CHAPTER-1

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The use of successive approximation to prove the existence and uniqueness of the differential equation is a well known phenomena. Picard method is one among them. In the year 1922, it was Banach [1] who put this method under abstract framework and enhanced its application beyond the scope of differential and integral equations. For many years, Banach contractive principle was confined to extensions and generalizations under the umbrella of metric fixed point theory. This theory had a point of departure by creating a new field known as nonlinear functional analysis with the fundamental work of Browder [12] in 1965. The theory of nonexpansive mapping within nonlinear analysis clearly emerged within a decade. In fact, the nonexpansive mappings are special case of Banach’s contraction mappings where Lipschitz constant is one. The class of nonexpansive mappings attracted analysts because, for proving the existence or approximation of fixed point one has to go beyond pure “metric” methods. The reason behind this was, the behaviour of the nonexpansive mappings, which exhibited entirely different from contraction mapping. In the case of contraction mapping only “metric completeness” was required for the existence of fixed point and Picard constructive method was sufficient for iteration of mapping.

In case of nonexpansive mapping however, examples [92] were found in which fixed point was not unique and in some cases they do not exists. In certain cases Picard iteration failed to converge to the fixed point. The theory of nonexpansive mapping flourished because these behaviour. Several type of nonexpansive mapping having different geometrical properties/structure came into existence and several iterative method were investigated for their
convergence. As per requirement of our thesis, let we confine our discussion to two aspects of nonexpansive mappings. First involving only existence results and second, involving iteration/convergence results.

Here, we point out some important class of nonexpansive mappings. Those are class of quasinonexpansive mappings, asymptotically nonexpansive mappings and Lipschitzian mappings. It would be unfair if we fail to mention here the class of asymptotically regular mapping introduced by Browder and Petryshyn [13]. It was the first result which concerned iterative convergence of the sequence of nonexpansive mapping. This class of nonexpansive mapping came in to existence in 1966. It was introduced due of non-constructiveness of the existence results given separately by Browder [11], Gohde [37] and Kirk [57]. Next class of nonexpansive mapping is quasinonexpansive mapping [27]. Although it was labelled by Dotson [27] but it was introduced by Diaz and Matcalf [26] with some other related ideas. Almost at the same time, asymptotically nonexpansive mapping [35] came into existence. Next year, other important class known as uniformly Lipschitzian mapping was introduced by Goebel & Kirk [36]. The existence result for the family of asymptotically nonexpansive mappings was studied by Passty [80].

Mann [73] derived an iteration scheme in 1953. Later, it was used fixed point iteration of the class of quasinonexpansive mapping. Krasnoselskii [60] has also tried the convergence of nonexpansive mapping. There were many authors who tried either Mann iteration or other to prove the convergence for nonexpansive mapping under different conditions. In this context, Halpern [47] algorithm is worth mentioning. Next is Groetsch [45] who proved convergence result for nonexpansive mapping extending the result of Browder and Petryshyn [13] by excluding the concept of asymptotically regularity in his hypothesis. Very soon, Ishikawa [51] introduced an iteration scheme and
proved its convergence for nonexpansive mappings. Originally, Ishikawa iteration scheme does not include Mann iteration. Later, it was modified to include Mann iteration. Finally, discussion on iteration of nonexpansive mappings would not be completed unless we mention the Kirk iteration scheme [57]. It was introduced prior to Ishikawa iteration scheme. Kirk [57] scheme was extended by Massa [74]. The Kirk scheme was based on concept of two mappings and its was suited to different constructive conditions.

Asymptotically regular mappings

Asymptotically regular mapping [13] is important for studying fixed point properties of nonexpansive mapping defined on a convex subset D of a Banach space. Ishikawa [50] observed that if D is bounded and convex and if T: D → D is nonexpansive than f = (I + T)/2 has the same fixed point set as T. The mapping f is asymptotically regular. Soon after introducing the asymptotic regular concept by Browder and Petryshyn [13], several results appeared in the literature of theory of nonexpansive mappings using asymptotically regular condition. In a paper, Khang [55] extended the first existence result of Browder and Petryshyn [13] to multivalued case. Som [104] extended such result for three mappings to get common fixed point. In a paper, Reich & Itai [88] proved convergence result for asymptotic regular mapping having Frechet differentiable condition. Further, Gornicki [38] proved an existence result for asymptotic regular mapping in L^p-space. Kruppel [62] proved such result in Hilbert space. Later, Gornicki [40] proved an existence result for asymptotically regular mapping in Banach space with normal structure. He also extended it improving the value of the constant k_p [41]. Benavides [4] proved an existence result of asymptotic regular maps for
the parameters forming infinite dimensional structure. He later extended his result for Orlicz function spaces [6].

