 - D^{\frac{1-R}{R}} \right). \tag{2.6.15}$$

Similarly in another case, (2.6.11) holds when $R > 1$. Hence theorem 2.6.2 is proved.

Thus (2.6.2) and (2.6.11) together give

$$H_R^\beta (P; U) \leq L_R^\beta (P; U) < D^{\frac{1-R}{R}} H_R^\beta (P; U) + \frac{R}{R - 1} \left( 1 - D^{\frac{1-R}{R}} \right), \tag{2.6.15}$$

which is Coding theorem for noiseless channel on ‘useful’ R-norm information measure of degree $\beta$. 

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**Particular cases:**

(i) When $\beta = 1$, (2.6.2) reduces to (2.5.1).

(ii) When $\beta = 1$, (2.6.11) reduces to (2.5.11).

**2.7 Conclusion**

R-norm information measure is defined and characterized when the probability distribution $P$ belongs to R-norm vector space. In the proposed study a new ‘useful’ R-norm information measure is generalized and characterized axiomatically.

The new mean codeword length is the generalization of mean codeword length due to Boekee and Lubbe (1980) and can be generalized further and the lower and upper bounds of the new generalized mean codeword length can be studied in terms of generalized ‘useful’ R-norm information measure given by (2.1.4).