CHAPTER IX

SOLUTIONS OF A CLASS OF INTEGRAL EQUATIONS BY FOX-H-TRANSFORM
9.1 INTRODUCTION

In the foregoing chapters we have demonstrated the use of transform techniques in finding the solution of boundary and initial value problems for ordinary and partial differential equations arising in various fields of physics and engineering. The theory of integral transforms also plays a predominating role in the solution of integral equations with which the remainder of the thesis is concerned.

Integral equations arise in virtually every field of scientific endeavour. Bellman, Jacquez, and Kalaba (1-3) have used integral equations in the development of mathematical models of chemotherapy. However, due to the non-deterministic nature of diffusion processes from blood plasma in to the body tissue, the stochastic versions of these equations were lateron used by Padgett and Tsokos (4-6). In deriving the solution of a certain aero-dynamical problem, Ta Li (7) was led to a class of integral equations involving Chebyshev polynomials in the kernel. Integral equations are frequently encountered in the field of elasticity. Wu and Chiu (8) have employed the integral equation technique for the solution of the contact problem of layered elastic bodies. A similar method has been used for melting and solidification problems in heat conduction by Boley (9) and Sikarskie and Boley(10). The prominent role of integral equations in the theory of automatic control is revealed by many writers (11,12) and is very well known. Numerous other applications of these equations to the problems in chemical kinetics, physiological systems, population growth, telephone engineering, turbulence, and systems theory may also be mentioned.
Study of integral equations has led to startling advances on practically every front in engineering and science and has exerted a profound impact on the modeling and analysis of physical problems. This impact will certainly continue to be felt.

While the ultimate interest in integral equations lies in its applications, the present work is concerned mainly with its formal aspects. Consequently, a mathematical and analytical viewpoint is emphasized throughout.

Notable contributions have been made by several authors (13-16) demonstrating the use of integral transforms in the field of integral equations. Widder (17, 18) has employed the Laplace transform to solve integral equations which were in the form of convolution with respect to it. Mellin transform has been utilized by Fuchsman (19) in solving the integral equation which could be reduced in the form of convolution with respect to Mellin transform. Shrivastava (20) has adopted the theory of Fourier and Hankel transform to solve certain integral equations. A finite Hankel transform has been used by Bhonsle (21) in this connection. Varma (22) has introduced a similar approach involving Mellin transform to obtain the inversion of a class of transforms with a difference kernel.

Solutions of certain integral equations are directly based on integral transforms and their inverses. Singh (23, 24) has solved two types of integral equations with different Bessel functions in the kernel. In this chapter we have obtained inversion integrals of a class of integral equations involving
The $F$-function in the kernel. The inversion embraces known cases besides giving interesting new results. The technique of Singh (23) is used for the purpose.

9.2 TPS KERNEL:

The $F$-function is defined as (cf. section (1.10))

$$H_{p,q,r}^m \left[ x \left| \left\{ (a_p, e_p) \right\} \right| \left\{ (b_q, f_q) \right\} \right] = \frac{1}{2\pi i} \left( \frac{\prod_{j=1}^{m} (b_j - f_j s) \prod_{j=1}^{n} (1 - a_j + e_j s)}{\prod_{j=m+1}^{q} (1 - b_j + f_j s) \prod_{j=n+1}^{r} (a_j - e_j s)} \right) x^s ds$$

(9.2.1)

Conditions for the existence of the above integral are assumed as stated by Singh (25).

The convergence of the integrals involving $F$-function can be investigated by considering the asymptotic expansion of the function.

According to Braaksma (26)

$$H_{p,q,r}^m \left[ x \left| \left\{ (a_p, e_p) \right\} \right| \left\{ (b_q, f_q) \right\} \right] = \begin{cases} O \left( |x|^{\nu_1} \right) & \text{for large } x \\ O \left( |x|^{\nu_2} \right) & \text{for small } x \end{cases}$$

where

$$\sum_{i=1}^{p} e_j - \sum_{j=1}^{q} f_j < 0, \sum_{i=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=m+1}^{w} f_j - \sum_{j=1}^{q} f_j > 0$$

$$\nu_1 = \max_i R \left( \frac{a_i - 1}{e_i} \right), \; (i = 1, 2, \ldots, n)$$

and

$$\sum_{j=1}^{p} e_j - \sum_{j=1}^{q} f_j < 0 \; \rho_2 = \min_k R \left( \frac{b_k}{f_k} \right) (k = 1, 2, \ldots, m)$$
We shall also assume that
\[ f(x) = \begin{cases} O(|x|^\alpha) & \text{for small } x \\ O(|x|^\beta) & \text{for large } x \end{cases} \]

The H-functions considered in the sequel carry these assumptions.

