CHAPTER V

MISCELLANEOUS RESULTS AND SOLUTIONS

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This chapter is devoted to the study of a few integral equations not discussed earlier in chapters II, III and IV.

The study of a class of integral equations was initiated by Tao Li (12). He solved the integral equation

\[ \int_{\pi}^{1} \left( \frac{t^2 - x^2}{2} \right)^{\frac{1}{2}} T_m \left( \frac{t}{\pi} \right) g(t) \, dt = f(x), \]

for the range \( 0 < c \leq x \leq 1 \),

where

\[ T_m(x) \text{ is a Chebyshev polynomial.} \]

Following Tao Li, in recent years, there appeared solutions of integral equations of the class

\[ \int_{\pi}^{1} k \left( \frac{t}{\pi} \right) g(t) \, dt = f(x) \]

whose kernels contain one of the classical orthogonal polynomials (1, 6, 7, 10).

Another class of integral equations

\[ \int_{\pi}^{1} k \left( \frac{t}{\pi} \right) g(t) \, dt = f(x) \]

has been introduced by S. Ilyin (5), when he considered the following integral equation

\[ \int_{\pi}^{1} \left( t^2 - x^2 \right)^{\frac{1}{2}} \left( \frac{t}{\pi} \right)^{-\frac{1}{2}} \left( \frac{t}{\pi} \right)^{-\frac{1}{2}} g(t) \, dt = f(x), \]

\( \lambda > 1, \mu > -\frac{1}{2}, 0 < a \leq x. \)
where \( P_n^m(x) \) represents the Legendre function.

It is obvious that the class of integral equations (5.1.3) has been defined by the class (5.1.2) by taking the upper limit as infinity.

In the present chapter we solve the integral equation which belong to the above class of integral equations considered by Erdelyi (5). It may also be seen that substitution of \(-1\) for the variable limit in the equation (5.1.2) creates another class of integral equations, viz:

\[
(5.1.5) \quad \int_{-1}^1 K\left(\frac{x}{t}\right) y(t) \, dt = f(x)
\]

We shall also consider this type of integral equation in what follows.

(5.2)

Our method depends upon the use of integral transforms defined by Sharma (11). The technique is similar as that of Prasad (2), who solved the integral equation (5.1.2) with Jacobi polynomial in the kernel, through finite Hankel transform.

We specify here a general theory to define the integral transforms for subsequent use.

If \( \varphi(x) \) is a set of orthonormal polynomials with respect to the weight function \( w(x) > 0 \) over the interval \( a < x < b \), then we have from Erdelyi (4, p 148).
\[(5.2.3) \quad \overline{V}(m) = \int_a^b V(x) \phi_m(x) \, dx \]

where

\[(5.2.4) \quad V(x) = \sum_{n=0}^{\infty} a_n W(x) \phi_n(x) \]

and

\[(5.2.5) \quad a_n = \frac{1}{\delta_m} \int_a^b V(x) \phi_n(x) \, dx = \frac{\overline{V}(m)}{\delta_m} \]

Hence in the light of (5.2.5), equation (5.2.4) gives the required inversion series of the transform (5.2.3) as

\[(5.2.6) \quad V(x) = \sum_{n=0}^{\infty} \frac{a_n}{\delta_m} W(x) \overline{V}(m) \]

Special cases of the above transform enumerated by Charne (11) include Legendre, Jacobi, Laguerre, Bessel, Hermite and Sonineanu transforms and their inversions in the form of infinite series.
The following results of Shama (11) are reproduced here which will be used in sequel.

If

\[
\overline{V}(m, \alpha) = \int_0^\infty V(x) L_{m+n}(x) \, dx,
\]

then

\[
\overline{V}(x) = \sum_{n=0}^{\infty} \frac{L_n(x)}{(m+n)!} \frac{(-x)^n}{(m+n)!(2n)!}.
\]

where \(L_n(x)\) is the Laguerre polynomial given by

\[
L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!(n-k)!}.
\]

