CHAPTER-3

SOME EXISTENCE RESULTS WITH
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In section-1 of this chapter, an existence result is proved in $L^p$ space and in section-2 its random version is studied under certain contractive condition with asymptotic regularity conditions.

SECTION-1

In this section an existence result is proved in $L^p$-space for asymptotically regular mapping $T$ satisfying:

\[ \| T^n x - T^n y \|^2 \leq a_n \| x - y \|^2 + b_n (\| x - T^n x \| \cdot \| y - T^n y \|) \]

\[ + c_n (\| x - T^n y \| \cdot \| y - T^n x \|) \]

where $a_n, b_n$ and $c_n$ are nonnegative constants with some conditions.

Let us recall from chapter 2, following definition:

Let $C$ be a nonempty (and, generally, bounded closed convex) subset of $E$. A mapping $T: C \to C$ is called Lipschitzian if there exists a positive number $k_n$ such that

\[ \| T^n x - T^n y \| \leq k_n \| x - y \| \tag{3.1.1} \]

for all $x, y$ in $C$ and $n \in \mathbb{N}$.

$T$ is uniformly $k$-Lipschitzian if $k_n = k \ \forall \ n \geq 1$, nonexpansive if $k_n = 1$, for all $n \geq 1$ and asymptotically nonexpansive [35] if $k_n \geq 1$, for all $n \geq 1$.

\[ \lim_{n \to \infty} k_n = 1. \]
We now introduce a type of inequality for the more general class of mappings whose $n^{th}$ iterate $T^n$ satisfy:

$$
\| T^n x - T^n y \|^2 \leq a_n \| x - y \|^2 + b_n (\| x - T^n x \| \cdot \| y - T^n y \|) + c_n (\| x - T^n y \| \cdot \| y - T^n x \|),
$$

for each $x, y \in K$ and integer $n \geq 1$ where $a_n, b_n$ and $c_n$ are nonnegative constants satisfying certain conditions. Such mappings are more general than nonexpansive mappings. Also by taking $b_n = c_n = 0$, it will be seen that the above class of mappings are more general than asymptotically nonexpansive mappings.

In chapter 1, we have mentioned that the concept of asymptotic regularity is due to Browder and Petryshyn [13]. Let us recall the definition:

A mapping $T:E \to E$ into itself is said to be asymptotically regular if

$$
\lim_{n \to \infty} \| T^{n+1} x - T^n x \| = 0
$$

for all $x$ in $E$.

It is well known that if $T$ is nonexpansive then $T_t = tI + (1-t)T$ is asymptotically regular for all $0 < t < 1$.

We have already mentioned that the first existence results on fixed point of a general type for nonlinear nonexpansive mapping in noncompact setting were those which obtained independently by Browder [12] and Gohde [37] in uniformly convex Banach spaces. Kirk [57] obtained the same result under the slightly weaker assumptions that the underlying space be reflexive with normal structure. It was observed that if one assume $T$ to be a Lipschitzian mapping with a Lipschitz constant $k > 1$, then we do not to have a fixed point, even in Hilbert space as $k$ is arbitrary close to 1. However, Kruppel [63] proved the following result:
THEOREM K: Let $E$ be a uniformly convex Banach space and $K$ be a closed convex and bounded subset in $E$, and let $T$ be mapping from $K$ into itself. If $T$ is asymptotically regular and $\lim_{n \to \infty} \| T^n \| \leq 1$ (where $\| T \|$ denotes the Lipschitzian norm of $T$), then $T$ has a fixed point in $K$.

Further, in [38], Gornicki proved the following result:

THEOREM G. Let $X$ be a $L^p$-space ($1 < p \leq 2$) and $C$ be a nonempty closed convex and bounded subset of $X$, and let $T$ be a mapping from $C$ into itself. If $T$ is asymptotically regular and $\lim_{n \to \infty} \| T^n \| < k_p$ (where $k_p > 1$ and $\| T \|$ denotes the Lipschitzian norm of $T$), then $T$ has a fixed point in $C$.

Now, we prove our existence result with inequality (3.1.2). Our result includes the Theorem G and K both.

Before we proceed further, we need some preliminaries:

A Banach space $E$ has uniformly normal structure [33] if

$$N(E) = \sup \{ r_K(K) : K \subset E \text{ is convex and diam } K = 1 \} < 1,$$

where $r_K(K) = \inf \{ \sup \{ \| x - y \| : y \in K \} : x \in K \}$.

It was proved in [15], [24] that $N(E) \leq 1 - \delta_E(1)$ thus $e_0(E) < 1$ implies uniformly normal structure, where $\delta_E(.)$ is the modulus of convexity of $E$ and $e_0(E)$ is the characteristic of convexity of $E$.

Yu [119] proved that if $E$ is a uniformly smooth space, then $E$ has a uniformly normal structure. Also, it was proved [118] that uniformly normal structure does not necessarily imply that the space has good geometric properties.
For \( p > 1 \), the functional \( \| \cdot \|^p \) is said to be uniformly convex [12] on the Banach space \( E \), if there exists a positive constant \( c_p \) such that for all \( \lambda \in [0,1] \) and \( x, y \in E \) the following inequality holds:

\[
\| \lambda x + (1-\lambda)y \|^p \leq \| x \|^p + (1-\lambda) \| y \|^p - W_p(\lambda)c_p \| x - y \|^p,
\]

where \( W_p(\lambda) = \lambda (1-\lambda)^p + \lambda^p (1-\lambda) \).

In a Hilbert space \( H \) following equality holds:

\[
\| \lambda x + (1-\lambda)y \|^2 = \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda (1-\lambda) \| x - y \|^2
\]

for all \( x, y \in H \) and \( \lambda \in [0,1] \).

