CHAPTER II

UNSTEADY FLOW OF NON-NEWTONIAN FLUIDS
2.1 INTRODUCTION

This chapter records the theoretical treatment of two classes of fluids. The first one consists of the semi-empirical, three parameter model proposed by Oloroyd (1) and has been chosen since it describes qualitatively most of the known material functions for elastic-viscous fluids whereas some of the models reported in the literature fail to give even a crude description of all the known material functions for such fluids. The fluids considered in the other class are the fluids first discussed by Rivlin and Ericksen (2). The main reason for discussing the Rivlin-Ericksen model is that its equations of state are sufficiently general to include, as special cases, many of the other equations of state which have appeared in the literature. Moreover, the theory is advanced from purely phenomenological considerations.

The study of flow properties of non-Newtonian fluids in different flow fields is of great practical importance. For example, the flow in a porous channel is important in transpiration cooling and gaseous diffusion processes (3). It is well known that many of the elastic properties of dilute polymer solutions can be detected and measured conveniently by observing suitable types of oscillatory flows. Pulsatile flow in a porous channel has been proved useful in the dialysis of blood in artificial kidneys (4). The usefulness of such investigations suggests, therefore, that the development of proper methods to obtain the exact solutions of the non-Newtonian fluids flow problems will be of interest.
A number of authors working in the field have considered the flow of a visco-elastic fluid, represented by Oldroyd's (1) three parameter model, through channels of various cross-sections. Devi Singh (5) has discussed the motion of a visco-elastic Maxwell fluid through two concentric circular cylinders when the implied pressure gradient is an exponential function of time. Frater (6) has investigated the flow of an elastic-viscous liquid between two torsionally oscillating discs. Unsteady flow of a linear viscoelastic Maxwell fluid through long circular as well as co-axial circular ducts has been treated by Sharma (7) assuming the axial pressure gradient as an arbitrary function of time. Leslie and Tanner (8) have attempted the problem of slow steady flow of a viscoelastic fluid past a sphere. The non-steady laminar flow of a Maxwell fluid near an oscillating porous infinite plane has been studied by Dube and Tewari (9).

A catalogue of several useful techniques exists for the analysis of flow problems such as described above. But many of these techniques are either difficult to apply or else their application is restricted to only a small class of problems. In the present chapter we have adopted a technique which permits more accurate treatment of a wider class of problems and leads to a substantial improvement over earlier presentations of the method. One of the features of this approach is that it does not require the use of two initial conditions on the flow variable to solve the problem.
As an application of this approach we have determined the velocity profiles for viscoelastic Maxwell fluid flowing through channels of rectangular cross-sections under the influence of a time varying pressure gradient. The corresponding problem for ordinary viscous fluids was considered by Fan & Chao (10) using a different method. Later, Ghosh (11) furnished the solution of the same problem for visco-elastic fluids by assuming a series for the flow variable. His solution was limited to homogeneous initial and boundary conditions. Moreover, he has imposed a restriction by assuming the pressure gradient impressed upon the system to be zero initially. This restriction represents a rather severe limitation on the usefulness of his approach. It may also be mentioned here that some of the particular solutions obtained by him do not satisfy the assumed initial conditions of the problem. The results reported by Ghosh (11) are shown to emerge as special cases of the solutions obtained here. The flow between two parallel plates is studied as a limiting case of the problem in hand.

The other boundary value problem solved in the present chapter is concerned with the flow of Rivlin-Ericksen fluids through rectangular ducts with pressure gradient as any function of time. Unsteady flow through channels bounded by two parallel flat plates is considered in this case also. The analysis is carried out by means of the following pair of transforms:
(i) the Laplace transform, and
(ii) the finite Fourier transform
defined in the subsequent section.

2.2 REQUIRED INTEGRAL TRANSFORMS;

The Laplace transform of any function \( f(t) \) is defined as
\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-pt} f(t) \, dt, \quad \Re p > 0 \tag{2.2.1}
\]
provided the above integral exists.

The convolution property of Laplace transform is
\[
\int_0^t f_1(u) f_2(t-u) \, du = \mathcal{L}\{f_1(f_2)\} \cdot \mathcal{L}\{f_2(f_1)\} \tag{2.2.2}
\]
where \( \mathcal{L}\{f_1(f_2)\} \) and \( \mathcal{L}\{f_2(f_1)\} \) are the Laplace transforms of \( f_1(t) \) and \( f_2(t) \) respectively.

If the function \( f(x,y) \) satisfies the Dirichlet conditions in the region \( 0 \leq x \leq a, \, 0 \leq y \leq b \) and if its finite sine transform is defined as
\[
\mathcal{F}_S\{f(x,y)\} = \int_0^a \int_0^b f(x,y) \sin \alpha x \sin \beta y \, dx \, dy \tag{2.2.3}
\]
then its inversion is given by (12)
\[
f(x,y) = \frac{4}{ab} \sum_{n=1}^{\alpha} \sum_{m=1}^{\beta} \mathcal{F}_S\{f(x,y)\} \sin \alpha x \sin \beta y \tag{2.2.4}
\]
where \( \alpha_n = \frac{\alpha n}{a} \) and \( \beta_m = \frac{\beta m}{b} \)
The following operational property is satisfied by (2.2.3) (17)

\[ \int_{a}^{b} \int_{0}^{\pi} \frac{\partial^2 f}{\partial x^2} \sin \alpha x \sin \beta y \, dx \, dy = \alpha \left[ f^*_s(m) + (-1)^n \beta^*_s(m) \right] - \alpha \left[ f^*_s(n,m) \right] \]

where \( f^*_s(m) \) and \( h^*_s(m) \) are finite sine transforms of the functions \( f(0, y) \) and \( f(a, y) \) respectively.