9.3 FOX-H-TRANSFORM:

The H-Transform used in this chapter is given below:

If
\[ \varphi(u) = \lambda u \int_0^\infty \mathcal{H}_{\alpha,\beta} \left[ \frac{u(x)}{\mathcal{H}_{\alpha,\beta}} \right] f(x) \, dx \]

then
\[ f(x) = \frac{C}{\lambda} \int_0^\infty \left[ \mathcal{H}_{\alpha,\beta} \left( \frac{\varphi(u)}{u} \right) \right] \left[ \mathcal{H}_{\alpha,\beta} \left( \frac{\varphi(u)}{u} \right) \right] \, dx \]

where
\[ \alpha_p = 1 - \alpha_p - \beta_p, \quad \beta_p = 1 - \beta_p - \beta_q \]

provided
(i) \( 0 \leq \eta \leq \beta \), \( 1 \leq \nu \leq q \), \( \sum \beta_j - \frac{q}{2} \beta_q < 0 \),
(ii) \( f(x) \) is continuous at \( x = u \), \( \alpha > 0 \), \( s = k + it \), \( 0 < t < \infty \)
(iii) \( \alpha + \nu + 1 > 0 \), \( \beta + s + 1 < 0 \)
(iv) \( 1 + s < k < 1 + s + 2 \)
(v) The H-function in (9.3.2) exists.

The result (9.3.2) is due to Singh (27).
7.4 REQUIRED RESULTS

The following results will be used in the proof of the theorem:

\[
\int_0^\infty (x-t)^{\mu-1} \sum_{m,n} \left[ a_{m,n} \left( \frac{a_{p}}{b_{q}} \right)^2 \right] \, dx
\]

\[
= \Gamma(\mu) \sum_{m,n} \left[ a_{m,n} \left( \frac{a_{p}}{b_{q}} \right)^2 \right] \left[ \frac{(a_{p}, e_{p})^2}{(0, 1)} \right] \left[ \frac{(b_{q}, f_{q})^2}{(-\mu, 1)} \right] \quad (7.4.1)
\]

\[
0 < R(\mu) < R \left( \frac{1-a_{j}}{e_{j}} \right) \quad 1 \leq j \leq n
\]

The results (7.4.1) and (7.4.2) are respectively known as Riemann-Liouville and Weyl fractional integrals for \( H \) function. These results can be easily obtained by using the definition of \( H \)-function and Beta function.

\[
\sum_{m,n} \left[ \frac{a_{m,n}}{b_{q}} \left( \frac{a_{p}}{b_{q}} \right)^2 \right] = \sum_{m,n} \left[ \frac{a_{m,n}}{b_{q}} \left( \frac{a_{p}}{b_{q}} \right)^2 \right] = \sum_{m,n} \left[ \frac{a_{m,n}}{b_{q}} \left( \frac{a_{p}}{b_{q}} \right)^2 \right] \quad (7.4.3)
\]

The above relation follows from the definition of \( H \)-function.

Since

\[
\sum_{m,n} \left[ \frac{a_{m,n}}{b_{q}} \left( \frac{a_{p}}{b_{q}} \right)^2 \right] = \frac{1}{c} \sum_{m,n} \left[ \frac{a_{m,n}}{b_{q}} \left( \frac{a_{p}}{b_{q}} \right)^2 \right] = \frac{1}{c} \sum_{m,n} \left[ \frac{a_{m,n}}{b_{q}} \left( \frac{a_{p}}{b_{q}} \right)^2 \right]
\]
We can take any power of $x$ as the argument of $H$-function used in (9.3.1). Therefore there is no loss of generality in taking $x$ as the argument.

