CHAPTER-4

CERTAIN ITERATIVE BEHAVIOUR OF NONEXPANSIVE MAPPING
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This chapter starts with an example to underline the relationship between nonexpansive mapping and quasinonexpansive mapping. Rest of the chapter is divided into two sections. In section 1, convergence of a Kirk type iteration scheme is introduced and its iterative convergence is proved for quasinonexpansive mappings. Section 2 is furnished with the convergence result for the class of nearly nonexpansive mappings. The condition of uniformly convexity space structure excluded from the hypothesis.

First we give following example which shows that the class of nonexpansive mapping is contained in the class of quasi-nonexpansive mappings:

EXAMPLE 4.1.1. Let $T: \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$T(x) = \begin{cases} 0, & \text{if } x=0 \\ x \sin \frac{1}{x}, & \text{if } x \neq 0 \end{cases}$$

Obviously $x = 0$ is the only fixed point of $T$, i.e., $T(0) = 0$.

Since if $y \in \mathbb{R}$, $p = 0$ then

$$|Ty - p| = |Ty - 0| = \left| y \sin \frac{1}{y} - 0 \right| = \left| y \sin \frac{1}{y} \right|$$

$$= |y| \left| \sin \frac{1}{y} \right| \leq |y| = |y-0| = |y-p|$$

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\[ |Ty - p| \leq |y - p| \text{ for all } y \in R. \]

Then \( T \) is continuous quasinonexpansive, but it is not nonexpansive. Let \( x = \frac{2}{\pi} \) and \( y = \frac{2}{3\pi} \). Then

\[ |Tx - Ty| = \left| \frac{2}{\pi} \sin \frac{\pi}{2} - \frac{2}{3\pi} \sin \frac{3\pi}{2} \right| = \left| \frac{2}{\pi} \sin \frac{\pi}{2} + \frac{2}{3\pi} \sin \frac{\pi}{2} \right| = \left| \frac{2}{\pi} + \frac{2}{3\pi} \right| = \frac{8}{3\pi} \]

where as \( |x - y| = \left| \frac{2}{\pi} - \frac{2}{3\pi} \right| = \frac{4}{3\pi} \).

**SECTION 1**

We have discussed in chapter-1, how various iteration schemes were developed to ensure the convergence of sequences of various class of nonexpansive mappings.

In a paper, Kirk [59] studied the iteration process given by

\[ x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + \ldots + \alpha_k T^k x_n, \]

where \( \alpha_i \geq 0, \alpha_0 > 0 \) and \( \sum_{i=1}^{k} \alpha_i = 1 \), and proved convergence result by approximating fixed points of nonexpansive mappings on convex subset of uniformly convex Banach space. Later, Maiti and Saha [72] extended the result of Kirk [59].

Let \( T_i : D \rightarrow D \) \((i = 1, 2, \ldots)\) be nonexpansive mappings and let

\[ S = \alpha_0 + \alpha_1 T_1 + \alpha_2 T_2 + \ldots + \alpha_k T_k, \]

where \( \alpha_i \geq 0, \alpha_0 > 0 \) and \( \sum_{i=1}^{k} \alpha_i = 1 \).

Recently, Liu et al. [70] have introduced a new iteration process \( x_{n+1} = S x_n \), where \( x_0 \in D \). They have showed that \( \{x_n\} \) converges to a common fixed point of \( T_i \) \((i = 1, 2, \ldots, k)\) in a Banach space under certain
property, say, condition A. This result has improved the corresponding result of Kirk [59], Maiti and Saha [72], Senter and Dotson [99].

Now, our intention is to introduce a new iteration process which generalizes the iteration scheme of Liu et al. [70]. Further, we present a necessary and sufficient condition for the convergence of newly introduced iteration process as an improvement over the result of Kirk [59], Maiti and Saha [72] and Liu et al. [70] for quasi-nonexpansive mappings in Banach space without uniform convexity.

First, we give the new iteration process as below:

Let \( x_0 \in D \) and let

\[
S = \alpha_{n,0} I + \alpha_{n,1} T_1 + \alpha_{n,2} T_2 + \ldots + \alpha_{n,k} T_k,
\]

where \( \alpha_{n,i} \geq 0, \alpha_{n,0} > 0 \) and \( \sum_{i=1}^{k} \alpha_{n,i} = 1 \).

Thus,

\[
x_{n+1} = Sx_n \tag{4.1.1}
\]

We require following lemma to prove our result.