In the case \( f(0, y) = f(a, y) = 0 \), we have

\[ \int_{a}^{b} \int_{0}^{\pi} \frac{\partial^2 f}{\partial x^2} \sin \alpha x \sin \beta y \, dx \, dy = -\alpha \left[ f^*_s(n,m) \right] \]

and similarly

\[ \int_{a}^{b} \int_{0}^{\pi} \frac{\partial^2 f}{\partial y^2} \sin \alpha x \sin \beta y \, dx \, dy = -\beta \left[ f^*_s(n,m) \right] \]

when the function \( f(x, y) \) vanishes on the boundaries \( y = 0 \) and \( y = b \).

2.3 FORMULATION OF THE PROBLEM AND GOVERNING EQUATIONS:

Consider a slow shearing motion of a viscoelastic Maxwell fluid through a large rectangular duct. We choose a system of rectangular cartesian coordinates \((x, y, z)\) with the \( z \) coordinate measured parallel to the axis of the duct. The rectilinear flow through the tube is characterized by

\[ u = v = 0, \quad w = w(x, y, t) \]
The equations of motion and continuity in the absence of extraneous forces are

\[ \mathbf{f} \left[ \frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_i \cdot \nabla \mathbf{v}_i \right] = -\mathbf{p}_i + \mathbf{S}_{ij} \mathbf{j} \]  \hspace{1cm} (2.3.2)

and

\[ \mathbf{v}_i \cdot \mathbf{n} = 0 \]  \hspace{1cm} (2.3.3)

where \( \mathbf{f} \) is the density and \( \mathbf{p} \) is the pressure.

The equation of continuity (2.3.3) shows that \( \mathbf{W} \) is independent of \( z \). The stress-strain relations for the present problem are as follows [Cf. § 1.2 (1)]

\[(1 + \lambda_1 \frac{\partial}{\partial t}) S_{zx} = \eta_0 (1 + \lambda_2 \frac{\partial}{\partial t}) \frac{\partial \mathbf{w}}{\partial x}\]  \hspace{1cm} (2.3.4)

\[(1 + \lambda_1 \frac{\partial}{\partial t}) S_{zy} = \eta_0 (1 + \lambda_2 \frac{\partial}{\partial t}) \frac{\partial \mathbf{w}}{\partial y}\]  \hspace{1cm} (2.3.5)

\[S_{zz} = S_{zx} = S_{zy} = S_{zy} = 0\]  \hspace{1cm} (2.3.6)

The equations of motion (2.3.2) can now be written as

\[0 = \frac{\partial \mathbf{p}}{\partial x}\]  \hspace{1cm} (2.3.7)_1

\[0 = \frac{\partial \mathbf{p}}{\partial y}\]  \hspace{1cm} (2.3.7)_2

and

\[ \mathbf{f} \frac{\partial \mathbf{w}}{\partial t} = -\frac{\partial \mathbf{p}}{\partial z} + \frac{\partial}{\partial x} S_{zx} + \frac{\partial}{\partial y} S_{zy}\]  \hspace{1cm} (2.3.7)_3

The quantities involved in the above equations are as defined in section (1.2)(i).
From (2.3.7)_1 and (2.3.7)_2, we notice that the fluid pressure at any instant t is constant on any cross-section perpendicular to its axis and that \( \frac{\partial p}{\partial z} \) is function of time alone. We choose

\[
- \frac{1}{f} \frac{\partial p}{\partial z} = f(t) \tag{2.3.8}
\]

The boundary and initial conditions are

\begin{align*}
W &= 0 \quad \text{at } x = 0, 2a; \quad t > 0 \tag{2.3.9}_1 \\
W &= 0 \quad \text{at } y = 0, 2b; \quad t > 0 \tag{2.3.9}_2 \\
W &= W_0 \quad \text{at } t = 0, 0 < x < 2a; \quad 0 < y < 2b \tag{2.3.9}_3
\end{align*}