9.5 **Theorem I**:

If the integral equations

\[
\alpha \int_0^x (x-y)^{\mu-1} \frac{dz}{z} \left[ \sum_{i=0}^{\infty} \left\{ \frac{\psi(y)}{y z} \right\} \frac{f(y)}{\psi(y)} \right] f(y) dy = \varphi(x, z)
\]

and

\[
\int_0^x x^{-\lambda} \varphi(x, z) dx = \psi(z)
\]

exist, then the solution of (9.5.1) is given by

\[
f(y) = \frac{2 d}{1 - \frac{1}{2}} \int_0^\infty \frac{\psi(z) z^{1 - \lambda - \mu} H_{p+1, q+1}^{\mu-1} \frac{y z}{\sqrt{z}} \left[ \sum_{i=0}^{\infty} \left\{ \frac{\psi(y)}{y z} \right\} \frac{f(y)}{\psi(y)} \right] f(y) dy \]

where

\[
A_k = 1 - a_k - \frac{\mu - \lambda + 1}{2} e_k, \quad E_k = \frac{1}{2} e_k
\]

\[
B_k = 1 - b_k - \frac{\mu - \lambda + 1}{2} f_k, \quad F_k = \frac{1}{2} f_k
\]

provided

(i) $\lambda + \sigma_2 + 1 > 0$

(ii) $0 < R(\mu) < R(\lambda) - \sigma_1$

(iii) $-\lambda - 2 \min \left\{ R(b_j/f_j) - 1 < R(\mu - \lambda) - \beta - 2 \max_{1 \leq j \leq \kappa} R(\alpha_{j-1}/e_j) \right\} - (0 \leq j \leq m) b_0 = \lambda - \mu, \quad f_0 = 1$

(iv) The $H$-function in (9.5.3) exists and the conditions of continuity of $f(y)$ as laid down in (9.3.2) are satisfied.
The above conditions are general and stringent. The particular results are valid under somewhat liberal conditions and therefore the conditions for special cases may be derived from consideration of analytic continuations.

**Proof of the Theorem:**

We suppose that both \( \Phi(x, z) \) and \( \Psi(z) \) exist. Now substituting the value of \( \Phi(x, z) \) from (9.5.1) in (9.5.2), interchanging the order of integration which is justified under the stated conditions, absorbing the power of \( x \) in the \( H \)-function with the help of (9.4.3) and using (9.4.2) to evaluate the inner integral, we get after a little manipulation

\[
\Psi(z) = \frac{a \Gamma(\mu)}{2^\mu \mu!} \int_0^\infty f(y) H_{\mu+1, \mu+1}^{\mu, \mu} \left[ \frac{y^\mu z^\mu}{\mu!} \right] dy
\]

which gives (9.5.3) on inversion by means of (9.3.2) under the stated conditions. Thus the theorem is proved.

**9.6 Particular Cases:**

Putting \( \mu = 1, \nu = 4, \eta = \beta = 0, m = 2 = q, \lambda = 1/2 \)

\( b_1 = \mu/2, f_1 = 1 = f_2, b_2 = -\mu/2, \lambda = 1/2 \)

we get the integral equations as

\[
\int_0^x k_\nu(z \sqrt{x y}) f(y) dy = \Phi(x, z)
\]

and

\[
\int_0^\infty z^{-\mu/2} \Phi(x, z) dx = \Psi(z)
\]

by virtue of the relation

\[
k_\nu(z \sqrt{x y}) = \frac{1}{2} H_{0,2}^{3,3} \left[ \frac{z^2}{2} \left( \frac{y}{2}, \frac{\nu}{2} \right) \right]
\]
Then from (9.5.4) after a little manipulation

$$
\Psi(z) = 2z^{-\frac{3}{2}} \int_0^\infty f(y) \frac{-(\nu_1 + \nu_2)}{2}(yz)^{\nu_2} K_{\nu_1}(yz) dy
$$

which can be inverted by using the technique given by CONOLLY (24) and the inversion is

$$
\frac{1}{C+i\infty} \int_{C-i\infty}^{C+i\infty} f(y) \frac{y^{\nu_1+1}}{2\pi i} \int_0^\infty I_{\nu_1-1}(yz) \frac{z^2 \Psi(z)^2}{\Psi(z)} dz (9.6.1)
$$

This is given by Singh (24).