Again, if

\[
\overline{V}_1(m, \alpha, \beta) = \int_0^\infty V(x) P_{m}^{(\alpha, \beta)}(x) \, dx
\]

then

\[
\overline{V}_1(x) = \sum_{m=0}^{\infty} \frac{P_{m}^{(\alpha, \beta)}(x)}{(m+n)!} \frac{(-x)^n}{(m+n)!(2n)!} \overline{V}(m, \alpha, \beta)
\]

where

\[
\overline{\delta}_m = \frac{2^{x+\beta+1}}{(1+x+\beta+\alpha \cdot m+1)} \frac{(1+\beta+\alpha \cdot m)}{m!(1+x+\beta+2m)(1+x+\beta+2m)}.
\]

and the function \(P_{m}^{(\alpha, \beta)}(x)\) is the Jacobi polynomial of degree \(m\) and orders \(\alpha, \beta \ (\alpha > 1, \beta > 1)\), defined by
\[ P^{(\alpha, \beta)}_n(x) = \binom{\alpha + \beta}{\alpha + n} P_{\alpha + n} \left( x \right) \]

The function \( P^{(\alpha, \beta)}_n(x) \) is a solution of the differential equation
\[
(1-x^2) \frac{d^2 y}{dx^2} + \left( \beta - \alpha - (\alpha + \beta + 2)x \right) \frac{dy}{dx} + (\alpha + \beta + 1) y = 0
\]

It may be mentioned that
\[
P^{(\alpha, \beta)}_n(1) = \binom{\alpha + \beta}{\alpha + n} = \frac{(\alpha + \beta)!}{n!}
\]
and
\[
\frac{2^n}{n!} P^{(\alpha, \beta)}_n(x) = (-1)^n (1-x)(1+x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} \left[ (1-x)(1+x) \right]
\]

Then \( \alpha = \beta = \gamma \), Jacobi polynomial reduces to Legendre polynomial defined as
\[
P^{(\gamma,\gamma)}_n(x) = 2^\gamma P_{\gamma + n} \left( x \right)
\]

The form (5.3.3) of Jacobi transform was introduced by Scott (2) and later on Debnath (3) introduced yet another transform described by the following statement:

If
\[
\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x) P^{(\alpha, \beta)}_n(x) dx
\]
then
\[
f(x) = \sum_{n=0}^{\infty} \frac{P^{(\alpha, \beta)}_n(x) f^{(\alpha, \beta)}_n}{\delta n}
\]
for (\( n = 0, 1, 2, \ldots \))
where \( \xi_n \) is defined by (5.3.5).

Lastly, if

\[
V_2(n) = \int_{-1}^{1} V_{2}(x) C''_n(x) dx
\]

then

\[
V_2(x) = \sum_{n=0}^{\infty} \frac{C''_n(x) m! (n+m)! \sqrt{n} \sqrt{2n+1}}{\sqrt{n} \sqrt{2n+1} \sqrt{2n+1}}
\]

where the function \( C''_n(x) \) is the Gegenbauer polynomial defined by

\[
C''_n(x) = \sum_{k=0}^{n} \frac{(-1)^k (2)_k (n-k) (2x)^{n-k}}{k! (n-k)!}
\]

We shall now invert some integral equations by means of the aforesaid transforms. Inversions figure as theorems. The integral equations exist since the functions involved are assumed to be single valued, finite and continuous in the ranges under consideration.

(5.4)

\textbf{Lemma} 1: If

The function \( f(x) \) is piece wise continuous for all \( x > 0 \), then the integral equation

\[
\int_{\rho}^{\infty} \frac{1}{y} \left( \frac{x}{y} \right)^{\alpha} \left( 1 - \frac{x}{y} \right)^{\beta-\alpha-1} f(y) dy = g(x),
\]

will have the solution
\[(5.4.2) \quad f(y) = \frac{\alpha^\frac{\beta}{2} y^{\frac{\beta}{2} - 1} \cdot \gamma \cdot \Gamma\left(\frac{\beta}{2}\right)}{1 - y^{\frac{\beta}{2}}} \cdot \frac{\Gamma\left(m + \frac{\beta}{2}\right)}{\Gamma\left(m + 1\right)} \cdot \frac{\Gamma\left(n - m + \frac{\beta}{2}\right)}{\Gamma\left(n - m + 1\right)} \cdot \frac{\Gamma\left(n + \frac{\beta}{2}\right)}{\Gamma\left(n + 1\right)},\]

where

\[(5.4.3) \quad g(\tau, \alpha) = \int_0^1 y(x) L_{m+\frac{\beta}{2}}(\alpha) \, dx\]

and \(\beta > \text{Re}\alpha > -1\).

