CHAPTER VII

JACOBI POLYNOMIALS AND ELASTIC VIBRATIONS OF A CIRCULAR PLATE
7.1 INTRODUCTION:

Many of today's structures are subjected to excitations. Examples range all the way from aircraft and missile structures subjected to acoustic and aerodynamic loads to civil engineering structures acted upon by earthquake and wind loads. Owing to the presence of such external loading, body forces, or heating the structure undergoes a deformation. In several cases the response statistics of such excited systems will be strongly time dependent. The study of the motion of deformation produced by the exciting forces in a mechanical system is generally referred to as the study of "mechanical vibrations". The term "vibrations" implies that the motion of deformation is, at least in part, oscillatory in nature.

The consequences of vibrations of a mechanical system are usually undesirable and are often an important design consideration. For example, the vibratory motion may result in a magnification of the stresses in a system, perhaps exceeding the ultimate strength of the material. Even if the ultimate strength is not exceeded, the repetitive nature of the vibratory stresses may in time produce fatigue failure of the material. In a machine, vibrations may cause improper operation or excessive wear. The vibratory motion transmitted to the surroundings may have adverse effects on other systems. On some occasions objectionable sound may be generated. As a consequence, considerable effort has been directed toward the development of proper analysis of vibrations and for determining the dynamic response statistics of such excited systems. An analysis of transient vibrations,

produced by an external force of arbitrary nature in an elastic thin circular plate, is presented in this chapter.

The theoretical treatment of specific boundary value problems associated with the small deflections of thin plates for which the transverse vibrations have been investigated, is quite old (1). In particular, the problem of elastic vibrations of circular plates subjected to lateral forces has received considerable attention in the literature because of its practical and academic interest. As an example of the forced vibrations produced in a circular plate by an external force, Sneddon (2) has considered the dynamic bending of the plate by normal impact. A similar treatment of the problem for concentrated loading has also been given by Nowacki (3). However, the small transverse vibration of a thin circular plate of finite radius under the influence of an arbitrary distribution of normal pressure has not received much attention.

It is therefore the purpose of this work to present analytical solutions for the transient displacements of the central plane, \( z = 0 \), of a thin circular plate simply supported on the boundary and subjected to pressure pulses which vary axially as well as with time. The pressure \( q \) is assumed to be axisymmetric and expressible as the product of a function of position and a function of time. Then

\[
q(r, t) = \Phi(r) \cdot \Psi(t)
\]  

(7.1.1)

where \( \Phi(r) \) will be called the load shape and \( \Psi(t) \) will be called the pulse shape. The arbitrary load shape is taken in the following form

\[
\Phi(r) = \kappa_0 \left[ \left(1 - \frac{\gamma^2}{a^2}\right) \sum_{n=1}^{\infty} \frac{\beta_n}{\beta_n} \left(1 - \frac{2\gamma^2}{\alpha_n^2}\right) \right]
\]  

(7.1.2)
where \( p_n^{(x,y)}(x) \) is the Jacobi Polynomials (4). Of ultimate interest in this problem is the behaviour of the family of load shapes \( \Phi(r) \) investigated in the last section. The nature or spatial distribution of pressure \( \Phi(r) \) over the surface area of the plate is seen to depend on the parameters involved therein. For purposes of illustration, the values of load shape parameters \( \lambda, \beta, \gamma \) and \( \tau \) are chosen as 0 and 1.

The pulse shapes for which results are presented are as follows:

(a) Rectangular Pulse:
\[
\psi(t) = \begin{cases} 
\psi_{\text{max}}, & 0 \leq t \leq t_0 \\
0, & t > t_0
\end{cases}
\]

(b) Triangular Pulse:
\[
\psi(t) = \begin{cases} 
\frac{2t}{t_0} \psi_{\text{max}}, & 0 \leq t \leq \frac{t_0}{2} \\
2\left(1 - \frac{t}{t_0}\right) \psi_{\text{max}}, & \frac{t_0}{2} \leq t \leq t_0 \\
0, & t > t_0
\end{cases}
\]