2.4 **Solution of the Problem**

Taking the Laplace transform with respect to t of the equations (2.3.4), (2.3.5) and (2.3.7)_3, we obtain on account of (2.3.9)_3

\[
(1 + \lambda_1 p) \overline{S}_{zx} = \eta_0 (1 + \lambda_2 p) \frac{\partial \overline{\omega}}{\partial x} - \eta_0 \lambda_2 \frac{\partial \omega_0}{\partial x}
\]

\[
(1 + \lambda_1 p) \overline{S}_{zy} = \eta_0 (1 + \lambda_2 p) \frac{\partial \overline{\omega}}{\partial y} - \eta_0 \lambda_2 \frac{\partial \omega_0}{\partial y}
\]

\[
\overline{p \omega} - \omega_0 = \overline{f(p)} + \frac{1}{f} \left( \frac{\partial}{\partial x} \overline{S}_{zx} + \frac{\partial}{\partial y} \overline{S}_{zy} \right)
\]

where

\[
\overline{S}_{zx} = \int_0^\infty e^{-pt} S_{zx} \, dt
\]

\[
\overline{S}_{zy} = \int_0^\infty e^{-pt} S_{zy} \, dt
\]

\[
\overline{f(p)} = \int_0^\infty e^{-pt} f(t) \, dt
\]
and $W_0$ is the initial value of $W$. It has been assumed that $S_{zx}$ and $S_{zy}$ vanish initially. Eliminating $S_{zx}$ and $S_{zy}$ from the above equations, we get

$$(b \overline{w} - w_0)(1 + \lambda_1 b) = (1 + \lambda_1 b) \overline{f}(b) + \mu (1 + \lambda_2 b)$$

and $S_{zx}$ and $S_{zy}$ vanish initially. Eliminating $S_{zx}$ and $S_{zy}$ from the above equations, we get

$$(b \overline{w} - w_0)(1 + \lambda_1 b) = (1 + \lambda_1 b) \overline{f}(b) + \mu (1 + \lambda_2 b)$$

where $\mu = \nu_0 / \rho$

Now applying the finite sine transform (2.2.3) to (2.4.1), we have, in view of relations (2.2.6) and (2.2.7)

$$\left[ \lambda_1 b^2 + (1 + \nu \lambda_2 \gamma_{mn}) b + \mu \gamma_{mn} \right] \overline{w}_s^* = (1 + \lambda_1 b) S_{mn} \overline{f}(b)$$

and $S_{zx}$ and $S_{zy}$ vanish initially. Eliminating $S_{zx}$ and $S_{zy}$ from the above equations, we get

$$\left[ \lambda_1 b^2 + (1 + \nu \lambda_2 \gamma_{mn}) b + \mu \gamma_{mn} \right] \overline{w}_s^* = (1 + \lambda_1 b) S_{mn} \overline{f}(b)$$

where

$$S_{mn} = \frac{16 ab}{mn \pi^2} \sin^2 \frac{m \pi}{2} \sin^2 \frac{n \pi}{2}$$

$$\overline{w}_s^* = \int_0^{2a} \int_0^{2b} \overline{w} \sin \alpha x \sin \beta y dx dy$$

and $L_m = \frac{m \pi}{2a}$, $\beta_n = \frac{n \pi}{2b}$
Initially the motion is steady in the channel. Eliminating $S_{z\alpha}$ and $S_{z\gamma}$ from (2.3.4), (2.3.5) and (2.3.7), the initial distribution of velocity $W_0$ is given by

$$
\frac{\partial^2 W_0}{\partial x^2} + \frac{\partial^2 W_0}{\partial y^2} = -\frac{f(c)}{\mu} \quad (2.4.3)
$$

where $f(c)$ is the initial value of $f(t)$.

The equation (2.4.3) is to be solved under the boundary conditions

$$
W_0 = 0 \quad \text{at} \quad x = 0, 2 a
$$
$$
W_0 = 0 \quad \text{at} \quad y = 0, 2 b
$$

In view of relations (2.2.6) and (2.2.7) we have

$$
W_{0S}^* = \frac{f(c)}{\mu} \cdot \frac{S_{m,M}}{\chi_{m,M}} \quad (2.4.4)
$$

Substituting the value of $W_{0S}^*$, it follows from equation (2.4.2) that

$$
\bar{W}_S(m,n,p) = \frac{(1+\lambda_1 b) S_{m,n} f(b) + \frac{f(c) S_{m,n}}{\mu \chi_{m,n}} [(1+\lambda_1 b) + \nu \lambda_2 \gamma_{m,n}]}{\lambda_1 b^2 + (1+\lambda_2 \gamma_{m,n}) b + \mu \gamma_{m,n}}
$$

By virtue of (2.2.4)

$$
\bar{W}(x,y,p) = \frac{1}{\alpha_0} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{W}_S(m,n,p) \sin \alpha_m x \sin \beta_n y
$$

(2.4.5)
Finally the Laplace inversion of (2.4.5) leads to the solution
\[\omega(x, y, t) = \frac{1}{\alpha \beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{S_{m,n}}{\lambda_1(\kappa - \beta)} \int_0^t \left[ (1 + \lambda_1\kappa) \int_0^t f(t - \tau) d\tau \right] e^{\beta \tau} d\tau\]
\[\left\{ -(1 + \lambda_1\beta) \int_0^t e^{\beta \tau} f(t - \tau) d\tau \right\} + \frac{f(0)}{\lambda_1\gamma_{m,n}} \left[ (1 + \lambda_1\kappa) e^{\beta t} \right] \]
\[- (1 + \lambda_1\beta) e^{\beta t} + \lambda_2 \gamma_{m,n} \left( e^{\beta t} - e^{\beta t} \right) \] \[\sin\alpha x \sin\beta y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{S_{m,n}}{\lambda_1(\kappa - \beta)} \int_0^t \left[ (1 + \lambda_1\kappa) \int_0^t f(t - \tau) d\tau \right] e^{\beta \tau} d\tau\]
\[\left\{ -(1 + \lambda_1\beta) \int_0^t e^{\beta \tau} f(t - \tau) d\tau \right\} + \frac{f(0)}{\lambda_1\gamma_{m,n}} \left[ (1 + \lambda_1\kappa) e^{\beta t} \right] \]
\[- (1 + \lambda_1\beta) e^{\beta t} + \lambda_2 \gamma_{m,n} \left( e^{\beta t} - e^{\beta t} \right) \] (2.4.6)