**Proof:**

Multiplying both sides of (5.4.1) by \(L_{m+\frac{\beta}{2}}(\alpha)\) and integrating with respect to \(x\) between 0 and \(\infty\), we get

\[
\int_0^\infty L_{m+\frac{\beta}{2}}(\alpha) \int_0^\infty y(x) L_{m+\frac{\beta}{2}}(\alpha) \, dx \, dy = \int_0^\infty g(x) L_{m+\frac{\beta}{2}}(\alpha) \, dx
\]

Interchanging the order of integration, we obtain

\[(5.4.4) \quad \int_0^\infty f(y) \left[ \int_0^\infty y(x) L_{m+\frac{\beta}{2}}(\alpha) \right] \, dx \, dy = \int_0^\infty g(x) L_{m+\frac{\beta}{2}}(\alpha) \, dx.
\]

From Table 4, page 53-5), by changing \(x\) to

\[(5.4.5) \quad \int_0^\infty y(x) L_{m+\frac{\beta}{2}}(\alpha) \, dx = A L_{m+\frac{\beta}{2}}(\alpha)
\]

where

\[(5.4.6) \quad A = \frac{\Gamma\left(m + \frac{\beta}{2}\right)}{\Gamma\left(m + 1\right)} \frac{\Gamma\left(\alpha + \frac{\beta}{2}\right)}{\Gamma\left(\alpha + 1\right)}
\]

and \(\beta > \text{Re}\alpha > -1\).
Using (5.4.5), the equation (5.4.4) reduces to the form:

\[ \dot{A} + f(1, \ldots, n) \dot{y} = \int f(x, \ldots, n) \, dx \]

or by (5.4.1) we set

\[ \Gamma = \int \, \dot{y} = \int f(\tau, x) \, d\tau \]

Inversion by means of (5.3.2) with \( m = 0 \) yields the desired result (5.4.2).

This completes the proof.

(5.5)

Theorem: If the function \( \phi(x) \) is piecewise continuous for all \( x > 0 \), then

\[ \int_0^\infty \frac{1}{y} \left( \frac{x}{y} \right)^{\beta} \left(1 - \frac{x}{y}\right)^{\lambda-1} \phi(y) \, dy = g_1(x) \quad (5.5.1) \]

has the solution

\[ \phi(y) = \frac{m! \, y^{\alpha+1} \, \Gamma \left( \frac{x + B + 1}{y} \right)}{\Gamma \left( \frac{x + B + m+1}{y} \right)} \sum_{m=0}^{\infty} \frac{(\delta + B + m \, y^\alpha \, \Gamma \left( \frac{x + B + m+1}{y} \right))}{\Gamma \left( \frac{x + B + m+1}{y} \right)} \]

where

\[ g_1(x) = \int_0^\infty \phi(y) \, \Gamma \left( \frac{x + B + m+1}{y} \right) \, dy \quad (5.5.2) \]

and (1) \( \Re \alpha > -1 \), (2) \( \Re \beta > -1 \).
\textbf{Note:}

From identity \((4, \text{ p. 293 - 4})\), we get

\begin{equation}
(5.5.4) \quad \int_{0}^{y} \frac{y}{y^{1}} (x)^{\alpha} (1 - \frac{x}{y})^{\beta} J_{\mu}(x) \cdot J_{\nu}(y - x) \, dx = \frac{\alpha + \beta + 1}{\Gamma(\alpha) \Gamma(\beta + \nu + 1)} \int_{0}^{y} \frac{y}{y^{1}} (x)^{\alpha} (1 - \frac{x}{y})^{\beta} J_{\nu}(x) \, dx
\end{equation}

where

\begin{equation}
(5.5.5) \quad B = \frac{(\alpha + \nu + 1) \Gamma(\alpha + \nu + 1)}{(\nu + 1) \Gamma(\alpha + \nu + 2)}
\end{equation}

and we \( x > y \), \( \alpha > y \)

where \( \alpha \) and \( \beta \) are non-negative integers.

