CHAPTER III

ON THE ORDER OF MAGNITUDE OF
THE MATRIX TRANSFORM OF
\( \{nB_n(x)\} \) SEQUENCE

3.1 Let \( f(t) \) be a periodic function with period \( 2\pi \), and integrable in the sense of Lebesgue over the interval \((-\pi, \pi)\). Let its Fourier series be

\[
(3.1.1) \quad f(t) \sim \frac{1}{2}a_0 + \sum (a_n \cos nt + b_n \sin nt)
= \frac{1}{2}a_0 + \sum A_n(t),
\]

then the conjugate series of \( (3.1.1) \) at \( t = x \) is given by:

\[
(3.1.2) \quad \sum (b_n \cos nx - a_n \sin nx) \equiv \sum B_n(x).
\]

We write

\[
\Psi(t) = f(x + t) - f(x - t).
\]

Suppose that \( T = (\lambda_{n,k}) \) is a triangular matrix

(i.e. \( \lambda_{n,k} = 0, \ k \geq n+1 \)) which is regular i.e. it satisfies the Silverman-Toeplitz conditions\(^1\).

If we choose

1. Hardy, G.H. (1).
\[
\lambda_{n,k} = \begin{cases} 
\frac{p_{n-k}}{p_n}, & \text{for } 0 \leq k \leq n, \\
0, & \text{for } k > n,
\end{cases}
\]
then the T-process reduces to the Norlund method of summation where

\begin{equation}
(3.1.4) \quad P_n = p_0 + p_1 + \cdots + p_n \quad (P_n \neq 0).
\end{equation}

If we choose

\[
\lambda_{n,k} = \frac{A_{n-k}}{A_{n}}, \quad \text{for } k \leq n,
\]
then the T-process reduces to the Cesaro method of summation of order \( \alpha \).

3.2 In the year 1909, Lebesgue investigated the order of partial sum of Fourier series. He proved that if \( f(x) \) is continuous then \( S_n = O(\log n) \); where \( S_n \) is the \( n \)-th partial sum of the Fourier series of \( f(x) \). In 1910, from the examples a, b, c given by Fejer, it was found that no more is true in the above statement, since if \( \varepsilon(n) \) is a function which decreases steadily to zero as \( n \to \infty \), however slowly, there is a Fourier series of a continuous function for which \( S_n > \varepsilon(n) \log n \) for arbitrary large value of \( n \). The result of Lebesgue for the order of partial sum of the Fourier series has been extended by Sunouchi in 1951, so as to applicable to
Cesaro mean of partial sums. Kumari, S.\textsuperscript{1} further extended the result of Sunouchi under less restrictive conditions.

In 1954, Mohanty and Nanda\textsuperscript{2} proved the result giving the order of Cesaro means of the first derived series of Fourier series, but this result can be easily derived either by putting the above mentioned of Lebesgue in a kernel used by Wang\textsuperscript{3} or by putting $r = 1$ the result of S. Kumari, for the $r$-th derived series of Fourier series. Lukacs and further Kumari, S.\textsuperscript{4} determined the order of $n$-th partial sum of conjugate series similar to Lebesgue.

The object of this chapter is to prove the following theorem on the order of magnitude of matrix transform of \{nB\textsubscript{n}(x)\} sequence.

**THEOREM**: Let $t\textsubscript{n}$ be a matrix transform of \{nB\textsubscript{n}(x)\} sequence satisfying

$$\sum_{k=1}^{n} k |\lambda_{n,k} - \lambda_{n,k+1}| = O(1), \text{ as } n \to \infty$$

and if

$$\psi(t) = O(1),$$

then

$$t\textsubscript{n}(x) = O(1).$$

3.3

**Proof of the Theorem.**

The matrix transform of \( \{nB_n(x)\} \) sequence is given by

\[
t_n = \sum_{k=1}^{n} \lambda_{n,k}B_k(x)
\]