where \(\alpha, \beta\) are the roots of
\[\lambda_1\beta^2 + (1 + \lambda_2\gamma_{m,n})\beta + \lambda_1\gamma_{m,n} = 0 \] (2.4.7)

and
\[\lambda_1(\alpha - \beta) = \left[ (1 + \lambda_2\gamma_{m,n})^2 - 4\lambda_1\mu\gamma_{m,n} \right]^{1/2}\]
(2.4.8)

where
\[\alpha(\alpha - \beta) = \left[ (1 + \lambda_2\gamma_{m,n})^2 - 4\lambda_1\mu\gamma_{m,n} \right]^{1/2}\]

The above result under homogeneous conditions becomes
\[\omega(x, y, t) = \frac{1}{\alpha \beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{S_{m,n}}{\lambda_1(\kappa - \beta)} \int_0^t \left[ (1 + \lambda_1\kappa) \int_0^t f(t - \tau) d\tau \right] e^{\beta \tau} d\tau\]
\[\left\{ -(1 + \lambda_1\beta) \int_0^t e^{\beta \tau} f(t - \tau) d\tau \right\} + \frac{f(0)}{\lambda_1\gamma_{m,n}} \left[ (1 + \lambda_1\kappa) e^{\beta t} \right] \]
\[- (1 + \lambda_1\beta) e^{\beta t} + \lambda_2 \gamma_{m,n} \left( e^{\beta t} - e^{\beta t} \right) \] \[\sin\alpha x \sin\beta y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{S_{m,n}}{\lambda_1(\kappa - \beta)} \int_0^t \left[ (1 + \lambda_1\kappa) \int_0^t f(t - \tau) d\tau \right] e^{\beta \tau} d\tau\]
\[\left\{ -(1 + \lambda_1\beta) \int_0^t e^{\beta \tau} f(t - \tau) d\tau \right\} + \frac{f(0)}{\lambda_1\gamma_{m,n}} \left[ (1 + \lambda_1\kappa) e^{\beta t} \right] \]
\[- (1 + \lambda_1\beta) e^{\beta t} + \lambda_2 \gamma_{m,n} \left( e^{\beta t} - e^{\beta t} \right) \] (2.4.9)
This expression can be made to agree with the result of Ghosh (11) after suitable adjustment. It will be of interest to point out that Ghosh has obtained the solution by a different approach.

Now if \( \lambda_1 \rightarrow 0, \lambda_2 \rightarrow 0 \), then by (2.4.8),

\[
\lambda_1(x-\beta) \rightarrow 0
\]

and also

\[
x \rightarrow -\mu \gamma_{n,m}, \beta \rightarrow -\infty
\]

Therefore (2.4.6) becomes

\[
\omega = \frac{1}{\alpha \beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} S_{m,n} \sin \lambda_m x \sin \beta_n y \left[ \int_0^t \mu \gamma_{m,n} f(t-\tau) d\tau \right] + \frac{f(\alpha)}{\mu \gamma_{n,m}} \left( -\mu \gamma_{m,n} f(t) \right)
\]

(2.4.10)

and (2.4.9) reduces to

\[
\omega(x,y,t) = \frac{1}{\alpha \beta} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} S_{m,n} \sin \lambda_m x \sin \beta_n y \int_0^t \mu \gamma_{m,n} f(t-\tau) d\tau
\]

(2.4.11)

These results correspond to the velocity profiles for ordinary viscous fluids flowing through rectangular channels.

2.5 FLOW UNDER OSCILLATING PRESSURE GRADIENT:

In the previous section we have determined the velocity profiles under the influence of an axial pressure gradient varying arbitrarily with time. Here we assume

\[
f(t) = A \cos \beta b \cdot t
\]

(2.5.1)

where \( A \) and \( b \) are real constants.
Substituting (2.5.1) in (2.4.6), we obtain

\[ \Omega(x, y, t) = \frac{A}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{S_{m,n}}{\lambda_i(x-\beta)} \left\{ \frac{(i+\lambda_1 \xi)}{\kappa^2 + b_i^2} (\kappa e^{-\xi t} - \xi c_0 b_i t + b_i \sin b_i t) \right. \\
\left. - \frac{(1+\lambda_1 \beta)}{\beta^2 + b_i^2} (\beta e^{-\beta t} c_0 b_i t + b_i \sin b_i t) + \frac{1}{\mu \gamma_{m,n}} \left[ (i+\lambda_1 \xi) e^{-\xi t} - \xi e^{-\xi t} \right] \right\} \sin \lambda_i x \sin \beta_i y \]

where-as (2.4.11) takes the form

\[ \Omega(x, y, t) = \frac{A}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{S_{m,n}}{\lambda_i(x-\beta)} \left\{ \gamma_{m,n} \mu c_0 b_i t + b_i \sin b_i t \right. \\
\left. - \mu \gamma_{m,n} e^{-\mu \gamma_{m,n} t} \right\} \sin \lambda_i x \sin \beta_i y \]

The expression (2.5.3) can be made to satisfy Fan and Chao's (10) result after a little adjustment.