By multiplying both sides of \((5.5.1)\) by \( J_{\mu}(x) \) and integrating with respect to \( x \) between the limits \( 0 \) to \( \infty \), we obtain

\[
\int_{0}^{\infty} J_{\mu}(x) \cdot \left\{ \int_{0}^{\infty} \frac{y}{y^{1}} (x)^{\alpha} (1 - \frac{x}{y})^{\beta} J_{\mu}(y - x) \, dy \right\} \, dx = \int_{0}^{\infty} J_{\mu}(x) \, dx.
\]

The change in the order of integration gives

\[
\int_{0}^{\infty} \phi(y) \cdot \left\{ \int_{0}^{\infty} \frac{y}{y^{1}} (x)^{\alpha} (1 - \frac{x}{y})^{\beta} J_{\mu}(x) \, dx \right\} \, dy = \int_{0}^{\infty} \phi(x) \, dx.
\]

which by \((5.5.4)\) reduces to the form

\[
B \int_{0}^{\infty} \phi(y) \cdot J_{\nu}(y) \, dy = \int_{0}^{\infty} \phi(x) \cdot J_{\nu}(x) \, dx.
\]

Then by \((5.5.1)\), we get
If the function $h(x)$ is piecewise continuous for all $x > 0$, then the integral equation

$$
\int_0^1 C_n \{ 1 - x^2 (1-y) \} \phi(y) dy = x_h(x)
$$

has the solution

$$
\phi(y) = \sum_{n=0}^{\infty} \frac{(1-y)^n (1+y)^{\frac{\beta}{\alpha}} \Gamma(\frac{\beta}{\alpha})}{\delta_n} \phi \left( \frac{\beta}{\alpha}, \frac{\beta}{\alpha} \right)
$$

where

$$
\delta_n = \frac{2 \left( (m+\nu+\frac{\beta}{\alpha}) \right) \int_0^1 x^{\nu} (1-x^2) \phi(x) dx}{(2\alpha \nu \Gamma(\nu+\frac{\beta}{\alpha})}
$$

and

1. $\Re \nu > -\frac{1}{2}$
2. $\Re \nu > 0$
3. $\alpha = \nu + \frac{1}{2}$
4. $\beta = \nu - \frac{1}{2}$

where $\delta_n$ is defined by (5.3.5).

From (4.6.20 : 21) we get

$$
\int_0^1 x^{2\nu} (1-x^2)^{\frac{\nu}{2}} C_n \{ 1 - x^2 (1-y) \} dx =
$$
\[
\int_0^1 x^{\nu}(1-x^2)^{\frac{T-1}{2}} \int_0^1 2^\nu \Gamma(\nu+\frac{1}{2})^{-1} p(x, y) \phi(y) dy dx = \int_0^1 x^{\nu}(1-x^2)^{\frac{T-1}{2}} h(x) dx.
\]

Then the use of transform formula (5.3.4) and inversion series (5.3.2) gives the required result (5.6.2).

Thus the theorem is proved.

(5.7)

Remark II: If

the function \( h(x) \) is piecewise continuous for all \( x > 0 \),

then

\[
(5.7.1) \int (1-y)(1+y)^{\nu} \int_0^1 \left( 1 - x^2(1-y)^2 \right)^{\frac{T-1}{2}} \phi(y) dy = h(x)
\]

\[\]

where

\( \nu > \frac{1}{2}, \ h > 0, \ x = \nu + \frac{1}{2}, \ \beta = \nu - \frac{1}{2}. \)
has the solution

\[ f(y) = \sum_{n=0}^{\infty} \frac{P_n(x)P_n(y)}{y^{2n+1}} \int_0^1 x^{2n}(1-x^2)^{\frac{T-1}{2}} \frac{h(x)}{\alpha} dx, \]

for \( -1 < y < 1 \).

where \( \delta_m \) is defined by (5.3.3) and \( \lambda, \mu, \sigma \) are functions of \( n, x, \beta \).