**Quasinonexpansive Mappings**

The concept of quasinonexpansive mapping was introduced by reducing the full force of non-expansiveness and limiting it to one fixed point only. Nonexpansive mapping with one fixed point is called quasinonexpansive mapping. As it was mentioned, the concept was labelled by Dotson [27] to prove an existence result. He also gave an example which was quasinonexpansive but not nonexpansive. Dai [23] proved another existence result in Hilbert space and than extended it to k-uniformly rotund space [22]. Later, Das and Debata [25] proved an existence result for pair of quasinonexpansive map. Lin [69] extended it to set-valued quasinonexpansive mapping.

On the other hand, Iteration of quasinonexpansive mapping was equally interesting. Dotson [27] used Mann iteration for proving strong convergence of quasinonexpansive mapping. Later, Singh and Nelson [103] used Kirk type iteration procedure for such purpose. Massa [75] further extended the result of Dotson to strictly and uniformly convex spaces. In a paper, Chidume [20] gave an example of quasinonexpansive mapping which was not uniformly asymptotically regular. Zhang & Fu [122] not only proved existence of common fixed points for family of quasinonexpansive mapping but also established convergence of Ishikawa iteration for such class. Men [76] introduced a generalized quasinonexpansive multivalued map. Maiti and Ghosh [71] established the convergence of Ishikawa iteration for quasinonexpansive map. Ghosh and Debnath [31] proved convergence of Ishikawa iteration of quasinonexpansive mapping under a different contractive condition.
Asymptotically nonexpansive mappings.

The concept of asymptotically nonexpansive mapping was introduced to study the fixed point behaviour of large iterates of such mapping, when a positive constant $k_n \in [1, \infty)$ considered as $k_n \geq 1$ and $\lim_{n \to \infty} k_n = 1$. Many existence results were proved for asymptotically nonexpansive mappings. In a paper, Samanta, S.K. [94] proved existence and approximation results for asymptotically nonexpansive asymptotically regular mapping by using Demiclosedness property. Yu and Dai [120] extended it to uniformly rotund space. Lim and Xu [66] proved both existence and approximation results in uniformly smooth Banach space for asymptotically nonexpansive mapping using Maluta coefficient. In another result, Lin, Tan and Xu [67] proved the existence of fixed point for asymptotically nonexpansive mapping using Opial condition. Kim and Kirk [56] proved existence result for asymptotically nonexpansive mapping satisfying Goebel’s Lipschitz condition. Vijayaraju [113] studied it in product space. Recently, Rouhani [90] has proved an existence result for almost asymptotically nonexpansive mapping defined on a Hilbert space.

On the other hand, several results for iterative sequence of asymptotically nonexpansive mappings were proved. In a paper, Gornicki [44] proved a weak convergence of asymptotically nonexpansive mappings satisfying Opial condition. Schu [95] proved weak and strong convergence of Mann Scheme for asymptotically nonexpansive mapping. Rhoades [89] proved an existence of fixed point of strongly convergence sequence of Mann type for asymptotically nonexpansive map. Sharma, Thakur & Cho [102] proved existence of common fixed point of Mann iterates for asymptotically nonexpansive sequence of map. Schu [96] proved the strong convergence for
asymptotically nonexpansive and asymptotically pseudocontractive map, respectively.