**Lemma 4.1.1.** Let \( D \) be a convex subset of a normed space \( E \) and \( T_i : D \to D \) be a quasinonexpansive mapping for all \( i = 1, 2, \ldots, k \). If for an arbitrary \( x_0 \in D \) a sequence \( \{x_n\} \) in \( D \) is defined by the iteration (4.1.1), then

\[
\|x_{n+1} - p\| \leq \|x_n - p\|
\]

for all \( n \geq 1 \) and \( p \in \bigcap_{i=1}^{k} F(T_i) \), where \( F(T) \) denotes the common fixed point of \( T_i \) (\( i = 1, 2, \ldots, k \)).
PROOF. From (4.1.1), we have

\[ \|x_{n+1} - p\| = \|Sx_n - p\| \]
\[ = \|(a_{n,0}x_1 + a_{n,1}T_1x_1 + a_{n,2}T_2x_2 + \ldots + a_{n,k}T_kx_k) - p\| \]
\[ \leq \alpha_{n,0}\|x_n - p\| + \alpha_{n,1}\|T_1x_n - p\| + \ldots + \alpha_{n,k}\|T_kx_n - p\| \]
\[ \leq (\alpha_{n,0} + \alpha_{n,1} + \ldots + \alpha_{n,k})\|x_n - p\| . \]

This completes the proof of Lemma 4.1.1.

**LEMMA 4.1.2.** Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of nonnegative numbers such that \( a_{n+1} \leq (1 + Mb_n)a_n \quad \forall \ n \geq 1 \) and \( \sum_{n=1}^{\infty} b_n < \infty \), where \( M \) is a nonnegative constant. If \( \lim \inf_{n \to \infty} a_n = 0 \), then \( \lim_{n \to \infty} a_n = 0 \).

**PROOF.** Since \( \{a_n\} \) is bounded, then

\[ a_{n+m} \leq (1 + Mb_{n+m-1})a_{n+m-1} \]
\[ \leq e^{Mb_{n+m-1}}a_{n+m-1} \]
\[ \leq e^{M(b_{n+m-1} + b_{n+m-2})}a_{n+m-2} \]

\[ \leq e^{M \sum_{i=n}^{n+m-1} b_i}a_n \leq e^{MK}a_n \quad \forall \ n, m \in \mathbb{N}, \]

where \( K = \sum_{n=1}^{\infty} b_n \).

Hence.
which implies that

\[ \limsup_{m \to \infty} a_m \leq e^{MK} \liminf_{n \to \infty} a_n = 0. \]

It follows that

\[ \lim_{n \to \infty} a_n = 0. \]

Now, we give our main result of this section:

**THEOREM 4.1.1.** Let \( D \) be a closed convex subset of a Banach space \( E \) and \( T_i : D \to D \) \( (i = 1, 2, \ldots, k) \) be quasi-nonexpansive mappings and \( F(T_i) \subset D \) is nonempty closed set. Let \( \{x_n\} \) be a sequence in \( D \) defined by (4.1.1). Then \( \{x_n\} \) converges strongly to a common fixed point of \( T_i \) \( (i = 1, 2, \ldots, k) \) if and only if

\[ \liminf_{n \to \infty} d\left(x_n, \bigcap_{i=1}^{k} F(T_i)\right) = 0. \]

**PROOF.** Suppose \( \{x_n\} \) converges to a common fixed point of \( T_i \) \( (i = 1, 2, \ldots, k) \), then

\[ \liminf_{n \to \infty} d\left(x_n, \bigcap_{i=1}^{k} F(T_i)\right) = 0. \]

Conversely, suppose \( \liminf_{n \to \infty} d\left(x_n, \bigcap_{i=1}^{k} F(T_i)\right) = 0. \)

Then from Lemma 4.1.1, we have

\[ \|x_{n+1} - p\| \leq \|x_n - p\| \]
which implies that
\[ d\left(x_{n+1}, \bigcap_{i=1}^{k} F(T_i) \right) \leq d\left(x_n, \bigcap_{i=1}^{k} F(T_i) \right) \] (4.1.2)

Since \( \liminf_{n \to \infty} d\left(x_n, \bigcap_{i=1}^{k} F(T_i) \right) = 0 \), it follows from inequality (4.1.2) and Lemma 4.1.2., that
\[ \lim_{n \to \infty} d\left(x_n, \bigcap_{i=1}^{k} F(T_i) \right) = 0. \] (4.1.3)

Since each \( F(T_i) \subset D \) is closed, it follows that \( \bigcap_{i=1}^{k} F(T_i) \) is also closed.