If \( 1 < p \leq 2 \), then we have for all \( x, y \in L^p \) and \( \lambda \in [0,1] \),

\[
(*) \quad \| \lambda x + (1-\lambda)y \|^2 \leq \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda (1-\lambda)(p-1) \| x - y \|^2
\]

We also need following lemma to prove our main result.

**Lemma 3.1.1** [63]. Let \( K \) be a nonempty closed convex subset of a Banach space \( E \) and let \( \{n_i\} \) be an increasing sequence of natural numbers. Assume that \( T:K \to K \) is an asymptotically regular mapping such that for some \( m \in \mathbb{N}, \ T^m \) is continuous. If

\[
\lim_{i \to \infty} \| z - T^{n_i}x \| = 0
\]

for some \( x \in K \) and \( z \in K \). Then \( Tz = z \).

Now, we state and prove our main results of this section:
THEOREM 3.1.1. Let E be a $l^p$-space ($1 < p \leq 2$), C a nonempty closed convex and bounded subset of E. If $T: C \to C$ an asymptotically regular mapping satisfying (3.1.2), where $a_n, b_n$ and $c_n$ are nonnegative constants. If \( \lim\inf_{n \to \infty} b_n \) exists and
\[
\left\{ \frac{\gamma + \sqrt{\gamma^2 + 4(p-1)(1+\gamma - 1)}(\alpha + 2\gamma)^{1/2}}{2(p-1)N} \right\} < 1
\]
where
\[
\alpha = \lim\inf_{n \to \infty} a_n
\]
\[
\gamma = \lim\inf_{n \to \infty} c_n,
\]
and N is the normal structure coefficient of E, then T has a fixed point in C.

PROOF. Let \( \{n_i\} \) be a sequence of natural numbers such that
\[
\left\{ \frac{\gamma + \sqrt{\gamma^2 + 4(p-1)(1+\gamma - 1)}(\alpha + 2\gamma)^{1/2}}{2(p-1)N} \right\} < 1
\]
where
\[
\alpha = \lim\inf_{n \to \infty} a_n = \lim_{i \to \infty} a_{n_i}
\]
\[
\gamma = \lim\inf_{n \to \infty} c_n = \lim_{i \to \infty} c_{n_i}
\]
Given an element $z_0 \in C$ and by Lemma 3.1.1, we can inductively construct a sequence \( \{z_m\} \) such that $z_m$ is the unique asymptotic centre of the sequence \( \{T^{n_i}z_{m-1}\}_{i \geq 1} \) with respect to the functional:
\[
\limsup_{i \to \infty} \|x - T^{n_i}z_{m-1}\|_p,
\]
over $x$ in C. Now for each $m \geq 1$, we set
Using the condition of asymptotic regularity of $T$, we have
\[ r_m \leq \frac{1}{N} \cdot (\alpha D_m^2 + 2\gamma D_m^2)^{1/2} \leq \frac{1}{N} (\alpha + 2\gamma)^{1/2} D_m, m = 0, 1, 2, \ldots \]  \hspace{1cm} (3.1.3)

Now for each fixed \( m \geq 1 \) and all \( n, n_j \) we have from (*),

\[
\| \lambda z_{m+1} + (1 - \lambda)T^{n_j}z_{m+1} - T^{n_i}z_m \|^2 + \lambda (1 - \lambda)(p - 1) \| z_{m+1} - T^{n_j}z_{m+1} \|^2 \\
\leq \lambda \| z_{m+1} - T^{n_i}z_m \|^2 + (1 - \lambda) \| T^{n_j}z_{m+1} - T^{n_j}z_m \|^2 \\
\leq \lambda \| z_{m+1} - T^{n_i}z_m \|^2 + (1 - \lambda) \| T^{n_j}z_{m+1} - T^{n_i+n_j}z_m \| \\
\quad + \| T^{n_i+n_j}z_m - T^{n_i}z_m \| \|^2 \\
\leq \lambda \| z_{m+1} - T^{n_i}z_m \|^2 + (1 - \lambda) \| T^{n_j}z_{m+1} - T^{n_i+n_j}z_m \| \\
\quad + \sum_{\ell=0}^{n_j-1} \| T^{n_i+\ell}z_m - T^{n_i+\ell}z_m \| \|^2 \\
\leq \lambda \| z_{m+1} - T^{n_i}z_m \|^2 + (1 - \lambda) \| T^{n_j}z_{m+1} - T^{n_i+n_j}z_m \| \\
\quad + b_{n_j} \| z_{m+1} - T^{n_j}z_{m+1} \| \| T^{n_i}z_m - T^{n_i+n_j}z_m \| \\
\quad + c_{n_j} \| T^{n_i+n_j}z_m \| \| T^{n_j}z_m - T^{n_i+n_j}z_m \| \| T^{n_i}z_m - T^{n_i+n_j}z_m \| \\
\quad + \sum_{\ell=0}^{n_j-1} \| T^{n_i+\ell}z_m - T^{n_i+\ell}z_m \| \|^2 \\
\leq \lambda \| z_{m+1} - T^{n_i}z_m \|^2 + (1 - \lambda) \| T^{n_j}z_{m+1} - T^{n_i+n_j}z_m \| \\
\quad + b_{n_j} \| z_{m+1} - T^{n_j}z_{m+1} \| \| T^{n_i+\ell}z_m - T^{n_i+\ell}z_m \| \\
\quad + c_{n_j} \| z_{m+1} - T^{n_i}z_m \| \| T^{n_i+\ell}z_m - T^{n_i+n_j}z_m \| \\
\quad \| T^{n_i}z_m - T^{n_i+n_j}z_m \| \\
\quad + \sum_{\ell=0}^{n_j-1} \| T^{n_i+\ell}z_m - T^{n_i+\ell}z_m \| \|^2.\]
Taking the limit superior as $i \to \infty$ on each side, by definition of $z_m$ and by the asymptotic regularity of $T$, we have

$$r_m^2 + \lambda (1 - \lambda) (p - 1) \| z_{m+1} - T^n z_{m+1} \|^2 \leq \lambda r_m^2 + (1 - \lambda) \| z_{m+1} - T^n z_{m+1} \| \| z_{m+1} - T^n z_{m+1} \|^{1/2}.$$  