\[
= (\pi)^{-1} \sum_{k=1}^{n} \lambda_{n,k} \int_{0}^{\pi} \varphi(t) \sin kt \, dt
\]

\[
= (\pi)^{-1} \sum_{k=1}^{n} \lambda_{n,k} \left( \int_{0}^{\pi/k} \varphi(t) \sin kt \, dt + \int_{\pi/k}^{\pi} \varphi(t) \sin kt \, dt \right)
\]

\[
= (\pi)^{-1} \sum_{k=1}^{n} \lambda_{n,k} \left( P_k + Q_k \right)
\]

\[
= I_1 + I_2, \text{ say.}
\]

Invirtue of \( |\sin kt| \leq kt \), we see that the integral

\[
P_k = k \int_{0}^{\pi/k} \varphi(t) \sin kt \, dt
\]

\[
= 0 \left( k^2 \right) \int_{0}^{\pi/k} t \, dt
\]

\[
= 0(1).
\]

Hence due to regularity of the method of summation, we have

\[
I_1 = O(1).
\]

* \( r \) is an arbitrary constant.*
Let
\[ G(k, t) = \sum_{\nu=1}^{k} \sin \nu t = \frac{\cos \frac{1}{2}t - \cos (k + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \]
and
\[ k L(k, t) = \sum_{\nu=1}^{k} G(\nu, t) = \frac{1}{2} k \cot \frac{1}{2}t - \frac{\sin kt}{4 \sin^2 \frac{1}{2}t} \]
we write the identity
\[ 2k L(2k, t) - k L(k, t) = \sum_{\nu=1}^{2k} G(\nu, t) \]
\[ = kG(k, t) + \sum_{\nu=0}^{k-1} (k - \nu) \sin (k + \nu + 1)t \]
Hence
\[ \frac{\sin 2kt - \sin kt}{4 \sin^2 \frac{1}{2}t} = \frac{k \sin (k + \frac{1}{2})t}{2 \sin \frac{1}{2}t} \]
\[ + \sum_{\nu=0}^{k-1} (k - \nu) \sin (k + \nu + 1)t \]
If
\[ \bar{D}_k(t) = \sum_{\nu=1}^{k} \nu \sin \nu t = \]
\[ \frac{\sin kt}{4 \sin^2 \frac{1}{2}t} \frac{ksin(k+\frac{1}{2})t}{2 \sin \frac{1}{2}t} \]
\[ \frac{\sin kt}{4 \sin^2 \frac{1}{2}t} \frac{\sin 2kt - \sin kt}{4 \sin^2 \frac{1}{2}t} \]
\[ + \sum_{\nu=0}^{k-1} (k - \nu) \sin (n + \nu + 1)t \]
(3.3.1)
Consider the matrix \((b_{n,k})\) such that \(b_{n,k} = 0\) for \(k = 1, 2, 3, \ldots, N\); and \(b_{n,k} = \lambda_{n,k}\) elsewhere, then the matrix \(b_{n,k}\) and \(\lambda_{n,k}\) are equivalent. Therefore we shall assume without loss of generality that \(\lambda_{n,k} = 0\) for \(k = 1, 2, 3, \ldots, N-1\); where \(n > N\).

Now,

\[
I_2 = (\pi)^{-1} \sum_{k=N}^{n} \lambda_{n,k} \left[ \int_{\frac{\pi}{k}}^{\pi} \psi(t) \left( \bar{B}_n(t) - \bar{B}_{k-1}(t) \right) dt \right]
\]

\[
= \sum_{k=N}^{n} \left( \lambda_{n,k} - \lambda_{n,k+1} \right) \int_{\frac{\pi}{k}}^{\pi} \psi(t) \bar{B}_k(t) dt
\]

\[
- \sum_{k=N+1}^{n} \lambda_{n,k} \left[ \int_{\frac{\pi}{k}}^{\frac{\pi}{k-1}} \psi(t) \bar{B}_{k-1}(t) dt \right].
\]

\[
= I_{21} - I_{22}, \text{ say.}
\]