Therefore from inequality (4.1.3), we conclude that \( \{x_n\} \) converges strongly to an element of \( \bigcap_{i=1}^{k} F(T_i) \). This completes the proof.

If we put \( \alpha_{n,i} = 0 \) for all \( i = 2, 3, k \), then \( \alpha_{n,0} + \alpha_{n,1} = 1 \). Again put \( \alpha_{n,1} = \alpha_n \) then \( \alpha_{n,0} = 1 - \alpha_n \) and put \( T_1 = T \), then we have the following corollary.

**COROLLARY 4.1.1.** Let \( D \) be a closed convex subset of a Banach space and \( T: D \to D \) be a quasinonexpansive mapping and \( F(T) \subset D \) be nonempty closed set. Let \( \{x_n\} \) be a sequence in \( D \) defined by
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n \] (4.1.4)

Then \( \{x_n\} \) converges strongly to a fixed point of \( T \) if and only if
\[ \liminf_{n \to \infty} d(x_n, F(T)) = 0. \]

**REMARK 4.1.1.** Theorem 4.1.1 improved the corresponding result of Kirk [59], Maiti and Saha [72] and Senter and Dotson [99].
SECTION 2

To scale the geometry of Banach spaces, most natural way is the nature of convexity. The definition of strictly convex spaces, which is fundamental for geometric theory of Banach spaces, is due to Clarkson [21]. It gained importance in metric fixed point theory when Browder and Gohde independently ensured the existence of fixed point for nonexpansive mapping in uniformly convex Banach space. At the same time, Kirk proved this existence result for nonexpansive mapping using the geometric structure called normal structure in reflexive Banach space.

In this section we consider a new concept known as nearly nonexpansive mappings [93] to prove necessary and sufficient conditions for a generalized Ishikawa Iteration scheme (Sharma and Sahu [101]) in Banach space converging to fixed point without the condition of uniform convexity on space. Our results generalized and improve the results of Ghosh and Debnath [32], Qihou [87] and Petryshyn and Williamson [81].

Let $D$ be a nonempty subset of a Banach space and $T$ be a self mapping in $D$. $T$ is called asymptotically quasi nonexpansive if there exists a positive constant $k_n \in [1, +\infty)$, $\lim_{n \to \infty} k_n = 1$, such that

$$||T^n x - p|| \leq k_n ||x - p|| \forall x \in E, \forall p \in F(T) \forall n \in N \quad (F(T)$$

denotes the set of fixed points of $T$).

$T$ is called asymptotically nonexpansive if there exists a positive constant $k_n \in [1, \infty)$ such that $\lim_{n \to \infty} k_n = 1$ and

$$||T^n x - T^n y|| \leq k_n ||x - y|| \forall x, y \in E, n \in N.$$

$T$ is called nonexpansive if

$$||Tx - Ty|| \leq ||x - y||$$

for all $x, y \in E$. 

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From the above definition, it follows. Following important facts that if
\( F(T) \) is nonempty, a nonexpansive mapping must be quasi-nonexpansive
asymptotically nonexpansive mapping must be asymptotically quasi-
nonexpansive. However, converse does not hold.

The mapping \( T \) is asymptotically nonexpansive type \([58]\) if

\[
\limsup_{n \to \infty} \sup_{y \in D} \{ \| T^n x - T^n y \| - \| x - y \| : x \in D \} \leq 0 \text{ for all } y \in D.
\]

The class of asymptotically nonexpansive type mappings is essentially
wider class than asymptotically nonexpansive mappings (see \([58]\)).

Petryshyn and Williamson \([81]\), in 1973, proved a sufficient and
necessary condition for Mann iterative sequence for quasinonexpansive
mapping converging to fixed point. In 1977, Ghosh and Debnath \([32]\]
extended the results of Petryshyn and Williamson \([81]\) and gave the sufficient
and necessary condition for Ishikawa iterative sequences of
quasinonexpansive mappings converging to fixed points.