Taking the limit superior as $j \to \infty$, we get

$$r_m^2 + \lambda (1 - \lambda) (p - 1) D_{m+1}^2 \leq \lambda r_m^2 + (1 - \lambda) (\alpha r_m^2 + \gamma r_m (r_m + D_{m+1}))^{1/2}$$  

implies

$$r_m^2 + \lambda (1 - \lambda) (p - 1) D_{m+1}^2 \leq \alpha r_m^2 + \gamma r_m (r_m + D_{m+1}).$$

Letting $\lambda \to 1$, we get

$$r_m^2 + (p - 1) D_{m+1}^2 \leq \alpha r_m^2 + \gamma r_m (r_m + D_{m+1}),$$

or

$$r_m^2 + (p - 1) D_{m+1}^2 \leq (\alpha + \gamma) r_m^2 + \gamma r_m D_{m+1},$$

or

$$\sqrt{r_m^2 + (p - 1) D_{m+1}^2} \leq \sqrt{\alpha r_m^2 + \gamma r_m (r_m + D_{m+1})},$$
\[(p-1)D_{m+1}^2 - \gamma r_m D_{m+1} - ((\alpha + \gamma) - 1)r_m^2 \leq 0,\]
or
\[F(t) = (p-1)t^2 - \gamma r_m t - ((\alpha + \gamma) - 1)r_m^2 \leq 0,\]
where \(t = D_{m+1}\).

It can be easily seen that
\[F(t) \leq 0 \text{ for all } t = \frac{\sqrt{\gamma^2 + 4(p-1)((\alpha + \gamma) - 1)}}{2(p-1)} r_m.\]

It follows from (3.1.3) that
\[D_{m+1} \leq \frac{\gamma + \sqrt{\gamma^2 + 4(p-1)((\alpha + \gamma) - 1)}}{2(p-1)}(\alpha + 2\gamma)^{1/2} D_m.\]

Hence,
\[D_{m+1} \leq AD_m, \quad m = 1, 2, \ldots,\]
where
\[A = \left[\frac{\gamma + \sqrt{\gamma^2 + 4(p-1)((\alpha + \gamma) - 1)}}{2(p-1)}(\alpha + 2\gamma)^{1/2}\right] \leq 1,
\]
by assumption of the theorem.

Since
\[\|z_{m+1} - z_m\| \leq \|z_{m+1} - T^{n_i}z_m\| + \|T^{n_i}z_m - z_m\| \leq r_m + D_m \leq 2D_m \leq \ldots \leq 2A^mD_0 \to 0\]
as \(m \to \infty\), it follows that \(\{z_m\}\) is a Cauchy sequence. Let
\[z = \lim_{m \to \infty} z_m.\]

Then, we have
\[\|z - T^{n_i}z\| \leq \|z - z_m\| + \|z - T^{n_i}z_m\| + \|T^{n_i}z_m - z\| \leq \|z - z_m\| + \|z_m - T^{n_i}z_m\| + \|a_{n_i}\|z_m - z\|^2 + b_{n_i}\|z_m - T^{n_i}z\| + c_{n_i}\|z_m - T^{n_i}z_m\|^2 \cdot \|z - T^{n_i}z\|\}^{1/2}\]
\begin{align*}
\leq & \|z - z_m\| + \|z_m - T^{n_i}z_m\| + \{a_{n_i}\|z_m - z\|^2 \\
+ & b_{n_i}(\|z_m - T^{n_i}z_m\|, \|z - T^{n_i}z\|) \\
+ & c_{n_i}(\|z_m - z\| + \|z - T^{n_i}z\|, (\|z - z_m\| \\
+ & \|z_m - T^{n_i}z_m\|)\})^{1/2}.
\end{align*}

Set \( p = \limsup_{n \to \infty} \|z - T^{n_i}z\| \).

Taking the limit superior as \( i \to \infty \) on each sides or above inequality, we have
\[
\mu = \limsup_{i \to \infty} \|z - T^{n_i}z\| \leq D_m + (\beta D_m + \gamma D_m^2)^{1/2} \\
\leq D_m + p^{1/2} (\beta + \gamma)^{1/2} D_m^{1/2} \to 0
\]
as \( m \to \infty \). Therefore \( Tz = z \) by Lemma 3.1.1. This completes the proof.

If we put \( b_n = 0, c_n = 0 \) in inequality (3.1.2) of Theorem 3.1.1, we get the following result.

**COROLLARY 3.1.1 [63].** Let \( C \) be a nonempty bounded closed convex subset of a Hilbert space \( H \). If \( T:C \to C \) is an asymptotically regular mapping such that
\[
\liminf_{n \to \infty} \|T^n\| < \sqrt{2},
\]
then \( T \) has a fixed point in \( C \).

**COROLLARY 3.1.2 [41, Corollary 3].** Let \( C \) be a nonempty bounded closed convex subset of \( L^p(1 < p \leq 2) \). If \( T:C \to C \) is an asymptotically regular mapping such that
\[
\]
THEOREM 3.2.1 [GSS4]. Let $C$ be a nonempty closed convex subset of a Banach space $E$ which has uniformly normal structure, i.e., $N(E) < i$. Let $T: C \rightarrow C$ be an asymptotically regular mapping which holds the inequality (3.1.2) such that

$$(\alpha + 2\gamma)^{1/2} A^+ N(E) < 1,$$

where \( \alpha = \liminf_{n \to \infty} a_n, \gamma = \liminf_{n \to \infty} c_n, \quad A^+ = \frac{\gamma^2 + 4\gamma + 4\alpha}{2} \) and \( \liminf_{n \to \infty} b_n \) exists. \( \{T^n z\} \) is bounded for some $z \in C$, then $T$ has a fixed point in $C$.