II Putting \( w = 1 = \mu, \eta = 0 = \rho, q = 2, a = 1, \lambda = \frac{\mu}{2} \)
\( d = 4, b_1 = \frac{\mu}{2}, b_2 = -\frac{\mu}{2}, f_1 = f_2 = 1 \)
we get the integral equations as

$$
\int_0^\infty x^{-\frac{\mu}{2}} \Phi(x, z) dx = \Psi(z)
$$

and the solution is given by

$$
\Phi(y) = \frac{y^{\frac{\mu}{2}+1}}{2} \int_0^\infty z^2 \Psi(z) J_{\nu-1}(yz) dz (9.6.2)
$$

after absorbing a power of \( z \) in the \( H \)-function and a little manipulation.

This again is given by Singh (25).

III Taking \( a = -x^{-\frac{1}{2}}, w = 2, \eta = 0, p = 1, q = 3, d = 1 \)
\( a_1 = \frac{1}{2}, e_1 = 1 = f_1 = f_2 = f_3, b_1 = \mu, b_2 = 0, b_3 = -1 \)
\( \lambda = \mu \), we get the integral equations as

$$
\int_0^\infty (x-y)^{\mu-1} J_{\mu}(x \sqrt{xy}) Y_{\mu}(x \sqrt{xy}) f(y) dy = \Phi(x, z)
$$

$$
\int_0^\infty x^\mu \Phi(x, z) dx = \Psi(z)
$$

by virtue of the relation

$$
J_{\mu}(x) Y_{\mu}(x) = -x^{-\frac{1}{2}} H_{1,3}^{2,0} \left[ x^2 \left( \frac{1}{2}, 1 \right) \right. (\mu, 1), (0, 1), (-\mu, 1) \]
and the inversion is
\[ f(y) = \frac{-2\pi^{1/2}}{\Gamma(\mu)} \int_0^\infty \psi(z) z^{\mu-1/2} H_{\mu,1}^{1,1} \left[ \frac{y}{z} \right] \left( \frac{\mu}{z^{1/2}}, \frac{1}{2} \right) \left( \frac{\mu + \mu}{z^{1/2}}, \frac{1}{2} \right) \left( \frac{-\mu + \mu}{z^{1/2}}, \frac{1}{2} \right) \, dz \]
(9.6.3)

If further we put \( \mu = \frac{1}{2} \), we find
\[ f(y) = \frac{-2\pi^{1/2}}{\Gamma(\mu)} \int_0^\infty \psi(z) z^{-1/2} H_{0,2}^{1,0} \left[ \frac{y}{z} \right] \left( \frac{3}{4} + \mu, \frac{1}{2} \right) \left( \frac{3}{4}, \frac{1}{2} \right) \, dz \]
\[ = -2\pi^{1/2} y \int_0^\infty \psi(z) z^{\mu+3/2} \int_0^\infty (2y/z) \, dz \]
(9.6.4)

This can be verified by using (29, p.206(41)) and Hankel transform inversion theorem.

9.7 **THEOREM II**

If the integral equations
\[ a \int_0^\infty (y-x)^{\mu-1} H_{p,q}^{m,n} \left[ \frac{y-x}{z} \right] \frac{z^{\mu+r}}{z^{\mu+1}} \frac{(a,b,c)}{(d,e,f)} \, dy = \phi_i(x, z) \]
(9.7.1)