Recently, Sharma and Sahu \([101]\) have introduced a new iteration
process as generalization of Ishikawa iteration process for asymptotically
nonexpansive mappings in the framework of uniformly convex Banach space
as below:

For \( x_1 \in D \), define a sequence \( \{x_n\} \) in \( D \) such that for each \( n \in \mathbb{N} \) and
fixed \( m \in \mathbb{N} \)

\[
x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n,
\]

\[
y_n = y_n^{(0)} = \beta_n^{(0)} T^n y_n^{(1)} + (1 - \beta_n^{(0)})x_n,
\]

\[
y_n^{(1)} = \beta_n^{(2)} T^n y_n^{(2)} + (1 - \beta_n^{(2)})x_n,
\]

\[
y_n^{(m-1)} = \beta_n^{(m)} T^n x_n + (1 - \beta_n^{(m)})x_n
\]

(4.2.1)
where \( \{a_n\}, \{\beta_n^{(1)}\}, \{\beta_n^{(2)}\}, \ldots, \{\beta_n^{(m-1)}\} \) and \( \{\beta_n^{(m)}\} \) be sequences of real numbers in \([0,1]\) satisfying the appropriate conditions.

More recently, Sahu et al. [93] introduced the concept of (nonlipschitzian) nearly nonexpansive mapping as below:

Let \( D \) be a nonempty subset of a Banach space \( E \) and \( T:D \to D \) a nonlinear mapping. Then \( T \) is said to be nearly nonexpansive if there exists a sequence \( \{u_n\} \) is \((0,\infty)\) with \( \lim_{n \to \infty} u_n = 0 \)

\[
\|T^n x - T^n y\| \leq \|x - y\| + u_n \quad \text{for} \quad x, y \in D \quad \text{and} \quad n \in \mathbb{N}.
\]

Now we propose to give the minimum condition for strong convergence of above generalized Ishikawa iteration process for nearly nonexpansive mappings in Banach space. The concept of uniformly convexity on underlying space is not used for such purpose. However, first let us recall the following Lemma to use for our main result:

**LEMMA 4.2.1** [105]. Let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of nonnegative numbers such that \( a_{n+1} \leq a_n + b_n, n \geq 0 \). If \( \sum_{n=0}^{\infty} b_n \) converges, then \( \lim_{n \to \infty} a_n \) exists.

**LEMMA 4.2.2.** Let \( D \) be a nonempty convex subset of a normed space \( X \) and \( T:D \to D \) be a nearly nonexpansive mapping with \( F(T) \neq \emptyset \) and sequence \( \{a_n\} \)
such that \( \sum_{n=0}^{\infty} a_n \) converges. Let \( \{a_n\}, \{\beta_n^{(1)}\}, \{\beta_n^{(2)}\}, \ldots, \{\beta_n^{(m-1)}\} \) and \( \{\beta_n^{(m)}\} \) be sequences of real numbers. Then, we have the following:

(a) \( \|x_{n+1} - p\| \leq \|x_n - p\| + (m+1)u_n \) for all \( n \in \mathbb{N} \) and \( p \in F(T) \).

(b) \( \lim_{n \to \infty} \|x_n - p\| \) exists for \( p \in F(T) \).
PROOF. (a) From (4.2.1), we have
\[
\|y_n^{(m-1)} - p\| \leq \beta_n^{(m)} \|T^n x_n - p\| + (1 - \beta_n^{(m)}) \|x_n - p\|
\]
\[
\leq \|x_n - p\|
\]
\[
\|y_n^{(m-2)} - p\| \leq \beta_n^{(m-1)} \|T^n y_n^{(m-1)} - p\| + (1 - \beta_n^{(m)}) \|x_n - p\|
\]
\[
\leq \beta_n^{(m-1)} \|x_n - p\| + (1 - \beta_n^{(m)}) \|x_n - p\| + 2u_n
\]
\[
\leq \|x_n - p\| + 2u_n
\]

So, in general
\[
\|y_n - p\| = \|y_n^{(0)} - p\| \leq \|x_n - p\| + mu_n
\]

We therefore conclude that
\[
\|x_{n+1} - p\| \leq \|x_n - p\| + (m + 1)u_n\text{ for all } n \in \mathbb{N}.
\]

Form Lemma 4.2.1, we obtain that \( \lim_{n \to \infty} \|x_n - p\| \) exists for all \( p \in F(T) \).

Now, we give our main result of this section.