PROOF. Follows from Theorem 3.1.1.

SECTION-2

One current important and interesting aspect of nonlinear functional analysis is to randomize deterministic fixed point theorems of nonlinear mappings. Random fixed point theorems are stochastic generalization of classical fixed point theorems, and are required for the theory of random equations. Research of this direction was initiated by the Prague School of probabilistic in connection with random operator theory. In recent years random fixed point theorem have received much attention (see e.g. [2], [7-9],

\[ G.S.Saluja and D.R.Sahu : \text{An existence result for asymptotically regular mappings, Pure and Appl.Math.Sci. (Accepted).} \]
In this section we prove random fixed point theorem for the following class of mappings:

Let $(\Omega, \Sigma)$ be a measurable space and $K$ a nonempty bounded closed convex separable subset of a $1^p$-space ($1 < p \leq 2$) $F$. $T: \Omega \times K \rightarrow K$ satisfying the condition:

For each $x, y \in K$, $w \in \Omega$ and integer $n \geq 1$,

$$\|T^n(\omega, x) - T^n(\omega, y)\|^2 \leq a_n(\omega) \|x - y\|^2 + b_n(\omega) (\|x - T^n(\omega, x)\| + \|y - T^n(\omega, y)\|)$$

$$+ c_n(\omega) (\|x - T^n(\omega, y)\| + \|y - T^n(\omega, x)\|)$$

(3.2.1)

where $a_n, b_n, c_n : \Omega \rightarrow [0, \infty)$ are functions satisfying certain conditions and $T^n(\omega, x)$ is the value at $x$ of the $n$th iterate of the mapping $T(\omega, \cdot)$.

Further, we establish random fixed point theorem for these mappings.

As a consequence of our main result, we extend and randomize the corresponding deterministic ones of Gornicki ([38], [41]), Xu [116] and others.

Before proceeding further we give some preliminaries:

Let $(\Omega, \Sigma)$ be a measurable space with $\Sigma$ a $\sigma$-algebra of subsets of $\Omega$. Let $(X, d)$ be a metric space. We denote by $CL(X)$ (resp. $CB(X)$, $K(X)$) the family of all nonempty closed (resp. closed bounded, compact) subsets of $X$ and by $H$ the Hausdorff metric on $CB(X)$ induced by $d$ i.e.

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$

for $A, B \in CB(X)$, where $d(x, B) = \inf \{d(x, y) : y \in B\}$.
is the distance from a point \( x \in X \) to a subset \( B \subset X \). A multifunction \( f: \Omega \to X \) is called \((\Sigma-)\) measurable if, for any open subset \( B \) of \( X \), the set \( f^{-1}(B) = \{ \omega \in \Omega : f(\omega) \cap B \neq \emptyset \} \subset \Omega \). Note that in Himmelberg [48], such multifunction is called weakly measurable. In the present paper, since only this type of multifunction is used, we omit the term 'weakly' for simplicity. Note also that if \( f(\omega) \in K(X) \) for all \( \omega \in \Omega \), then \( f \) is measurable if and only if \( f^{-1}(F) \in \Sigma \) for all closed subsets \( F \) of \( X \). A measurable operator \( x: \Omega \to X \) is called a measurable selector of a measurable multifunction \( f: \Omega \to X \) if \( x(\omega) \in f(\omega) \). Let \( M \) be a nonempty closed subset of \( X \). Then a mapping \( f: \Omega \times M \to X \) is called a random operator if for each \( x \in M \), the mapping \( f(., x): \Omega \to X \) is measurable. An operator \( x: \Omega \to X \) is said to be a random fixed point of \( f \) if \( x \) is measurable and \( x(\omega) \in f(\omega, (x(\omega))) \) for all \( \omega \in \Omega \).

A random mapping \( f: \Omega \times K \to K \) is said to be more general than nonexpansive if, for a fixed \( \omega \in \Omega \), the mapping \( f(\omega, .): K \to K \) has the above Property \((3.2.1)\). A random mapping \( f: \Omega \times K \to K \) is said to be asymptotically \((\omega, .)\) regular if

\[
\lim_{n \to \infty} \| T^{n+1}(\omega, x) - T^n(\omega, x) \| = 0
\]

for each \( x \in K \) and \( \omega \in \Omega \).

The following theorem is needed for our main result:

**Theorem A** [117]. Let \((\Omega, \Sigma)\) be a measurable space, \( X \) a separable metric space and \( Y \) a metric space. If \( f: \Omega \times X \to Y \) is measurable in \( \omega \in \Omega \) and continuous in \( x \in X \), respectively, and if \( x: \Omega \to X \) is measurable, then \( f(., x(.)): \Omega \to Y \) is measurable.

The normal structure coefficient \( N(X) \) of \( X \) is defined (Bynum [15]) by
\[ N(X) = \inf \left\{ \frac{\text{diam} K}{r_K(K)} \right\} \]

where the infimum is taken over all bounded convex subsets \( K \) of \( X \) consisting of more than one point, \( \text{diam}(K) = \sup\{||x-y|| : x, y \in K\} \) is the diameter of \( K \) and \( r_K(K) = \inf_{x \in K} \left\{ \sup_{y \in K} ||x-y|| \right\} \) is the Chebyshev radius of \( K \) relative to itself. A space \( X \) is said to have uniformly normal structure if \( N(X) > 1 \). It is known that every uniformly convex Banach space has uniformly normal structure (Danès [24]) and \( N(H) = \sqrt{2} \) for all Hilbert space \( H \). Recently, Pichugorov [82] (Prus [83]) calculated that \( N(L^p) = \min\{ \frac{1}{1+p}, \frac{1}{p-1}\} \), \( 1 < p < \infty \). Some estimates for normal structure coefficient in other Banach spaces may be found in Prus [84].