\textbf{Lemma 1:}

We have the following results from Erdelyi (4, p 113 - 117):

\[ \int_0^1 x^{2n}(1-x^2)^{\frac{T-1}{2}} \{ 1-x^2(1-y)^2 \} dx = A P_n(x), \]

where

\[ A = \frac{(2n)! \Gamma(n+\frac{1}{2}) \Gamma(m)}{2 \Gamma(n+2m+\sigma+\frac{1}{2})}. \]

and

(i) \( n > -\frac{1}{2} \); (ii) \( \Re \sigma > 0 \)

(iii) \( \lambda = \mu + \sigma - \frac{1}{2} \); (iv) \( \beta = \mu - \sigma - \frac{1}{2} \).

Now multiplying both sides of (5.7.1) by \( x^{2n}(1-x^2)^{\sigma-1} \)

and integrating with respect to \( x \) between the limits 0 to 1, we get

\[ \int_0^1 x^{2n}(1-x^2)^{\sigma-1} \{ \int_0^1 (1-y)(1+y) C_n \{ 1-x^2(1-y)^2 \} dx \} \, dy \]

\[ = \int_0^1 x^{2n}(1-x^2)^{\frac{T-1}{2}} \frac{h(x)}{\alpha} dx. \]

Change in the order of integration gives
\[
\int_0^1 (1-x)(1+y)^{P-1} x^{\alpha-1} \frac{d y}{y} = \int_0^1 x^{2\beta} (1-x^2)^{\beta-1} K(x) \, dx.
\]

Using (5.7.3) and (5.3.8) to (5.7.5), we have
\[
\int_0^1 (x, \beta) = \int_0^1 x^{2\beta} (1-x^2)^{\beta-1} K(x) \, dx.
\]

Finally, using the inversion series (5.3.7) we obtain the desired result (5.7.2).

This completes the theorem.

(5.8)

In the present section we attempt to find an inversion of the following integral equation
\[
\int_0^1 L_{\beta, \gamma}^{\alpha, \beta} \left[ \frac{(1-y)^{\alpha-1} y^{\beta-1}}{2} \right] f(y) \, dy = g(x)
\]

where the kernel is the Meijer $G$-function defined by
\[
L_{\beta, \gamma}^{\alpha, \beta} \left[ \frac{(1-y)^{\alpha-1} y^{\beta-1}}{2} \right] = \frac{\pi i}{2} \int_0^\infty \frac{\Gamma(b_j-s) \Gamma(m-s) \Gamma(1-a_j+s) x^s \, ds}{\Gamma(n-s) \Gamma(1-b_j+s) \Gamma(1-a_k+s) x^s}
\]

where an empty product is interpreted as 1, $1 \leq m \leq n$, $1 \leq n \leq p$, and the parameters are such that no pole of $\Gamma(b_j-s)$, $j = 1, 2, \ldots, m$ coincides with any pole of $\Gamma(1-a_k+s)$, $k = 1, 2, \ldots, n$.

If $p + q < 2(m + n)$ and $|arg x| < (m + n \cdot \frac{p}{2} - \frac{q}{2}) \pi$,
First we deduce a result which we require for the purpose.

From, [ref. 1 (4, p 417 -1)] we have

\[ \int_0^1 x^{\gamma-1} \left( 1-x \right)^{\alpha-1} \left( \frac{ax}{b} \right)^m \left( \frac{b}{c} \right)^n \left( \frac{d}{e} \right)^p \, dx \]

(5.8.1)

\[ = \Gamma \left[ \beta \mu \right] \left( \frac{a}{b} \right)^m \left( \frac{b}{c} \right)^n \left( \frac{d}{e} \right)^p \]

which leads to:

\[ \int_0^1 x^{\gamma+2\lambda-1} \left( 1-x \right)^{\alpha-1} \eta_{1,1}^{1,1} \left[ \left( \frac{y}{2} \right)^m \left( \frac{y}{2} \right)^n \right] \, dx \]

(5.8.2)

\[ = \Gamma \left[ \frac{1}{2} - \lambda \right] \eta_{2,1}^{1,2} \left[ \left( \frac{y}{2} \right)^m \left( \frac{y}{2} \right)^n \right] \]

also, we know that Ultraspherical polynomials is defined by

\[ C_n^\lambda (y) = \frac{1}{\Gamma (\lambda + \frac{1}{2})} \frac{\sqrt{\pi \Gamma (\lambda + 1)}}{\Gamma (\lambda + \frac{1}{2})} \frac{\Gamma (\lambda + \frac{1}{2})}{\Gamma (\lambda + 1)} \frac{\left( \frac{y}{2} \right)^m \left( \frac{y}{2} \right)^n}{\Gamma (\lambda + 1)} \]