(c) Harmonic Pulse:
\[
\psi(t) = \psi_{\text{max}} \sin \lambda t, \quad 0 \leq t \leq \frac{2\pi \kappa}{\lambda}
\]
\[
\psi(t) = 0, \quad t > \frac{2\pi \kappa}{\lambda}
\]

(d) Exponential Decay Pulse:
\[
\psi(t) = \psi_{\text{max}} e^{-\frac{t}{t_0}}, \quad 0 \leq t < \infty
\]

Results in case of irregular shaped pulses with multiple peaks may also be obtained but are not given here. The driving force, governing the transverse vibrations of the plate, is characterised by the behaviour of pulse shape.
The analysis is carried out by using the technique of integral transforms. Use of integral transform techniques in elasticity has been extensive, both for solving a variety of specific boundary value problems and in the development of general approaches valid for larger classes of problems. For this, the work of Sneddon (2) and Nowaki (3) is of special pertinence. The merit of the present analysis lies in the fact that separate treatments for a number of cases may be avoided by this single treatment of the problem in hand. A distinguishing feature of this problem is that it exhibits the use of special functions in the field of elasticity probably for the first time.

7.2 A FINITE HANKEL TRANSFORM:

In the present section we shall obtain the finite Hankel transform of \( \Phi (r) \) given by (7.1.2). Using Erdelyi (5, p. 284,3)) and the definition of Bessel functions, we get on interchanging the order of summation and integration

\[
\int_{0}^{r} y^{\alpha+1} \Phi (r) J_{\nu}(xy) \, dy = \frac{1}{\Gamma(n+1)} \frac{1}{(2\pi)^{1/2}} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} \sum_{\lambda=-\infty}^{\infty} \frac{(\lambda+1)^{\lambda} (x^{2}+y^{2})^{\lambda}}{\Gamma(\lambda+\nu+1)} \frac{\Gamma(\lambda+\nu+\alpha+1)}{\Gamma(\lambda+\alpha+2)} \cdot
\]

\[
\times 3 \Gamma_{2} \left[ \begin{array}{c}
-n, 1+\alpha+\beta+n, m'+(\nu+\alpha+2) \cdot \\
1+\alpha, m'+n+\nu+\alpha+2 \cdot
\end{array} \right]
\]

where

\[
\begin{align*}
\Re \alpha &> -1, \quad \Re \beta > -1, \quad \Re m' > -1, \quad \Re \nu > -1 \quad \text{and} \quad \Re \alpha > -1 / 2.
\end{align*}
\]

The factor

\[
\frac{1}{\Gamma(n+1) \Gamma(1+\alpha)}
\]


seems to be missing from the result given by Brielyi ((5)p. 284(3)). Incidentally the above result (7.2.1) generalizes the results obtained by Khonie (6). On taking \( m^1 = \frac{d}{2}, \sigma = \beta, \mu = \kappa \) and writing \( s = r + n \), the relation (7.2.1) reduces, on application of Saalschütz's theorem, to

\[
\iint_0^1 y^{x+1} (1-y^2)^\beta P_n^{(x \beta)} (1-2y^2) J_\kappa (xy) \, dy = \frac{2^\beta \Gamma (\kappa + n + 1)}{\kappa! \kappa^{x+\beta}} J_{x+\beta + 2n+1} (x)
\]

Let the finite Hankel transform of \( \varphi (r) \) be

\[
H \left[ \varphi (r) \right] = \int_0^1 \varphi (r) J_\kappa (r \gamma \xi_i) \, d\gamma = \overline{\varphi} _H (\xi_i) \quad (7.2.2)
\]

Then we easily see on specialization of parameters that

\[
H \left[ (1- \frac{r^2}{a^2})^\sigma P_n^{(x \beta)} (1-2r^2/a^2) \right] = \frac{a^2 \Gamma (\sigma + 1) \Gamma (1+\kappa + n)}{2 \Gamma (\kappa + 1)} \sum_{\beta = 0}^\infty (-1)^\beta \left( \frac{a \xi_i}{2} \right)^{2 \beta} \frac{\Gamma (\beta + \sigma + 2)}{\beta!}
\]

\[
\times \, \frac{\Gamma (\kappa + 1)}{\Gamma (1+\kappa)} \frac{\Gamma (\kappa + n + 1)}{\Gamma (1+\kappa + n)} \int \left[ \begin{array}{c} -n, 1+\kappa + \beta + n, \beta + 1; \\
1+\kappa, \sigma + \beta + 2; 1 \end{array} \right] \right] \quad (7.2.3)
\]