Now recall that the modulus of convexity of Banach space \( X \) is the function \( \delta(.) \) defined on \([0,2] \) by
\[
\delta(\varepsilon) = \inf \left\{ 1 - \frac{1}{2} ||x+y|| : ||x|| \leq 1, ||y|| \leq 1, ||x-y|| \geq \varepsilon \right\}
\]

\( X \) is said to be uniformly convex if \( \delta(\varepsilon) > 0 \) for all \( 0 < \varepsilon \leq 2 \).

Let \( p > 1 \) and denote by \( \lambda \) the number in \([0,1] \) and by \( W_p(\lambda) \) the function
\[
\lambda(1-\lambda)^p + \lambda^p (1-\lambda).
\]

The functional \( ||.||^p \) is said to be uniformly convex (Zalinescu [121]) on the Banach space \( X \) if there exists a positive constant \( c_p \) such that for all \( \lambda \in [0,1] \) and \( x, y \in X \),
\[
||\lambda x + (1-\lambda)y||^p \leq \lambda||x||^p + (1-\lambda)||y||^p - W_p(\lambda) c_p ||x-y||^p
\]
In a Hilbert space $H$, this equality holds:

$$\|\lambda x + (1 - \lambda)y\|^2 - \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 \leq \lambda(1 - \lambda)(\|x - y\|^2),$$  \hspace{1cm} (3.2.2)

If $1 < p \leq 2$, then we have for all $x, y$ in $L^p$ and $\lambda \in [0, 1]$,

$$\|\lambda x + (1 - \lambda)y\|^2 - \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 \leq \lambda(1 - \lambda)(p - 1)(\|x - y\|^2)$$  \hspace{1cm} (3.2.3)

In this section, we always assume that $K$ is a nonempty bounded closed convex subset of a Banach space $X$ and $T: \Omega \times K \to K$ is a class of random mapping satisfying the condition (3.2.1) where $a_n, b_n, c_n : \Omega \to [0, \infty)$ are functions such that there exists an integer $n_0$ satisfying $a_n(\omega) + 2c_n(\omega) \geq 1$ for all $n \geq n_0$ and $T^n(\omega, x)$ is the value at $x$ of the $n$-th iterate of the mapping $T(\omega, \cdot)$.

To prove our main result, we need the following lemma:

**LEMMA 3.2.1** [116]. Suppose that $(\Omega, \Sigma)$ be a measurable space with $\Sigma$ a $\sigma$-algebra of subsets of $\Omega$. Let $X$ be a Banach space with uniformly normal structure, $K$ a nonempty bounded closed convex separable subset of $X$ and that $\{x_n(\omega)\}$ is a sequence of measurable function $x_n : \Omega \to K$. Then there exists a measurable function $\zeta : \Omega \to K$ satisfying:

(a) $\|\zeta(\omega) - z\| \leq \limsup \limits_{n \to \infty} \|x_n(\omega) - z\|$ for all $z \in X$ and $\omega \in \Omega$;

(b) $\limsup \limits_{n \to \infty} \|x_n(\omega) - \zeta(\omega)\| \leq \frac{1}{N(X)} \limsup \limits_{n \to \infty} \{\|x_i(\omega) - x_j(\omega)\| : i, j \geq n\}$ for all $\omega \in \Omega$.

Now, we are in a position to give our main result:
**THEOREM 3.2.1 [GSS3]**. Let \((\Omega, \Sigma)\) be a measurable space and \(K\) a nonempty bounded closed convex separable subset of \(L^p\)-space \((1 \leq p < 2)\) \(X\). Suppose \(T: \Omega \times K \to K\) is a continuous asymptotically \((\omega, \cdot)\) regular mapping satisfying the condition (3.2.1). If \(\lim\inf b_n\) exists and

\[
\left[ \gamma(\omega) + \sqrt{\gamma^2(\omega) + 4(p-1)((\alpha(\omega) + \gamma(\omega)) - 1)} \right] \frac{2\gamma(\omega)}{2(p-1)} < 1,
\]

where \(\alpha(\omega) = \liminf_{n \to \infty} a_n(\omega)\), \(\gamma(\omega) = \liminf_{n \to \infty} c_n(\omega)\) and \(N(X)\) is the normal structure coefficient of \(X\).

Then \(T\) has a random fixed point.

**PROOF.** Let \(\{n_i\}\) be a sequence of natural numbers such that

\[
\left[ \gamma(\omega) + \sqrt{\gamma^2(\omega) + 4(p-1)((\alpha(\omega) + \gamma(\omega)) - 1)} \right] \frac{2\gamma(\omega)}{2(p-1)} < 1,
\]

where

\[
\alpha(\omega) = \liminf_{n \to \infty} a_n(\omega) = \lim_{i \to \infty} a_{n_i}(\omega),
\]

\[
\gamma(\omega) = \liminf_{n \to \infty} c_n(\omega) = \lim_{i \to \infty} c_{n_i}(\omega).
\]

Choose an arbitrary \(x_0 \in K\) and let \(x_0(\omega) = x_0\).

This \(x_0\) is obviously measurable. In view of Lemma 3.2.1 and Theorem A, we can inductively construct a sequence \(\{x_{m_n}(\omega)\}\) of measurable functions.