**THEOREM 4.2.1.** Let \( D \) be a closed convex subset of a Banach space \( X \) and \( T:D \to D \) be an nearly nonexpansive mapping sequence \( \{a_n\} \) such \( F(T) \neq \emptyset \) is nonempty closed set and \( \sum_{n=0}^{\infty} a_n < \infty \). Let for fixed \( m \in \mathbb{N} \), \( \{\alpha_n\}, \{\beta_n^{(1)}\}, \{\beta_n^{(2)}\}, \ldots, \{\beta_n^{(m-1)}\} \) and \( \{\beta_n^{(m)}\} \) be the sequences of real numbers in \([0,1]\).
For \( x_1 \in D \), from the iterative sequence of generalised Ishikawa iteration process (4.2.1). Then \( \{x_n\} \) converges to some fixed point of \( T \) if and only if
\[
\liminf_{n \to \infty} d(x_n, F(T)) = 0.
\]

Proof. Suppose \( \{x_n\} \) converges to some fixed point \( z \) of \( T \). Then
\[
\liminf_{n \to \infty} d(x_n, F(T)) = 0.
\]
Conversely, suppose \( \liminf_{n \to \infty} d(x_n, F(T)) = 0 \). From Lemma 4.2.1 (a), we have
\[
\|x_{n+1} - p\| \leq \|x_n - p\| + (m + 1) \sum_{i=n}^{n+r-1} u_i \quad \text{for all } n \in \mathbb{N},
\]
which implies
\[
d(x_{n+r}, F(T)) \leq d(x_r, F(T)) + (m + 1)u_n, \quad \text{for all } n, r \in \mathbb{N}.
\]
Hence,
\[
\limsup_{r \to \infty} d(x_n, F(T)) \leq M_1 \liminf_{r \to \infty} d(x_r, F(T)).
\]
Hence, we have \( \lim d(x_n, F(T)) \) exists. Since \( \liminf_{n \to \infty} d(x_n, F(T)) = 0 \), it follows that
\[
\lim_{n \to \infty} d(x_n, F(T)) = 0.
\]
For \( \varepsilon > 0 \) there exists a natural number \( n_0 \) such that
\[
d(x_n, F(T)) < \frac{\varepsilon}{3M_1}, \quad \text{for all } n \geq n_0,
\]
Then for given \( \varepsilon > 0 \), there exists a natural number \( n_0 \) such that
\[
d(x_n, F(T)) < \varepsilon / 2 \quad \text{for all } n \geq n_0,
\]
Since for all \( n, m \geq n_0 \)
\[
\|x_n - x_m\| \leq \|x_n - p\| + \|x_m - p\|< \varepsilon / 2 + \varepsilon / 2 = \varepsilon, \quad p \in F(T).
\]
Hence \( \{x_n\} \) is a Cauchy sequence. Let \( x_n \to z \in D \).

Since \( \lim_{n \to \infty} x_n = z \), \( F(T) \) is closed and \( \lim_{n \to \infty} d(x_n, F(T)) = 0 \), then we concluded that \( \{x_n\} \) converges strongly to fixed point of \( T \).

**REMARK 4.2.1.** Theorem extends Theorem 4.2.1 of Sahu et al. [93] for generalized Ishikawa iteration process.

It is important to note that an iterative sequence need not be approximate fixed point sequence. Interestingly, Sharma and Sahu [101] proved that the iterative sequence \( \{x_n\} \) defined by (4.2.1) is an approximate fixed point sequence for asymptotically nonexpansive mappings in a uniformly convex Banach space.

In next result, we shall consider our iterative sequence \( \{x_n\} \) is an approximate fixed point sequence and establish another convergence theorem for nearly nonexpansive mappings.

**THEOREM 4.4.2.** Let \( D \) be a closed convex subset of a Banach space \( X \) and \( T: D \to D \) be a nearly nonexpansive mapping with sequence \( \{a_n\} \) such that

\[ \sum_{n=1}^{\infty} a_n < \infty \]  
and \( F(T) \neq \emptyset \). Let for fixed \( m \in \mathbb{N} \), \( \{\alpha_n\} \), \( \{\beta_n^{(1)}\} \), \( \{\beta_n^{(2)}\} \), ..., \( \{\beta_n^{(m-1)}\} \) and \( \{\beta_n^{(m)}\} \) be the sequences of real numbers in \([0,1]\). For \( x_1 \in D \), from the iterative sequence of generalised Ishikawa iteration process (4.2.1), \( T \) satisfies the following:

(i) \( \{x_n\} \) is an approximate fixed point sequence, i.e.,

\[ \lim_{n \to \infty} \|x_n - Tx_n\| = 0. \]

(ii) For each approximate fixed point sequence \( \{x_n\} \)

\[ \liminf_{n \to \infty} d(x_n, F(T)) = 0. \]
Then \( \{x_n\} \) converges strongly to fixed point of \( T \).