(5.8.3)

which can be transformed into the form

\[ C_n^\lambda (y) = \frac{(2\lambda)m!}{m!} \frac{1}{\Gamma (\lambda + \frac{1}{2})} \frac{\Gamma (\lambda + 1)}{\Gamma (\lambda + \frac{1}{2})} \frac{\left( \frac{y}{2} \right)^m \left( \frac{y}{2} \right)^n}{\Gamma (\lambda + 1)} \]

(5.8.4)

Therefore from (5.8.2) and (5.8.4) we obtain

\[ \int_0^1 x^{\gamma+2\lambda-1} \left( 1-x \right)^{\alpha-1} \eta_{1,1}^{1,1} \left[ \left( \frac{y}{2} \right)^m \left( \frac{y}{2} \right)^n \right] \, dx \]

(5.8.5)

\[ = A C_n^\lambda (y) \]

where
(5.8.6) \[ A = \frac{\Gamma \left( \frac{1}{2} - n - \lambda \right) \Gamma (n + 2\lambda - 1) \Gamma (n - m - n + 1)}{\Gamma (n + 1)} \]

and \( \left| \text{arg} \left( \frac{1}{2} - n - \lambda \right) \right| < \pi \), \( n \left( \frac{1}{2} - n - \lambda \right) > 0 \). 

Hence we arrive at the following

**Example:** If

the function \( f(x) \) is piecewise continuous for all \( x > 0 \), then

the integral equation

(5.8.6) \[ \int_{1}^{x} \frac{g(t)}{t^{m+1}} \left( \frac{y}{t} \right)^{\frac{m+1}{m}} dt = g(x) \]

has the solution

(5.8.7) \[ f(y) = \sum_{n=0}^{\infty} \frac{\Gamma \left( \frac{1}{2} - n - \lambda \right) \Gamma (n + 2\lambda - 1) \Gamma \left( \frac{1}{2} - m - \lambda \right)}{(2\lambda n)!} \int_{0}^{1} C_{m}^{\lambda} (y) (1-y^{2})^{\frac{1}{2}} f(y) dy \]

where

(5.8.8) \[ f(y) = \frac{1}{A} \int_{0}^{1} x^{m+2\lambda-1} (1-x)^{-\frac{1}{2}-m-\lambda} g(x) dx \]

and \( A \) is given by (5.8.6).

**Note:**

Multiplying both sides of (5.8.6) by \( x^{m+2\lambda-1} (1-x)^{\frac{1}{2}-m-\lambda} \) and integrating with respect to \( x \) between the limits 0 and 1, we obtain

(5.8.9) \[ \int_{0}^{1} x^{m+2\lambda-1} (1-x)^{-\frac{1}{2}-m-\lambda} \left[ \int_{0}^{1} \frac{g(t)}{t^{m+1}} \left( \frac{y}{t} \right)^{\frac{m+1}{m}} dt \right] dx = \int_{0}^{1} x^{m+2\lambda-1} (1-x)^{-\frac{1}{2}-m-\lambda} g(x) dx. \]
Changing the order of integration and using the result (5.8.5), we get

\[ A \int_{-1}^{1} f(y) C_{n}^{\lambda} (y) \, dy = \int_{0}^{1} x^{\lambda+2\lambda-1} (1-x)^{\frac{1}{2}-n-\lambda} g(x) \, dx \]

which by transform relation (5.3.8) reduces to the form

\[ \bar{f} (n, \lambda) = \frac{1}{\alpha} \int_{0}^{1} x^{\lambda+2\lambda-1} (1-x)^{\frac{1}{2}-n-\lambda} g(x) \, dx \]

Finally using inversion series solution (5.3.9) we have the desired solution (5.8.7).

This establishes the proof of the theorem.
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<td>Selme, L.</td>
<td>1961</td>
<td>Annali dell'Univ. di Ferrara sezione VII Scienze Mat. Vol IX No.12, p 149.</td>
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