CONCLUSION:

In the above discussion we have taken arbitrary values of \( \alpha \) and \( \beta \) where \( \alpha, \beta \) are the roots of

\[ \lambda_1 \beta^2 + (1 + \mu \lambda_2 \gamma_{m,n}) \beta + \mu \gamma_{m,n} = 0 \]

The roots \( \alpha \) and \( \beta \) are imaginary when

\[ (1 + \mu \lambda_2 \gamma_{m,n})^2 < 4 \lambda_1 \mu \gamma_{m,n} \]

ie. when the values of \( m, n \) are such that the inequality

\[ T. 13 \]
\[ \left( 2 \lambda_1 - \lambda_2 \right) - 2 \left[ \lambda_1 \left( \lambda_1 - \lambda_2 \right) \right] < m \lambda_z^2 \gamma_{m, \lambda} \leq \left( 2 \lambda_1 - \lambda_2 \right) + 2 \left[ \lambda_1 \left( \lambda_1 - \lambda_2 \right) \right] \]

is satisfied. Hence the expression for \( \omega(x, y, t) \) corresponding to these values, will contain sine and cosine terms which shows that unlike ordinary viscous fluid, in viscoelastic fluid periodic oscillations are set up whose amplitudes decay with time.

2.6 UNSTEADY FLOW BETWEEN PARALLEL PLATES

We shall now investigate the laminar flow of a visco-elastic Maxwell fluid through a channel bounded by two parallel infinite flat plates subjected to an axial pressure gradient varying arbitrarily with time. This flow may be studied as a limiting case of the problem discussed in the earlier sections of the current chapter. Taking into consideration the result (2.4.6), we have

\[
\omega(x, y, t) = \frac{16}{\pi \gamma} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\sinh \eta_j x \sin \eta_j y \frac{\lambda_1}{\lambda_2}}{\lambda_1 \left( 2i+1 \right) \left( 2j+1 \right)} \left( 1 + \lambda_1 \mu_1 \right) \int_{0}^{t} e^{\frac{\lambda_2 \eta_j}{\gamma} (\tau - t)} d\tau \\
- \left( 1 + \lambda_1 \mu_2 \right) \int_{0}^{t} e^{\frac{\lambda_2 \eta_j}{\gamma} (\tau - t)} d\tau + \frac{f(0)}{\mu k_{ij}} \left[ \left( 1 + \lambda_1 \mu_1 \right) e^{\lambda_1 \eta_j t} \right. \\
- \left. \left( 1 + \lambda_1 \mu_2 \right) e^{\lambda_2 \eta_j t} + \mu \lambda_2 k_{ij} \left( e^{\lambda_1 \eta_j t} - e^{\lambda_2 \eta_j t} \right) \right] \right] 
\]

(2.6.1)

where

\[
k_{ij} = \frac{(2i+1)^2 \lambda_2^2}{4a^2} + \frac{(2j+1)^2 \lambda_2^2}{4b^2} \\
h_i = \frac{(2i+1) \lambda}{2a}, \quad h_j = \frac{(2j+1) \lambda}{2b}\]
and
\[ \mu_1, \mu_2 = \frac{-(1 + \mu \lambda k_i j) \pm \left[ (1 + \mu \lambda k_i j)^2 - 4 \lambda \mu k_i j \right]^{1/2}}{2 \lambda_1} \]

Now changing the origin to the middle point of the rectangular channel, (2.6.1) may be expressed as

\[ \omega(x, y; t) = \frac{16}{\pi^2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \frac{\cos \phi_i \cos \phi_j}{\lambda_1 (2i+1)(2j+1)(\mu_1 \mu_2)} \int_0^t \int_0^{\mu_1 \tau} e^{\mu_2 t} f(t - \tau) d\tau \]

\[ - (1 + \lambda \mu_2) \int_0^{\mu_2 \tau} e^{\mu_2 t} f(t - \tau) d\tau \] \[ + \frac{f(0)}{\mu_1 k_i j} \left[ (1 + \lambda \mu_1) e^{\mu_1 t} - \mu_2 h_i j (e^{\mu_1 t} - e^{\mu_2 t}) \right] \]

\[ (\text{2.6.2}) \]

In the case when \( b \to \infty \), we have

\[ k_i j = h_i j \to 0 \]

and

\[ \mu_1, \mu_2 = \frac{-(1 + \mu \lambda h_i j)^2 \pm \left[ (1 + \mu \lambda h_i j)^2 - 4 \lambda \mu h_i j \right]^{1/2}}{2 \lambda_1} \]