Now, we derive another strong convergence theorem of generalized Ishikawa process for asymptotically nonexpansive mapping in general Banach space.

**Theorem 4.2.3.** Let \( D \) be a closed convex subset of a Banach space \( X \) and \( T : D \rightarrow D \) be asymptotically nonexpansive mapping with sequence \( \{k_n\} \) such that \( F(T) \neq \emptyset \) and \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \). Let for fixed \( m \in \mathbb{N} \), \( \{\alpha_n\} \), \( \{\beta_n^{(1)}\} \), \( \{\beta_n^{(2)}\} \), ..., \( \{\beta_n^{(m-1)}\} \) and \( \{\beta_n^{(m)}\} \) be sequences of real numbers in \([0,1]\). For \( x_1 \in D \), the iterative sequence of generalised Ishikawa iteration process defined by (4.2.1). Let \( \{x_n\} \) be an approximate fixed point sequence and let there exists a constant \( c > 0 \) such that

\[
\|x_n - Tx_n\| \geq c d(x_n, F(T)), \quad \forall \ n \in \mathbb{N}.
\]

Then \( \{x_n\} \) converges strongly to a fixed point of \( T \).

**Corollary 4.2.1.** Let \( D \) be a closed convex subset of a Banach space \( X \) and \( T : D \rightarrow D \) be an asymptotically nonexpansive mapping with sequence \( \{k_n\} \) such that \( F(T) \neq \emptyset \) and \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \). Let for fixed \( m \in \mathbb{N} \), \( \{\alpha_n\} \), \( \{\beta_n^{(1)}\} \), \( \{\beta_n^{(2)}\} \), ..., \( \{\beta_n^{(m-1)}\} \) and \( \{\beta_n^{(m)}\} \) be sequences of real numbers in \([0,1]\). For \( x_1 \in D \), the iterative sequence of generalised Ishikawa iteration process defined by (4.2.1). Then \( \{x_n\} \) converges strongly to a fixed point of \( T \) if and only if

\[
\liminf_{n \to \infty} d(x_n, F(T)) = 0.
\]
PROOF. Since \( F(T) \neq \emptyset \), therefore \( T \) is an asymptotically quasi-nonexpansive mapping. Hence the result follows from **Theorem 4.2.1**.

**COROLLARY 4.4.2.** Let \( D \) be a nonempty closed convex subset of a Banach space \( X \) and \( T:D \to D \) be an asymptotically nonexpansive mapping with sequence \( \{k_n\} \) such that \( F(T) \neq \emptyset \) and \( \sum_{n=1}^{\infty} (k_n - 1) < \infty \). Let for \( m \in \mathbb{N} \), \( \{a_n\}, \{a_n^{(1)}\}, \{a_n^{(2)}\}, \ldots, \{a_n^{(m-1)}\} \) and \( \{\beta_n^{(m)}\} \) be nonnegative constants such that \( 0 < a < 1, \ 0 \leq \beta^{(i)}, \) for all \( i = 1, 2, \ldots, m \). For \( x \in D \), define the sequence of iterative process such that

\[
x_{n+1} = T_{\alpha,\beta,m,n} x = T_{\alpha,\beta,m,n} x_n \quad \forall \ n \in \mathbb{N}
\]

where

\[
T_{\alpha,\beta,m,n} x = (1 - \alpha) x + \alpha T^n U_{n,0} x,
\]

\[
U_{n,0} x = (1 - \beta^{(1)}) x + \beta^{(1)} T^n U_{n,1} x,
\]

\[
U_{n,1} x = (1 - \beta^{(2)}) x + \beta^{(2)} T^n U_{n,2} x,
\]

\[
U_{n,r-1} x = (1 - \beta^{(r)}) x + \beta^{(r)} T^n U_{n,r} x,
\]

\[
U_{n,m-1} x = (1 - \beta^{(m)}) x + \beta^{(m)} T^n x \quad \forall \ n \in \mathbb{N}.
\]
Then, \( \{x_n\} \) converges strongly to fixed point of \( T \) if and only if
\[
\liminf_{n \to \infty} d(x_n, F(T)) = 0.
\]

**PROOF.** It follows from **Theorem 4.2.1.**

**REMARK 4.2.2.** If we take \( m = 1 \) in **Theorem 4.2.1**, then we get Theorem 1 of Qihou [87]. So, **Theorem 4.2.1** is a good improvement of results of Ghosh and Debnath [32] and Petryshyn and Williamson [81].

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