and
\[ \int_0^\infty x^\lambda \phi_i(x, z) \, dx = \psi_i(z) \]
(9.7.2)
exist, then the solution of (9.7.1) is given by
\[ f(y) = \frac{2}{a d \frac{\lambda + \mu + 1}{2}} \int_0^\infty \psi_i(z) z^{\lambda+\mu} H_{p,q+1,\eta}^{1,0} \left[ \frac{y}{z} \right] \frac{z^{\mu+1}}{z^{\mu+1, \eta+1}} \left( \frac{a}{z^1} \right) \left( \frac{b}{z^{1/2}} \right) \left( \frac{c}{z^{1/2}} \right) \left( \frac{d}{z^{1/2}} \right) \left( \frac{e}{z^{1/2}} \right) \left( \frac{f}{z^{1/2}} \right) \, dz \]
\[ = \frac{2}{a d \frac{\lambda + \mu + 1}{2}} \int_0^\infty \psi_i(z) z^{\mu+1} H_{p,q+1,\eta}^{1,0} \left[ \frac{y}{z} \right] \left( \frac{a}{z^1} \right) \left( \frac{b}{z^{1/2}} \right) \left( \frac{c}{z^{1/2}} \right) \left( \frac{d}{z^{1/2}} \right) \left( \frac{e}{z^{1/2}} \right) \left( \frac{f}{z^{1/2}} \right) \, dz \]
(9.7.3)
where
\[ A_p = 1 - a_p - \frac{\lambda + \mu + 1}{2} e_p, \quad E_p = \frac{1}{2} e_p \]
\[ B_p = 1 - b_p - \frac{\lambda + \mu + 1}{2} f_p, \quad F_p = \frac{1}{2} f_p. \]
powered

 provided

(i) \( \beta + \alpha_i + 1 < 0 \)

(ii) \( -\min R\left( \frac{b_h}{f_h} \right) - 1 < R(\lambda) < \frac{e_i}{e_i} - 1 - \max R\left( \frac{a_i}{e_i} \right) \quad 1 \leq i \leq n \)

(iii) \( 0 < R(\mu) < \frac{e_i}{e_i} - \max R\left( \frac{a_i}{e_i} \right) - \beta, \quad 1 \leq i \leq n \)

(iv) \( -\lambda - 2 \min R\left( \frac{b_h}{f_h} \right) - 1 < R(\mu + \lambda) < -\beta - 2 \max R\left( \frac{a_j}{e_j} \right) - 1 \quad 0 \leq j \leq n \)

\( e_0 = 1, \quad a_0 = -\lambda \)

(iv) The \( H \) function in (9.7.3) exists and the conditions of continuity of \( f(y) \) as laid down in (9.3.2) are satisfied.

**PROOF:**

Substituting the value of \( \Phi(x,z) \) from (9.7.1) in (9.7.2), interchanging the order of integration which is justified under the stated conditions, absorbing the power of \( x \) in the \( H \) function with the help of (9.4.3) and evaluating the inner integral using (9.4.1) we get after a little manipulation

\[ \psi_1(z) = \frac{a \Gamma(\mu) z^{\lambda - \mu}}{2 c_0^{\mu + \lambda}} \int_0^\infty f(y) H_{p+1,q+1} \left[ \frac{y z}{c_0} \right] \left( \frac{\mu - \lambda}{2}, 1/2 \right) \left( \frac{a_p + \mu + \lambda}{2} e_p, e_{p+1}/2 \right)^2 \left( b_q + \frac{\lambda + \mu + 1}{2} f_q, f_{q+1}/2 \right)^2 \left( -\frac{\mu + \lambda}{2}, 1/2 \right) \right] dy \quad (9.7.4) \]

which gives (9.7.3) on inversion by means of (9.3.2) under the stated conditions. This proves the theorem.
9.8 PARTICULAR CASES:

I

Putting \( \alpha = \mu = 1 = \nu = p, \gamma = 3, d = 4, \lambda = \nu/2 \)
\( a_1 = 1/2 + \nu/2, \quad e_1 = 1 = f_1 = f_2 = f_3, \quad b_1 = 1/2 + \nu/2, \quad b_2 = -\nu/2 \)
\( b_3 = \nu/2 \), we get the integral equations (9.7.1) and (9.7.2) as
\[
\int_{x}^{\infty} H_{\nu}(z x^{-1/2}) f(y) dy = \Phi_{1}(x, z), \quad -\frac{3}{2} < \text{Re}(\nu) < -\frac{1}{2}
\]
and
\[
\int_{0}^{\infty} x^{1/2} \Phi_{1}(x, z) dx = \psi_{1}(z)
\]
on account of the relation
\[
H_{\nu}(x) = H_{1/3}^{1,1} \left[ \frac{x^2}{4} \left( \frac{1+\nu}{2}, 1 \right), \left( \frac{-\nu-1}{2}, 1 \right), \left( \frac{-\nu+1}{2}, 1 \right) \right]
\]
The required inversion is
\[
f(y) = \frac{1}{4} y^{1-\nu/2} \int_{0}^{\infty} x^{1/2} \psi_{1}(x) x^{1/2} \left[ \frac{x^2}{4} \left( \frac{-\nu-1}{2}, 1 \right), \left( \frac{-\nu+1}{2}, 1 \right), \left( \frac{-\nu+1}{2}, 1 \right) \right]
\]
and
\[
(9.8.1)
\]
after a little manipulation and absorption of power of \( z \) in
H-function