Because of the inversion theorem (2), we shall have

\[
(1- \frac{r^2}{a^2})^\sigma P_n^{(x \beta)} (1-2r^2/a^2) = \frac{\Gamma (\sigma + 1)}{\Gamma (1+\kappa + n)} \sum_{\beta = 0}^\infty (-1)^\beta \left( \frac{a \xi_i}{2} \right)^{2 \beta} \frac{\Gamma (\beta + \sigma + 2)}{\beta!}
\]

\[
\times \, \frac{\Gamma (\kappa + 1)}{\Gamma (1+\kappa)} \frac{\Gamma (\kappa + n + 1)}{\Gamma (1+\kappa + n)} \int \left[ \begin{array}{c} -n, 1+\kappa + \beta + n, \beta + 1; \\
1+\kappa, \sigma + \beta + 2; 1 \end{array} \right] \right] J_\kappa (r \xi_i \xi_i) \frac{\Gamma (\kappa + n + 1)}{\Gamma (\kappa + n + 1)} \quad (7.2.4)
\]

where \( \xi_i \) runs through the positive roots of

\[
J_\kappa (a \xi_i) = 0 \quad (7.2.5)
\]
The result (7.2.4) will prove useful in the verification of the solution. Throughout the analysis, we have chosen that $v > 0$.

### 7.3 Statement of the Problem

Consider the small transverse vibrations of an isotropic homogeneous thin elastic plate of radius $a$, thickness $h$ and flexural rigidity $D$. Following Nowacki (3), the differential equation governing the transverse displacement $\omega$, measured positively downward, of the middle surface of the plate is

$$c^2 \left( \frac{\partial^2}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \omega(r,t) + \frac{\partial^2}{\partial t^2} \omega(r,t) = \frac{q(r,t)}{\rho h} \quad (7.3.1)$$

where $q(r,t)$ is the lateral force intensity, $\rho$ is the mass density of plate material and $c^2 = D/\rho h$.

The boundary and initial conditions of the problem are

$$\omega = 0, \quad \frac{\partial \omega}{\partial r} + 0 \cdot \frac{\partial \omega}{\partial r} = 0; \quad \text{for } r = a \quad (7.3.2)$$

$$\omega = 0, \quad \frac{\partial \omega}{\partial t} = 0; \quad \text{at } t = 0 \quad (7.3.3)$$

The boundary conditions (7.3.2) imply that the plate is held along the outer edge $r = a$ in such a way that the displacement of the plate is zero and also so that the mean curvature vanishes there.

### 7.4 Solution of the Problem

We apply finite Hankel transform to obtain the solution of (7.3.1). Its solution is obtained as (3),

$$\omega(r,t) = A \sum_i \frac{f(\xi_i)}{\omega_i} R(\xi_i t) \quad (7.4.1)$$
where
\[ A = \frac{[(\tau + 1)]^2 (1 + x + \eta)}{\eta! [(1 + \alpha)]^2} \int_0^\infty \sum_{\beta=0}^\infty \frac{(-1)^\beta \left( \frac{a \xi_i}{2} \right)^{2\beta}}{\beta! [(\beta + \tau + 2)]^3} \left[ \begin{array}{c} -\eta, 1 + x + \beta + \eta, \beta + 1; \\ 1 + \alpha, \sigma + \beta + 2; \end{array} \right] \]
\[ \times \frac{J_0(\gamma \xi_i)}{[J_1(a \xi_i)]^2} \]
\( f(x \xi_i) = \int_0^t \psi(z) \sin \omega_i(t-z) \, dz \)
\[ R(\xi_i, t) = \int_0^t \psi(z) \sin \omega_i(t-z) \, dz \]
\[ \omega_i = \omega \xi_i^2 \]
and the sum is taken over all the positive roots of the equation
\[ J_0(a \xi_i) = 0 \]