**Lipschitzian Mappings**

The name of uniformly Lipschitzian mapping is derived from the name of the mapping called Lipschitzian mapping. It was introduced by Goebel & Kirk [36] as natural extension of nonexpansive mapping in $n^{th}$ iterate with Lipschitz constant $k$ greater than 0. They are characterized by non-expansiveness with at least one equivalent metric. All periodic Lipschitzian mappings are found uniformly Lipschitzian. The first existence result for this class of mapping involved the geometrical structure called uniform convexity [46]. The second result in this direction was due to Downing and Turret [28]. It contained an existence result in an arbitrary Banach space involving Lipschitzian mappings. The third existence result on this line involved uniform normed structure [16]. In a paper Guay and Singh [46] extended the result of Goebel & Kirk [36] replacing the uniform convexity by certain different assumption. Later, Gornicki [43] obtained an extension by using $m$-uniformly $k$-Lipschitzian mapping on this line. It was further extended by Lim [64] using a geometric structure called asymptotic density. In a paper, Gornicki by giving generalization [39] of his preceding paper, gave the $L^p$ and $H^p$ space version of his result. Benavides and Xu [3] proved an important result in this direction by using sequentially and weakly compact subset and subsequently using asymptotically regular mapping. Recently, Kaczor et.al. [54] proved an existence result for uniformly Lipschitzian mapping for unbounded sets.

In chapter 2 of this thesis, our object is to define a new class of mapping known as partially generalized Lipschitzian mapping (PGLM). This mapping has two important features One - we can obtain improved version of
the variants of Lipschitzian mappings from the proposed class of mapping. Second - fixed point exists for said class of mapping under demi-continuous (without continuity) assumption. For this purpose we have divided chapter 2 into two sections. In section 1, the class of mapping known as partially generalized Lipschitzian mapping (PGLM) worth comparable with known class of Lipschitzian mapping is defined. Further, the existence of fixed point is proved for said class of mapping in Banach space. In section-2 of this chapter, the strong convergence of partially generalized asymptotically nonexpansive mapping (PGANM) is discussed because PGANM is more general than PGLM.

The concept of asymptotic regularity with reference to nonexpansive mappings is discussed in the preceding paras. Another currently important and interesting aspect of nonlinear functional analysis is to study the random versions of deterministic fixed point theorems of nonlinear mappings because those could be useful for the theory of random equations. In chapter 3, we have proved some existence results with asymptotic regularity and having a specific contractive condition are proved. However, looking to the need of randomization of existence results, chapter 3 is divided in two sections. In section 1, an existence result is proved in \( L^p \) space and in section 2, its random version is studied in measurable space under certain conditions.

It is told in the preceding paras that Krasnoselskii [60] was one among many who tried to establish the iteration of the sequence of nonexpansive mappings. It was extended by Kirk [57]. Recently, Liu et al. [70] have introduced a new iteration process. This result has improved the corresponding result of Kirk [59], Maiti and Saha [72], Senter and Dotson [99]. In chapter 4, our intention is to introduce a new iteration process, which generalizes the iteration scheme of Liu et al. [70]. Further, we present a
necessary and sufficient condition for the convergence of newly introduced iteration process as an improvement over the result of Kirk [59], Maiti and Saha [72] and Liu et al. [70] for quasinonexpansive mappings in Banach space. In addition, also we study the convergence for nearly asymptotically quasinonexpansive mappings in Banach space. For this purpose, Chapter 4 is divided into two sections. In section 1 of chapter 4, Kirk type iteration scheme is introduced. Later, its convergence is proved for quasinonexpansive mapping whereas section 2 of this chapter is provided with the convergence result for the class of nearly nonexpansive mappings. We drop the condition of uniformly convexity from space structure. An example is furnished in this chapter to underline the relationship between nonexpansive mapping and quasinonexpansive mappings.

In a paper, Qihou [86] introduced (L-α) uniform Lipschitz asymptotically quasinonexpansive mapping to study the convergence of its Ishikawa iterative sequence with error term. In section 1 of last chapter 5, a convergence result using a generalized Ishikawa iterative sequence given by Sharma & Sahu [101] for (L-α) uniformly Lipschitz asymptotically quasi nonexpansive mapping in uniformly convex Banach space is proved. Our result extends the result of Tan & Xu [106], Sharma & Sahu [101] and Qihou [86]. In section 2 of this chapter certain convergence results for the class of relatively asymptotically quasi-nonexpansive mappings are proved. Some examples are given in support. Last but never the least a bibliography is given together with the list of publication of authors at the end of the thesis.