The velocity profiles in this case are then given by

\[ \omega(x, y; t) = \frac{4}{\pi} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} \frac{\cos \phi_i \cos \phi_j}{\lambda_1 (2i+1)(2j+1)(\mu_1 \mu_2)} \int_0^t \int_0^{\mu_1 \tau} e^{\mu_2 t} f(t - \tau) d\tau \]

\[ - (1 + \lambda \mu_2) \int_0^{\mu_2 \tau} e^{\mu_2 t} f(t - \tau) d\tau \] \[ + \frac{f(0)}{\mu_1 h_i j} \left[ (1 + \lambda \mu_1) e^{\mu_1 t} - \mu_2 h_i j (e^{\mu_1 t} - e^{\mu_2 t}) \right] \]

\[ (\text{2.6.3}) \]
The above equation describes the unsteady flow of a visco-elastic liquid confined between two parallel plates separated by a distance $2a$. The corresponding result for the flow under harmonically oscillating pressure gradient is

$$
\omega(x,t) = \frac{4A}{\kappa} \sum_{i=0}^{\infty} \frac{(-1)^i \alpha_i b_i x}{\lambda_i (2i+1)(1-\lambda_i)} \left[ \left( \frac{1+\lambda_i \mu_i}{\mu_i^2 + b_i^2} \right) \left( \frac{e^{\mu_i t} - \lambda_i \mu_i e^{\mu_i t}}{e^{\mu_i t} - e^{\mu_i t}} \right) + b_i \sin b_i t \right] - \frac{1}{\mu_i^2} \left[ \left( \frac{1+\lambda_i \mu_i}{\mu_i^2 + b_i^2} \right) \left( \frac{e^{\mu_i t} - \lambda_i \mu_i e^{\mu_i t}}{e^{\mu_i t} - e^{\mu_i t}} \right) \right]
$$

(2.6.4)

2.7 FLOW OF RIVLIN-ERICKSEN FLUIDS THROUGH RECTANGULAR DUCTS:

So far we have described the flow of a visco-elastic fluid, specified through the constitutive equations given by Oldroyd (1), by making the use of integral transforms. Now we investigate the flow properties of the Rivlin-Ericksen (2) fluids flowing through tubes of rectangular cross-sections under the influence of an axial pressure gradient varying arbitrarily with time. The constitutive equation for a viscoelastic fluid as given by Rivlin and Ericksen (2) has the following form:

$$
\Gamma_{ij} = P_{ij} \delta_{ij} + P_{ij} \hspace{2cm} (2.7.1)_1
$$

$$
P_{ij} = \Phi_i E_{ij} + \Phi_2 D_{ij} + \Phi_3 E_{im} E_{mj} \hspace{2cm} (2.7.1)_2
$$

$$
E_{ij} = U_{ij} + U_{ji} \hspace{2cm} (2.7.1)_3
$$

$$
D_{ij} = A_{ij} + A_{ji} + 2 U_{mi} U_{mj} \hspace{2cm} (2.7.1)_4
$$

$$
A_i = \frac{\partial U_i}{\partial t} + U_m U_{i,m} \hspace{2cm} (2.7.1)_5
$$
where the symbols have their usual meanings and a suffix following a comma denotes covariant differentiation.

The equations of motion and continuity have the following forms

\[ \frac{\partial \mathbf{A}_i}{\partial t} = \nabla_f \mathbf{A}_i \]  
\[ \mathbf{u}_i = 0 \]

where \( \rho \) is the density of the fluid.

For the present problem we have

\[ \mathbf{u} = 0, \quad \mathbf{v} = 0, \quad \omega = \omega(x, y, t) \]

where \( \mathbf{u}, \mathbf{v}, \omega \) are the physical components of the velocity vector. The equation of continuity and (2.7.3) give

\[ \frac{\partial \omega}{\partial z} = 0 \]

Under the conditions (2.7.3)\(_1,2\) the equation of motion governing the flow can be written as

\[ \frac{\partial \omega}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \kappa \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) + \beta \frac{\partial}{\partial t} \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \]

where \( \kappa = \eta / \rho \) and \( \beta = \eta_2 / \rho \) are the kinematical coefficient of viscosity and viscoelasticity respectively.

The equation (2.7.4) determines the velocity field \( \omega(x, y, t) \) and is to be solved under the following initial and boundary
Further, the motion considered is under the influence of a pressure gradient varying arbitrarily with time. We may safely assume

\[ \frac{1}{f} \frac{\partial p}{\partial x} = f(t) \]

Equation (2.7.4) now takes the form

\[ \frac{\partial w}{\partial t} = f(t) + \kappa \left( \frac{\partial w}{\partial x^2} + \frac{\partial w}{\partial y^2} \right) + \beta \frac{\partial}{\partial t} \left( \frac{\partial w}{\partial x^2} + \frac{\partial w}{\partial y^2} \right) \]

The solution of the above boundary value problem may be furnished by employing the transforms defined in the section (2.2).