7.5 VERIFICATION OF THE SOLUTION:

With the help of (7.4.1) and ((7)p.100, (5.24, (5.25)), we can obtain the values of
\[ \frac{1}{\gamma} \frac{\partial}{\partial \gamma} \left( \gamma \frac{\partial \omega}{\partial \gamma} \right) \]
and
\[ \frac{1}{\gamma} \frac{\partial}{\partial \gamma} \left( \gamma \frac{\partial \omega}{\partial \gamma} \right) \left( \frac{\partial^2 \omega}{\partial \gamma^2} + \frac{1}{\gamma} \frac{\partial \omega}{\partial \gamma} \right) \]
Using (7.1.1), (7.1.2) and (7.2.4), the value of \( y(x, t) \) may be obtained and from (7.4.1) we can find the value of \( \frac{\partial^2 \omega}{\partial t^2} \). Substituting these values in (7.3.1), we see that the equation is satisfied.

The boundary conditions (7.3.2) are satisfied, because
\[ J_0(a \xi_i) \] which is present in every term of \( \psi(a, t) \) and
\[
\frac{1}{\sqrt{y}} \frac{\partial}{\partial y} \left( \frac{\partial}{\partial y} \right) w(a, t) \text{ is zero. The initial conditions (7.3.3) are satisfied because } R(\xi, 0) \text{ and } \frac{\partial}{\partial t} R(\xi, 0) \text{ both vanish.}
\]

7.6 PARTICULAR CASES:

The results for some well known pulse shapes are recorded in the present section. For:

(i) Rectangular Pulse:
\[
\psi(t) = \begin{cases} 
\psi_{\text{max}}, & 0 \leq t \leq t_o, \\
0, & t > t_o
\end{cases}
\]

we have from (7.4.1)
\[
W(y, t) = \psi_{\text{max}} \sum_i \frac{f(v \xi_i)}{w_i^2} \left( 1 - \cos(w_i t) \right), 0 \leq t \leq t_o
\]

\[
= A \psi_{\text{max}} \sum_i \frac{f(v \xi_i)}{w_i^2} \left[ \cos(w_i(t-t_o)) - \cos(w_i t) \right] \quad (7.6.3)
\]

similarly for

(ii) Triangular Pulse:
\[
\psi(t) = \begin{cases} 
\frac{2t}{t_o} \psi_{\text{max}}, & 0 \leq t \leq \frac{t_o}{2} \\
2 \left( 1 - \frac{t}{t_o} \right) \psi_{\text{max}}, & \frac{t_o}{2} \leq t \leq t_o \\
0, & t > t_o
\end{cases}
\]

\[
W(y, t) = A \psi_{\text{max}} \sum_i \frac{f(v \xi_i)}{w_i^2} \left[ \frac{2t}{t_o} \frac{\sin(w_i t)}{t_o w_i} - \frac{2 \sin(w_i t)}{t_o w_i} \right], 0 \leq t \leq \frac{t_o}{2}
\]

\[
= A \psi_{\text{max}} \sum_i \frac{f(v \xi_i)}{w_i^2} \left[ \frac{2(t_o - t)}{t_o} \frac{\sin(w_i t)}{t_o w_i} + \frac{4 \sin(w_i(t-t_o))}{t_o w_i^2} \right] \quad (7.6.5)
\]

\[
= A \psi_{\text{max}} \sum_i \frac{f(v \xi_i)}{w_i^2} \left[ \frac{4 \sin(w_i(t-t_o))}{t_o w_i} - \frac{1}{2} t_o \frac{\sin(w_i t)}{t_o w_i} \right] \quad (7.6.5)
\]

\[
= A \psi_{\text{max}} \sum_i \frac{f(v \xi_i)}{w_i^2} \left[ \frac{2 \sin(w_i t)}{t_o} - \frac{2 \sin(w_i(t-t_o))}{t_o w_i} \right], t > t_o
\]
(iii) **Harmonic Pulse** : 

\[ \psi(t) = \psi_{\text{max}} \sin \omega t, \quad 0 \leq t \leq \frac{2\omega \tau}{\lambda} \]
\[ \psi(t) = 0, \quad t > \frac{2\omega \tau}{\lambda} \]  