Since
\[
\gamma_{\nu+1}(x) = H_{1/3}^{2,0} \left[ \frac{x^2}{4} \left( \frac{-\nu-1}{2}, 1 \right), \left( \frac{-\nu+1}{2}, 1 \right), \left( \frac{-\nu+1}{2}, 1 \right) \right]
\]
the solution is as given by Singh (24).

II

Again
\[
\int_{\mu}^{\infty}(Z x^{-1/2}) f(y) dy = \Phi_{1}(x, z), \quad -\frac{3}{2} < \text{Re}(\nu) < -\frac{1}{2}
\]
so putting
\( \alpha = \mu = 1 = \nu, \quad h = 0 = p, \quad q = 2 \)
\( b_1 = \nu/2, \quad b_2 = -\nu/2, \quad f_1 = 1 = f_2, \quad \lambda = \nu/2, \quad d = 4 \)
we get the relations (9.7.1) and (9.7.2) in the form
\[
\int_{x}^{\infty} J_{\nu}(z^{-1/2}) f(y) dy = \Phi_{1}(x, z), \quad -1 < \nu < -1
\]
and \[ \int_0^\infty x^{\nu/2} \varphi_1(x, z) \, dx = \psi_1(z) \]

and the inversion is in the form
\[
\begin{align*}
\psi_1(z) &= \frac{1}{2} \psi_1^\nu \int_0^\infty \psi_1(z) z^2 \mathcal{L}_{\nu+1}(z) \, dz \\
&= \frac{1}{2} \psi_1^\nu \int_0^\infty \psi_1(z) z^2 \mathcal{L}_{\nu+1}(z) \, dz \\
\end{align*}
\]

(9.8.2)

which again is given by Singh (23).

III  
Taking \( a = 1/2 \), \( b = 0 \), \( d = 4 \), \( m = 2 = q \)
\[ b_1 = \nu_2, \quad b_2 = -\nu_2, \quad f_1 = f_2 = 1, \quad \lambda = \frac{\nu_1 - \nu_2}{2} \]

the integral equations become
\[
\begin{align*}
\int_0^\infty f(y) (y-x)^\nu \kappa_4(z, \frac{y}{z}) \, dy = \varphi_1(x, z) \\
\int_0^\infty \frac{x^\nu}{x} \varphi_1(x, z) \, dx = \psi_1(z) \\
\end{align*}
\]

If we put \( \mu = \nu + 1/2 \), we find
\[
\begin{align*}
\psi_1(z) &= \frac{1}{(\mu)} \left( \frac{z}{2} \right)^{-3/2} \int_0^\infty \left( \frac{y}{2} \right)^{1/2} f(y) \frac{1}{2} \mathcal{L}_{1,3} \left[ \frac{y^2}{4} \left( \frac{1}{2}, 1 \right) \right] \\
&= \frac{1}{(\mu)} \left( \frac{z}{2} \right)^{1/2} \int_0^\infty f(y) \left( \frac{y}{2} \right)^{1/2} \mathcal{L}_{1,3} \left[ \frac{y^2}{4} \left( \frac{1}{2}, 1 \right) \right] \\
\end{align*}
\]

which can be verified from (23), p. 198 (86) and then
\[
\begin{align*}
f(y) &= \frac{2y^{-3/2}}{(\mu)} \int_0^\infty \psi(z) \mathcal{L}_{1,3} \left[ \frac{y^2}{4} \left( \frac{1}{2}, 1 \right) \right] \\
\end{align*}
\]

(9.8.3)

Similar new results can be obtained by specializing the parameters suitably.
# REFERENCES

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