CHAPTER IX

ON THE ALMOST EVERYWHERE

CONVERGENCE OF DOUBLE

FOURIER SERIES

9.1 Let \( f(x,y) \) be a function integrable in the Lebesgue sense over the square \([-\pi,\pi][-\pi,\pi]\) and periodic with period \(2\pi\) in each variable.

Suppose

\[
A_{mn}(x,y) = a_{mn}\cos mx \cos ny + b_{mn}\sin mx \cos ny +

c_{mn}\cos mx \sin ny + d_{mn}\sin mx \sin ny.
\]

The series

\[
\sum_{m,n=0}^{\infty} A_{mn}(x,y)
\]

is called the double Fourier series of the function \( f(x,y) \)

where

\[
a_{mn} + ib_{mn} = \lambda_{mn}^{-2} \int_{Q} f(x,y)e^{imx}\cos ny \, dx \, dy
\]

\[
a_{mn} + id_{mn} = \lambda_{mn}^{-2} \int_{Q} f(x,y)e^{imx}\sin ny \, dx \, dy
\]

such that
\[ \lambda_{mn} = \begin{cases} 
\frac{1}{4}, & \text{for } m = 0, n = 0. \\
\frac{1}{2}, & \text{for } m = 0, n > 0; n = 0, m > 0. \\
1, & \text{for } m > 0, n > 0. 
\end{cases} \]

We define

\[ p_{mn}, \quad q_{mn}, \quad \phi = \phi_{mn}, \quad \theta = \theta_{mn} \]

by

\[ a_{mn} = p_{mn} \cos \phi, \quad b_{mn} = p_{mn} \sin \phi, \]

\[ c_{mn} = q_{mn} \cos \theta, \quad d_{mn} = q_{mn} \sin \theta, \]

so that

\[ p_{mn}^2 = a_{mn}^2 + b_{mn}^2, \quad q_{mn}^2 = c_{mn}^2 + d_{mn}^2, \]

and

\[ (9.1.1) \quad A_{mn}(x,y) = p_{mn} \cos(mx - \phi) \cos ny + q_{mn} \cos(mx - \theta) \sin ny. \]

Further suppose

\[ (9.1.2) \quad \Delta_{i,j}^{(i,j)} f(x,y) = \sum_{k=0}^{i} \sum_{r=0}^{j} (-1)^{k+r} c_{i}^{k} c_{j}^{r} f[x+(i-2k)t, y+(j-2r)u] \]
(9.1.3) \[ L^{(1,j)}(h,s,x,y,f) = (hs)^{-1} \int_0^h \int_0^s (h,j)f(x,y) \, dx \, dy \]
and

(9.1.4) \[ L^{(1,j)}(h,s,f) = \sup_{0 \leq x \leq \pi, 0 \leq y \leq \pi} \left| L^{(1,j)}(h,s,x,y,f) \right|, \]

the quantity \( L^{(1,j)}(h,s,f) \) is called the L-modulus of smoothness of order \( i, j \) of the function \( f(x,y) \).

Regarding the question of almost convergence of Fourier series of an \( L^2 \) function, Carleson \(^1\) proved that the Fourier series of an \( L^2 \) function converges almost everywhere. Fefferman \(^2\) generalized the Carleson's result on double Fourier series of \( L^2 \) function. Zuk \(^3\) obtained generalization of a number of classical results on absolute convergence of Fourier series, by using the concept of L-modulus of smoothness which obviously a more general concept than that of modulus of continuity.

It is the purpose of this chapter to prove a theorem on almost convergence of double Fourier series of \( f(x,y) \), involving the concept of \( L^{(1,j)}(h,s,f) \).

**Theorem** - If \( f(x,y) \) satisfy the condition

(9.1.5) \[ \sum_{\kappa, \nu=1}^{\infty} \left[ L^{(1,j)}(\pi/2, \kappa \nu, f) \right]^2 < \infty, \]

1. Carleson, L. (1).
2. Fefferman, C. (1).
3. Zuk, V.V. (1).
then the Fourier series of \( f(x, y) \) converges almost everywhere.

9.2 Proof of the Theorem.

It can be easily obtain when

\((9.2.1)\) \(i\) and \(j\) are both even

\[
\Delta_{t,u}^{(i,j)} f(x,y) = (-1)^{\frac{i}{2}(i+j)} 2^{i+j} \sum_{m,n=1}^{\infty} \left[ (p_{mn} \cos(mx - \theta) \cos ny + q_{mn} \cos(mx - \theta) \sin ny) \sin^i mt \sin^j nu \right]
\]

\((9.2.2)\) \(i\) and \(j\) are both odd

\[
\Delta_{t,u}^{(i,j)} f(x,y) = (-1)^{\frac{i}{2}(i+j+2)} 2^{i+j} \sum_{m,n=1}^{\infty} \left[ (p_{mn} \sin(mx - \theta) \sin ny - q_{mn} \sin(mx - \theta) \cos ny) \sin^i mt \sin^j nu \right]
\]