---

\[ x_m : \Omega \times K \rightarrow K \] such that, for each \( \omega \in \Omega \) and \( m \geq 0 \), \( x_{m+1}(\omega) \) is the asymptotic centre of sequence \( \{T^n(\omega, x_m(\omega))\} \) in \( K \) i.e.

\[
\limsup_{n \rightarrow \infty} \| T^n(\omega, x_m(\omega)) \| \geq \inf_{y \in K} \limsup_{n \rightarrow \infty} \| T^n(\omega, x_m(\omega)) - y \|.
\]

Set, for each \( \omega \in \Omega \) and integer \( m \geq 0 \),

\[
r_m(\omega) = \limsup_{i \rightarrow \infty} \| x_{m+1}(\omega) - T^{n_i}(\omega, x_m(\omega)) \|;
\]

and

\[
D_m(\omega) = \limsup_{i \rightarrow \infty} \| x_m(\omega) - T^{n_i}(\omega, x_m(\omega)) \|.
\]

Since \( T \) satisfies condition (3.2.1), we have, for each \( \omega \in \Omega \) and \( x, y \in K \)

\[
\| T^S(\omega, x) - T^t(\omega, y) \| \leq \| T^S(\omega, x) - T^{t+s}(\omega, y) \| + \| T^{t+s}(\omega, y) - T^t(\omega, y) \| \leq \left[ a_1(\omega) \| x - T^t(\omega, y) \|^2 \right.
\]

\[
+ b_1(\omega) (\| x - T^S(\omega, x) \| \cdot \| T^t(\omega, y) - T^{t+s}(\omega, y) \|)
\]

\[
+ c_1(\omega) (\| x - T^{t+s}(\omega, y) \| \cdot \| T^t(\omega, y) - T^S(\omega, x) \|)^{1/2}
\]

\[
+ \| T^{t+s}(\omega, y) - T^t(\omega, y) \|
\]

\[
\leq \left[ a_1(\omega) \| x - T^t(\omega, y) \|^2 \right.
\]

\[
+ b_1(\omega) (\| x - T^S(\omega, x) \| \cdot \| T^t(\omega, y) - T^{t+s}(\omega, y) \|)
\]

\[
+ c_1(\omega) (\| x - T^{t+s}(\omega, y) \| + \| T^t(\omega, y) - T^{t+s}(\omega, y) \|)
\]

\[
\leq 3 \| T^t(\omega, y) - x \| + \| T^t(\omega, y) - x \|
\]

\[
(3.2.4)
\]

By Lemma 3.2.1, the inequality (3.2.4) and the asymptotic \((\omega, \cdot) - \)regularity of \( T \), we have
\[ r_m(\omega) \leq \lim_{N(X) \to \infty} \lim_{n \to \infty} \{ \sup \{ ||T^{n_i}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_m(\omega))|| : i, j > n \} \}
\]
\[ \leq \lim_{N(X) \to \infty} \lim_{n \to \infty} \left\{ \sup \{ ||T^{n_i}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_m(\omega))|| : i, j \geq n \} \right\} \]
\[ \leq \lim_{N(X) \to \infty} \limsup_{i \to \infty} \limsup_{j \to \infty} \left\{ \sup \{ ||T^{n_i}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_m(\omega))|| : i, j \geq n \} \right\} \]
\[ + b_{n_i}(\omega) \cdot (||x_m(\omega) - T^{n_i}(\omega, x_m(\omega))|| + ||T^{n_j}(\omega, x_m(\omega)) - T^{n_i+n_j}(\omega, x_m(\omega))||) \]
\[ + c_{n_i}(\omega) \cdot (||x_m(\omega) - T^{n_j}(\omega, x_m(\omega))|| + ||T^{n_i}(\omega, x_m(\omega)) - T^{n_i+n_j}(\omega, x_m(\omega))||) \]
\[ \cdot (||T^{n_j}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_m(\omega))|| + ||T^{n_i}(\omega, x_m(\omega)) - T^{n_i+n_j}(\omega, x_m(\omega))||) \]
\[ + c_{n_i}(\omega) \cdot (||x_m(\omega) - T^{n_i}(\omega, x_m(\omega))|| + \sum_{\ell=0}^{n_i-1} ||T^{n_j+\ell+1}(\omega, x_m(\omega)) - T^{n_j+\ell}(\omega, x_m(\omega))||) \]
\[ + c_{n_i}(\omega) \cdot (||x_m(\omega) - T^{n_j}(\omega, x_m(\omega))|| + \sum_{\ell=0}^{n_j-1} ||T^{n_j+\ell+1}(\omega, x_m(\omega)) - T^{n_j+\ell}(\omega, x_m(\omega))||) \]
\[ \cdot (||T^{n_j}(\omega, x_m(\omega)) - T^{n_j}(\omega, x_m(\omega))|| + ||T^{n_i}(\omega, x_m(\omega)) - T^{n_i+n_j}(\omega, x_m(\omega))||) \]
\[ + c_{n_i}(\omega) \cdot (||x_m(\omega) - T^{n_i}(\omega, x_m(\omega))|| + \sum_{\ell=0}^{n_i-1} ||T^{n_j+\ell+1}(\omega, x_m(\omega)) - T^{n_j+\ell}(\omega, x_m(\omega))||) \]
\[ + c_{n_i}(\omega) \cdot (||x_m(\omega) - T^{n_j}(\omega, x_m(\omega))|| + \sum_{\ell=0}^{n_j-1} ||T^{n_j+\ell+1}(\omega, x_m(\omega)) - T^{n_j+\ell}(\omega, x_m(\omega))||) \]

which implies that
and for each fixed \( m \geq 1 \) and \( n_i, n_j \), from (3.2.3), (3.2.4) and (3.2.5), it follows that