Applying the finite sine transform (2.2.3) to the equation (2.7.6) taking into account (2.7.5)\textsubscript{1,2}, we obtain in view of the relations (2.2.6) and (2.2.7)

\[ \frac{d W^*_s}{dt} = f(t) S_{ij} - \lambda^2 \gamma_{ij} W^*_s - \beta \gamma_{ij} \frac{d W^*_s}{dt} \]

where

\[ S_{ij} = \frac{16ab}{l_i l_j \pi^2} \sin^2 \frac{ix}{2} \sin^2 \frac{ij}{2} \]

\[ \gamma_{ij} = \lambda^2 + \beta^2 \]

\[ W^*_s(i,j,t) = \int_0^{2a} \int_0^{2b} w(x,y,t) \sin \lambda_i x \sin \beta_j y \, dx \, dy \]
and
\[ a_i = \frac{i \alpha}{2a}, \quad \beta_j = \frac{j \alpha}{2b} \]
on taking the Laplace transform of the above equation and using \( \omega_s^* = \omega_{0s}^* \) initially, we have
\[ \begin{cases} (1 + \beta_{ij})\phi + \alpha \gamma_{ij} \int \tilde{\omega}_s^* = \bar{f}(b) S_{ij} + (1 + \beta_{ij}) \omega_{0s}^* \end{cases} (2.7.7) \]
where
\[ \tilde{\omega}_s^*(i,j,b) = \int_0^\infty e^{-bt} \omega_s^*(i,j,t) \, dt \]
\[ \bar{f}(b) = \int_0^\infty e^{-bt} f(t) \, dt \]
and
\[ \omega_{0s}^* = \int_0^{2a} \int_0^{2b} \omega_0(x,y) \sin \alpha x \sin \beta y \, dx \, dy \]

Initially the motion is steady in the channel, the initial distribution of velocity \( \omega_0 \) is given by
\[ \frac{\partial^2 \omega_0}{\partial x^2} + \frac{\partial^2 \omega_0}{\partial y^2} = -\frac{f(0)}{\alpha} (2.7.8) \]
where \( f(0) \) is the initial value of \( f(t) \).

The equation \( (2.7.8) \) is to be solved under the boundary conditions
\[ \begin{align*} \omega_0 &= 0 \text{ at } x = 0, 2a \quad \omega_0 &= 0 \text{ at } y = 0, 2b \end{align*} \]
By virtue of the relations (2.2.6) and (2.2.7) we have

\[ W_{0S}^* = \frac{f(0)}{\alpha} \cdot \frac{\sin j}{\gamma i j} \]  

(2.7.9)

Substituting the above value of \( W_{0S}^* \) in (2.7.7), it follows that

\[ W_{S}^* = \frac{\sin j}{1 + s \gamma i j} \frac{f(0)}{\alpha} \cdot \frac{\sin j}{\gamma i j} + \frac{f(0) \sin j}{\alpha \gamma i j} \cdot \frac{1}{1 + s \gamma i j} \]  

(2.7.10)

In view of the convolution theorem (2.2.2), the Laplace inversion of (2.7.10) yields

\[ W_{S}(i,j,t) = \frac{\sin j}{1 + s \gamma i j} \int_{0}^{t} e^{-h \tau} f(t - \tau) d\tau + \frac{f(0) \sin j}{\alpha \gamma i j} e^{-ht} \]

where

\[ h = \frac{\alpha \gamma i j}{1 + s \gamma i j} \]

Inverting the sine transform we arrive at the final formula

\[ W(x, y, t) = \frac{16}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin \lambda_m x \sin \lambda_n y}{(2m+1)(2n+1)(1+s \lambda_m \lambda_n)} \int_{0}^{t} f(t - \tau) d\tau \]

\[ + \frac{16}{\alpha \pi^2} f(0) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin \lambda_m x \sin \lambda_n y}{(2m+1)(2n+1) \lambda_m \lambda_n} e^{-ht} \]

(2.7.11)
where

\[ K_{m,n} = \lambda_m + \lambda_n \]

\[ \lambda_m = \frac{(2m+1)\alpha}{2a}, \quad \lambda_n = \frac{(2n+1)\alpha}{2b} \]

\[ \mu = \frac{\alpha k_{mn}}{1 + \beta k_{mn}} \]

COROLLARY:

Corresponding expression of the velocity profiles for ordinary viscous fluids can be obtained by making the elastic parameter \( \beta \) tend to zero.

If \( \beta \to 0 \), then \( \mu \to \alpha k_{mn} \) and the relation (2.7.11) reduces to

\[
W(x, y, t) = \frac{16}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin ax \sin by}{(2m+1)(2n+1)} \int_0^t e^{-\mu t} f(t-\tau) d\tau
\]

\[
+ \frac{16}{\pi^2} f(0) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin ax \sin by}{(2m+1)(2n+1)k_{mn}} e^{-\mu t}
\]

(2.7.12)

It should be noted that the above result is the same as obtained in section (2.4) for the present case.
2.8 **FLOW UNDER A LINEAR PRESSURE GRADIENT**

Let us now assume that

\[ f(t) = A_0 + B_0 t \]

where \( A_0 \) and \( B_0 \) are real constants.