Since

\[
\bar{B}_{n-1}(t) = 0 (k/t)
\]

and using the second mean value theorem, we have

\[
\int_{\frac{\pi}{k}}^{\frac{\pi}{k-1}} \psi(t) \bar{B}_{n-1}(t) dt \leq (k/rn)^{\alpha} \int_{\frac{\pi}{k}}^{\frac{\pi}{k-1}} t^{-1} \bar{B}_{k-1}(t) dt
\]

\[
\leq 0 \left[ k(k/rn)^{\alpha} \right] \int_{\frac{\pi}{k}}^{\frac{\pi}{k-1}} t^{-1} dt
\]

\[
= 0 (k).\]
Hence

\[ I_{22} = 0(1) \sum_{k=n+1}^{n} |\lambda_{n,k}| \]

\[ = 0(1). \]

Further from (7.3.1), we have

\[
R_k = \int_{\frac{\pi}{k}}^{\pi} \psi(t) \frac{\sin kt}{2\sin^2 \frac{1}{2}t} \frac{\sin 2kt}{4\sin^2 \frac{1}{2}t} \]

\[ = \int_{\frac{\pi}{k}}^{\pi} \psi(t) \left\{ \frac{\sin kt}{2\sin^2 \frac{1}{2}t} - \frac{\sin 2kt}{4\sin^2 \frac{1}{2}t} \right\} dt \]

\[ + \sum_{\nu=1}^{k-1} \int_{\frac{\pi}{k}}^{\pi} \psi(t) \left\{ (k - \nu) \sin (k + \nu + 1)t \right\} dt \]

\[ = \int_{\frac{\pi}{k}}^{\pi} \psi(t) \left\{ \frac{\sin kt}{2\sin^2 \frac{1}{2}t} - \frac{\sin 2kt}{4\sin^2 \frac{1}{2}t} \right\} dt \]

\[ + \int_{\frac{\pi}{k}}^{\pi} \psi(t) \left\{ \sum_{\nu=1}^{k-1} (k - \nu) \sin (k + \nu + 1)t \right\} dt \]

\[ = J_1 + J_2 + J_3, \text{ say.} \]

Now,

\[
J_1 = \int_{\frac{\pi}{k}}^{\pi} \psi(t) \frac{\sin kt}{2\sin^2 \frac{1}{2}t} dt \]
The integral $J_{11}$ is numerically less than

$$A \int_{\frac{\pi}{k}}^{\pi} \frac{dt}{\sin^2 \frac{t}{2}} = O(k).$$
It is easy to see that
\[ |J_{12}| \leq n^{-1} \left[ \frac{2 \sin(\pi/k)}{2 \sin(\pi/k)} \right]^2 = o(k). \]

and
\[ |J_{13}| \geq n^{-1} \left[ \frac{2 \sin(\pi/k)}{2 \sin(\pi/k)} \right]^2 = o(1/k). \]

On account of the expression
\[ \frac{1}{2 \sin^2 t} - \frac{1}{2 \sin^2 (t - \pi/k)} \]
which is equal to \( o(k^{-1} t^{-3}) \), we see that
\[ J_{14} = o(k^{-1}) \int_{\pi/k}^{\pi} (t^{-3})dt = o(k). \]

Similarly we can prove that
\[ J_2 = o(k). \]

The last integral
\[ J_3 = \left| \sum_{\nu=1}^{k-1} (k - \nu) \int_{\pi/k}^{\pi} \psi(t) \sin(k+\nu+1)t dt \right| \]
\[ = \left| \sum_{\nu=1}^{k-1} \frac{k - \nu}{k + \nu + 1} \right| \]
\[ = o(k). \]

Combining \( J_1, J_2 \) and \( J_3 \), we have
\[ \sum_{\nu=1}^{n} k! \left( \lambda_{n,k} - \lambda_{n,k+1} \right) = o(1). \]

This completes the proof of the Theorem.