\[
\begin{align*}
\| \lambda x_{m+1}(\omega) + (1-\lambda)T_{n+1}^j(\omega, x_{m+1}(\omega)) - T_{n+1}^{n_i+1}(\omega, x_m(\omega)) \| &
\leq \lambda \| x_{m+1}(\omega) - T_{n+1}^{n_i}(\omega, x_m(\omega)) \|^2 + (1-\lambda) \| T_{n+1}^j(\omega, x_{m+1}(\omega)) - T_{n+1}^{n_i}(\omega, x_m(\omega)) \|^2 \\
+ \lambda(n_{n+1} + \lambda - 1) \| x_m(\omega) - T_{n+1}^{n_i}(\omega, x_m(\omega)) \|^2 + (1-\lambda) \| T_{n+1}^j(\omega, x_{m+1}(\omega)) - T_{n+1}^{n_i}(\omega, x_m(\omega)) \|^2 \\
\end{align*}
\]
\[ \lambda \| x_{m+1}(\omega) - T_n^{1j}(\omega, x_m(\omega)) \|^2 + (1 - \lambda) \{ (a_{n_j}(\omega), \| x_{m+1}(\omega) - T_n^{1j}(\omega, x_m(\omega)) \|^2 + (1 - \lambda) \} \sum_{\ell=0}^{n_j-1} \| T_n^{1j+\ell+1}(\omega, x_m(\omega)) \| \} \]

\[ + c_{n_j}(\omega) \| x_{m+1}(\omega) - T_n^{1j}(\omega, x_m(\omega)) \|^2 + \sum_{\ell=0}^{n_j-1} \| T_n^{1j+\ell+1}(\omega, x_m(\omega)) \| \} \]

Taking the limit superior as \( i \to \infty \) on each side, by the definition of \( x_m(\omega) \) and the asymptotic \((\omega, \cdot)\)-regularity of \( T \), we have

\[ r_m^2(\omega) + \lambda (1 - \lambda) (p - 1) \| x_{m+1}(\omega) - T_n^{1j}(\omega, x_m(\omega)) \|^2 \leq \lambda r_m^2(\omega) + (1 - \lambda) \{ (a_{n_j}(\omega), r_m^2(\omega) + c_{n_j}(\omega), r_m(\omega) \}

Taking the limit superior as \( j \to \infty \), we have

\[ r_m^2(\omega) + \lambda (1 - \lambda) (p - 1) D_m^2(\omega) \leq \lambda r_m^2(\omega) + (1 - \lambda) \{ \alpha(\omega), r_m^2(\omega) + \gamma(\omega), r_m(\omega) (r_m(\omega) + D_{m+1}(\omega)) \}

implies

\[ r_m^2(\omega) + \frac{\lambda (1 - \lambda)}{1 - \lambda} (p - 1) D_m^2(\omega) \leq \alpha(\omega) r_m^2(\omega) + \gamma(\omega) r_m(\omega) (r_m(\omega) + D_{m+1}(\omega)) \]

Letting \( \lambda \to 1 \), we get
\[ r_m^2(\omega) + (p-1)D_{m+1}^2(\omega) \]
\[ < \alpha(\omega)r_m^2(\omega) + \gamma(\omega)r_m^2(\omega) + \gamma(\omega)r_m(\omega)D_{m+1}(\omega) \]

Or
\[ (p-1)D_{m+1}^2(\omega) - \gamma(\omega)r_m(\omega)D_{m+1}(\omega) - \{\alpha(\omega) + \gamma(\omega) - 1\}r_m^2(\omega) \leq 0 \]

Or
\[ F(t) = (p-1)t^2 - \gamma(\omega)r_m(\omega)t - \{\alpha(\omega) + \gamma(\omega) - 1\}r_m^2(\omega) \leq 0, \]

where \( t = D_{m+1}(\omega) \)

It can be easily seen that
\[ F(t) \leq 0 \text{ for all } t = D_{m+1}(\omega) \]

It follows from (3.2.5) that
\[ D_{m+1}(\omega) \leq \frac{\gamma(\omega) + \sqrt{\gamma^2(\omega) + 4(p-1)(\alpha(\omega) + \gamma(\omega) - 1)}}{2(p-1)} \cdot r_m(\omega) \cdot N(X) \]

Hence,
\[ D_{m+1}(\omega) \leq A(\omega).D_m(\omega), \quad m = 1,2,... \]

where
\[ A(\omega) = \left[ \frac{\gamma(\omega) + \sqrt{\gamma^2(\omega) + 4(p-1)(\alpha(\omega) + \gamma(\omega) - 1)}}{2(p-1)} \cdot \frac{(\alpha(\omega) + 2\gamma(\omega))^{1/2}}{N(X)} \right] < 1, \]

by the assumption of the theorem. Since
\[ ||x_{m+1}(\omega) - x_m(\omega)|| \leq ||x_{m+1}(\omega) - T^1(\omega, x_m(\omega))|| + ||T^1(\omega, x_m(\omega)) - x_m(\omega)|| \]
\[ \leq r_m(\omega) + D_m(\omega) \]
as \( m \to \infty \), it follows that \( \{x_m(\omega)\} \) is a norm Cauchy sequence and hence convergent, whose limit is denoted by \( x(\omega) \). Thus we have

\[
\|x(\omega) - T_n^i(\omega, x(\omega))\| \leq \|x(\omega) - x_m(\omega)\| + \|x_m(\omega) - T_n^i(\omega, x_m(\omega))\|
\]

\[
+ \|T_n^i(\omega, x_m(\omega)) - T_n^i(\omega, x(\omega))\|
\]

\[
\leq \|x(\omega) - x_m(\omega)\| + \|x_m(\omega) - T_n^i(\omega, x_m(\omega))\|
\]

\[
+ \{a_n(\omega) \|x(\omega) - x_m(\omega)\|^2
\]

\[
+ b_n(\omega) \|x_m(\omega) - T_n^i(\omega, x_m(\omega))\| \|x(\omega) - T_n^i(\omega, x(\omega))\|
\]

\[
+ c_n(\omega) \|x_m(\omega) - x(\omega)\| \|x(\omega) - T_n^i(\omega, x(\omega))\|
\]

\[
\leq \|x(\omega) - x_m(\omega)\| + \|x_m(\omega) - T_n^i(\omega, x_m(\omega))\|
\]

\[
+ \{a_n(\omega) \|x(\omega) - x_m(\omega)\|^2
\]

\[
+ b_n(\omega) \|x_m(\omega) - T_n^i(\omega, x_m(\omega))\| \|x(\omega) - T_n^i(\omega, x(\omega))\|
\]

\[
+ c_n(\omega) \|x_m(\omega) - x(\omega)\| \|x(\omega) - T_n^i(\omega, x(\omega))\|
\]

\[
= \{\|x(\omega) - x_m(\omega)\| + \|x_m(\omega) - T_n^i(\omega, x_m(\omega))\|\}^{1/2}
\]

Set \( \limsup_{i \to \infty} \|z(\omega) - T_n^i(\omega, x(\omega))\| \).