\((9.2.3)\) \(i\) is even and \(j\) is odd

\[
\Delta_{t,u}^{(i,j)} f(x,y) = (-1)^{\frac{i}{2}(i+j+1)} 2^{i+j} \sum_{m,n=1}^{\infty} \left[ (p_{mn} \cos(mx - \theta) \sin ny - q_{mn} \sin(mx - \theta) \cos ny) \sin^i mt \sin^j nu \right]
\]

\((9.2.4)\) \(i\) is odd and \(j\) is even

\[
\Delta_{t,u}^{(i,j)} f(x,y) = (-1)^{\frac{i}{2}(i+j+1)} 2^{i+j} \sum_{m,n=1}^{\infty} \left[ (p_{mn} \sin(mx - \theta) \cos ny + q_{mn} \sin(mx - \theta) \sin ny) \sin^i mt \sin^j nu \right]
\]
and hence using (9.2.1),

\[(9.2.5) \quad L^{(i,j)}(h,s,x,y,f) \sim (-1)^{(i+j)} 2^{1+j} \sum_{m,n=1}^{\infty} \left[ (p_{mn} \cos(mx-\theta) \cos ny + q_{mn} \cos(mx-\theta)) \sin ny \right] \]

\[\cdot \frac{h}{s} \int_{0}^{\infty} \sin^{1}mt \, dt \int_{0}^{\infty} \sin^{1}nu \, du \]

Similar expressions can be obtained for (9.2.2), (9.2.3) and (9.2.4).

Since \( L^{(i,j)}(h,s,x,y,f) \in L_{2} \) in the interval \((-\pi, \pi)\)
we have by Parseval's relation

\[ \pi^{2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left[ L^{(i,j)}(h,s,x,y,f) \right]^{2} dx \, dy = \]

\[ = c \sum_{m,n=1}^{\infty} (p_{mn}^{2} + q_{mn}^{2}) (hs)^{-1} \int_{0}^{\infty} \sin^{1}mt \, dt \int_{0}^{\infty} \sin^{1}nu \, du \]

where \( c \) denotes a constant depending only on \( i \) and \( j \).

Now putting

\[ h = \frac{\pi}{2^{p}}, \quad s = \frac{\pi}{2^{q}} \]

and we observe that \( \sin mt > 0, \quad \sin nu > 0 \) which are increasing function of \( t \) and \( u \) respectively.

For \( -\frac{\pi}{2^{n+1}} \leq t \leq -\frac{\pi}{2^{n+1}} \),
\[
\sin^2\left(\frac{mn/2^{m+1}}{\pi/8}\right) > \sin^2\left(\frac{\pi}{8}\right).
\]

We have

\[
\sum_{m,n=1}^{\infty} (p_{mn}^2 + q_{mn}^2)(2pn^{-1})^{\frac{\pi}{2p}} \int_0^{\pi/2} \sin^4mt \, dt (2qnn^{-1})^{\frac{\pi}{2q}} \int_0^{\pi/2} \sin^4nu \, du
\]

\[
\geq \sum_{m=1}^{p} \sum_{n=1}^{q} (p_{mn}^2 + q_{mn}^2)(2pn^{-1})^{\frac{\pi}{2p}} \int_0^{\pi/2} \sin^4mt \, dt (2qnn^{-1})^{\frac{\pi}{2q}} \int_0^{\pi/2} \sin^4nu \, du
\]

\[
\geq (16)^{-1} \sin^{2+2j\pi/8} \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} (p_{mn}^2 + q_{mn}^2) \left(\frac{\pi}{2^{m+1}} \right)^{2j\pi/8} \left(\frac{\pi}{2^{n+1}} \right)^{2j\pi/8}.
\]
Thus we get
\[
\sum_{m,n=2}^{\infty} \sum_{\nu=1}^{\infty} (p_{mn}^2 + q_{mn}^2) \leq C \int \int \left[ L^{4,1}(\pi/2, \pi/2, x, y, f) \right]^2 dx \, dy
\]
and hence
\[
\sum_{m,n=2}^{\infty} (p_{mn}^2 + q_{mn}^2) = C \sum_{\nu=1}^{\infty} \left\{ \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} (p_{mn}^2 + q_{mn}^2) \right\}
\]
\[
= C \sum_{\nu=1}^{\infty} \left[ L^{4,1}(\pi/2, \pi/2, f) \right]^2
\]
i.e.
\[
(9.2.6) \sum_{m,n=2}^{\infty} (p_{mn}^2 + q_{mn}^2) \leq C \sum_{\nu=1}^{\infty} \left[ L^{4,1}(\pi/2, \pi/2, f) \right]^2
\]
\[
< \infty.
\]
This completes the proof of the Theorem.

Corollary - For \( n = 0 \) and \( j = 0 \), (9.2.6) reduces to
then the Fourier series of \( f(x,y) \) converges almost everywhere. This generalizes the theorem of Yadav\(^1\).