Substituting it in (2.7.11), we obtain

\[
\frac{\alpha}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin m\alpha \sin n\alpha}{(2m+1)(2n+1) k_{mn}} + \frac{16B_0 \alpha}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\sin m\alpha \sin n\alpha}{(2m+1)(2n+1) k_{mn}} \\
- \frac{16B_0 \alpha}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1+\beta k_{mn})\sin m\alpha \sin n\alpha}{(2m+1)(2n+1) k_{mn}} \\
+ \frac{16B_0 \alpha}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(1+\beta k_{mn})\sin m\alpha \sin n\alpha e^{-\lambda t}}{(2m+1)(2n+1) k_{mn}}
\]

Putting \( B_0 = 0 \), we get the expression corresponding to constant pressure gradient.

Using (2.7.12), similar expressions for the velocity profiles can be found in the case of ordinary viscous fluids flowing through rectangular channels when the axial pressure gradient varies linearly with time and also when it is constant.

2.9 **FLOW BETWEEN TWO PARALLEL PLATES**

In this section, the unsteady flow between two infinite parallel plates, situated at \( x = \pm a \) separated by a distance
2a, has been deduced as a limiting case of the problem treated in section (7.7). The purpose may be accomplished simply by shifting the origin to the middle point of the rectangular channel and then taking the distance parameter \( b \) to be infinite. Change of origin to the middle point of the channel changes the relation (2.7.11) to the following form

\[
\omega(x, y, t) = \frac{16}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} \cos \beta \lambda m x \cos \lambda n y}{(2m+1)(2n+1)(1+\beta k m n)} \int_0^t \frac{e^{-\frac{t}{z}}}f(t-z) d\tau 
\]

\[ + \frac{16}{\alpha \pi^2} \int_0^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} \cos \beta \lambda m x \cos \lambda n y}{(2m+1)(2n+1) k m n} e^{-\mu t} \]

(2.9.1)

Velocity profiles for the present case become known as \( b \to \infty \) in (2.9.1).

when \( b \to \infty, \lambda_n \to 0, k m n \to \lambda_m \) and

\[ \mu \to \alpha \lambda_m^2 / (1+\beta \lambda_m^2) = \sigma \] (say)

The flow between two parallel plates \( x = \pm \frac{a}{2} \) separated by a distance \( 2a \) is then given by

\[
\omega(x,t) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m \cos \beta \lambda_m x}{(2m+1)(1+\beta \lambda_m^2)} \int_0^t \frac{e^{-\frac{t}{z}}}f(t-z) d\tau 
\]

\[ + \frac{4 \int_0^{\infty}}{\pi^2} \sum_{m=0}^{\infty} \frac{(-1)^m \cos \beta \lambda_m x}{(2m+1) \lambda_m^2} e^{-\sigma t} \]

(2.9.2)
When \( f(t) = A_0 + B_0 t \), the expression (2.9.3) for the velocity distribution becomes

\[
W(x, t) = \frac{16 A_0 a^2}{x^3} \sum_{m=0}^{\infty} \frac{(-1)^m \cos \beta \lambda_m x}{(2m+1)^3} \]

\[
+ \frac{16 a^2 B_0 t}{x^3} \sum_{m=0}^{\infty} \frac{(-1)^m \cos \beta \lambda_m x}{(2m+1)^3} \]

\[
- \frac{64 B_0 a^4}{x^5} \sum_{m=0}^{\infty} \frac{(-1)^m (1+\beta \lambda_m^2)}{(2m+1)^5} e^{-t} \cos \beta \lambda_m x \]

\[
+ \frac{64 B_0 a^4}{x^5} \sum_{m=0}^{\infty} \frac{(-1)^m (1+\beta \lambda_m^2)}{(2m+1)^5} e^{-t} \cos \beta \lambda_m x \] (2.9.3)

By making the use of the results obtained by Dube and Singh (13) and Dube (14) the above equation simplifies to

\[
W(x, t) = \frac{A_0}{2x} \left( a^2 - x^2 \right) + \frac{B_0 t}{2x} \left( a^2 - x^2 \right) \]

\[
- \frac{B_0}{24 L^2} \left[ 5 \left( a^2 - x^2 \right) \left( a^2 - x^2 \right) - \frac{B_0 B}{2x} \left( a^2 - x^2 \right) \right] \]

\[
+ \frac{64 B_0 a^4}{x^5} \sum_{m=0}^{\infty} \frac{(-1)^m (1+\beta \lambda_m^2)}{(2m+1)^5} e^{-t} \cos \beta \lambda_m x \] (2.9.4)

The solution (2.9.4) is in complete agreement with that of Dube (14) who furnished it through a different approach. Moreover, all the conclusions drawn by Dube (14) regarding
the flow properties of Rivlin-Ericksen fluids would hold equally well in this case.

When $\beta \rightarrow 0$, we have the case of the flow of ordinary viscous fluids through a straight channel bounded by two parallel plates $\chi = \pm a$ separated by a distance $2a$. This particular problem has been studied by Dube (15) in an earlier investigation and the solution obtained by him agree well with the limiting form $(\beta \rightarrow 0)$ of our result (2.9.4).
REFERENCES