Taking the limit superior as \( i \to \infty \) on each side, we have
\[
\limsup_{i \to \infty} \| x(\omega) - T^{n_i}(\omega, x(\omega)) \| \leq D_m(\omega) + \{ \beta(\omega) + \gamma(\omega) \} D_m(\omega)^{1/2}
\]

\[
\leq D_m(\omega)^{1/2} + \{ \beta(\omega) + \gamma(\omega) \} D_m(\omega)^{1/2} \to 0
\]
as \( m \to \infty \). Therefore \( T^{n_i}(\omega, x(\omega)) \to x(\omega) \) as \( i \to \infty \) and so

\[
\| T^{n_i} + m(\omega, x(\omega)) - x(\omega) \|
\]

\[
\leq \| T^{n_i} + m(\omega, x(\omega)) - T^{n_i}(\omega, x(\omega)) \| + \| T^{n_i}(\omega, x(\omega)) - x(\omega) \|
\]

\[
\leq \sum_{\ell = 0}^{m-1} \| T^{n_i} + \ell + 1(\omega, x(\omega)) - T^{n_i} + \ell(\omega, x(\omega)) \| + \| T^{n_i}(\omega, x(\omega)) - x(\omega) \|
\]

and, from asymptotic \((\omega, \cdot)\)-regularity of \( T \), \( T^{n_i} + m(\omega, x(\omega)) \to x(\omega) \) as \( i \to \infty \). Since \( T \) is continuous, we have

\[
T^m(\omega, x(\omega)) = T^m(\lim_{i \to \infty} T^{n_i}(\omega, x(\omega)) = \lim_{i \to \infty} T^{n_i} + m(\omega, x(\omega)) = x(\omega).
\]

It is easily verified (by induction) that

\[
T^{s+1}(\omega, x(\omega)) = x(\omega) \quad \text{for } s = 1, 2, \ldots.
\]

Then we have

\[
\| T(\omega, x(\omega)) - x(\omega) \| = \| T^{s+1}(\omega, x(\omega)) - T^{s}(\omega, x(\omega)) \| \to 0 \quad \text{as } s \to \infty \]

and so \( T(\omega, x(\omega)) = x(\omega) \) for each \( \omega \in \Omega \). This \( x(\omega) \) is obviously measurable and thus a random fixed point of \( T \). This completes the proof.

In a Hilbert space \( H \), the following equality holds:

\[
\| \lambda x + (1-\lambda)y \|^2 = \lambda \| x \|^2 + (1-\lambda) \| y \|^2 - \lambda (1-\lambda) \| x-y \|^2
\]

(3.2.6)

for all \( x, y \) in \( H \) and \( \lambda \in [0,1] \).

By (3.2.6) and Theorem 3.2.1, we immediately obtain the following:
THEOREM 3.2.2 Let \((\Omega, \Sigma)\) be a measurable space and \(K\) a nonempty bounded closed convex separable subset of Hilbert space \(H\). Suppose \(T: \Omega \times K \to K\) is a continuous asymptotically \((\omega, .)\) regular mapping satisfying the condition (3.2.1) \(\lim \inf \|T_{\omega}\| = k(\omega) = \sqrt{2}\) for all \(\omega \in \Omega\).

where \(\alpha(\omega)\) and \(\gamma(\omega)\) are as in Theorem 3.2.1, then \(T\) has a random fixed point.

If we put \(b_n(\omega) = c_n(\omega) = 0\) in Theorem 3.2.2, we obtained the following result:

COROLLARY 3.2.1 [116, Corollary 1]. Let \((\Omega, \Sigma)\) be a measurable space. Let \(K\) a nonempty bounded closed convex separable subset of Hilbert space \(H\). If \(T: \Omega \times K \to K\) is an asymptotically \((\omega, .)\) regular mapping such that

\[
\lim_{n \to \infty} \|T^n\| = k(\omega) = \sqrt{2}\quad \text{for all } \omega \in \Omega.
\]

Then \(T\) has a random fixed point.

Now consider James space \(E_\beta = (\ell^2, \| \cdot \|_\beta)\) where \(\| \cdot \|_\beta = \max \{\| \cdot \|_2, \| \cdot \|_\omega\}\). It is known that \(E_\beta\) has uniformly normal structure if and only if \(1 \leq \beta < \sqrt{2}\) (see [98]).

We therefore have the following result:

COROLLARY 3.2.2 [116, Corollary 4]. Let \((\Omega, \Sigma)\) be a measurable space \(\Sigma\) a sigma algebra of subsets of \(\Omega\). Suppose \(1 \leq \beta < \sqrt{2}\), \(K\) is a bounded closed
convex subset of $F_{\beta}$ and $T: \Omega \times K \to K$ is a random uniformly Lipschitzian mapping such that

$$\liminf_{n \to \infty} \|T^n\|_{k(\omega)} < 2^{1/4} \beta^{1/2}$$

for all $\omega \in \Omega$.

Then $T$ has a random fixed point.

**REMARK 3.2.1.** Our results improve the corresponding results of Thakur et al. [111].

**REMARK 3.2.2.** Our result extends the corresponding result of Xu [116].