(7.6.6)

\[ w(r, t) = A \psi_{\text{max}} \sum_{i} \frac{f(r \xi_i)}{\omega_i^2 - \lambda^2} \left[ \frac{\sin \omega t - \lambda \sin \omega \tau}{\omega_i} \right] \]

end for

(iv) **Exponential Decay Pulse** :

\[ \psi(t) = \psi_{\text{max}} e^{-t/t_0}, \quad 0 \leq t < \infty \]  

(7.6.8)

\[ \psi(t) = A \psi_{\text{max}} \sum_{i} \frac{f(r \xi_i)}{(\omega_i^2 + \lambda^2)} \left[ e^{-\lambda t} + \frac{\alpha_i}{\omega_i} \sin \omega t - \alpha_i \omega t \right] \]

(7.6.9)

where \( \alpha_i = \frac{1}{t_0} \)

and \( f(r \xi_i) \), \( A \) and \( \omega_i \) are as defined in section (7.4).

7.7 **Behaviour of the Family of Load Shapes \( \Phi(\gamma) \)** :

From (7.1.2), we have

\[ \Phi(\gamma) = k_0 \left(1 - \frac{r^2}{a^2}\right)^{\sigma} \frac{\Gamma_{\alpha} \left(\gamma \beta\right)}{\Gamma_{\alpha} \left(\gamma \right)} \left(1 - \frac{2 \gamma^2}{a^2}\right) \]

\[ \Phi(\gamma) = k_0 \left(1 + \frac{\gamma^2}{a^2}\right)^{-\sigma} \frac{\Gamma_{\alpha} \left(\gamma \beta + 1\right)}{\Gamma_{\alpha} \left(\gamma \right)} \left(1 - \frac{2 \gamma^2}{a^2}\right) \]

(7.7.1)

We shall consider three cases here:

(i) Let \( n = 1 \), and \( \sigma > 0 \); then

\[ \Phi(\gamma) = \left(1 - \frac{\gamma^2}{a^2}\right)^{\sigma} \left[1 - \frac{\gamma^2 + \beta + 1}{a^2}\right] k_0 \left(1 + \frac{\gamma^2}{a^2}\right) \]  

(7.7.2)
We have
\[ \Phi(r) = 0 \text{ when } r = a \left( \frac{1 + \lambda}{\lambda \beta + 2} \right)^{1/2} \text{ and } r = s. \]

This shows the dependency of \( r \) on \( \lambda \) and \( \beta \). It also indicates that the load shape \( \Phi(r) \) changes sign at
\[ \gamma = a \left( \frac{1 + \lambda}{\lambda \beta + 2} \right)^{1/2} \]
as is clear from (7.7.2), since
\[ \Phi(r) = +\omega \text{ when } 0 < \gamma < a \left( \frac{1 + \lambda}{\lambda \beta + 2} \right)^{1/2} \quad (7.7.3) \]
and
\[ \Phi(r) = -\omega \text{ when } a \left( \frac{1 + \lambda}{\lambda \beta + 2} \right)^{1/2} < \gamma < a. \]

It is interesting to observe that the spatial distribution of the pressure pulse over the plate area bounded by the circle
\[ \text{of the plate surface is subjected to a load acting in the negative direction of the } Z \text{ axis. If } \lambda \text{ is kept constant, increase in the value of } \beta \text{ increases the area of the annular region and for fixed } \beta, \text{ increase in the value of } \lambda \text{ decreases the annular region. Thus the parameters } \lambda \text{ and } \beta \text{ determine the radius of the inner circle.} \]

Stated in an alternative manner, there is a bifurcation point as the load shape is reduced to negative values maintaining the pulse shape positive; a bifurcation point is approached at
\[ r = a \left( \frac{1 + \lambda}{\lambda \beta + 2} \right)^{1/2} \]. When the pulse shape becomes negative, then the loading in different regions of the plate will interchange the roles. It demonstrates the impact of the pulse shape over the load shape considered.

(ii) When \( n = 0, \lambda = \beta = 0, \sigma > 0 \), we have
\[ \Phi(\gamma) = \lambda_0 \left( 1 - \frac{\gamma^2}{\sigma^2} \right)^{1/2} \quad (7.7.4) \]
This represents an axially symmetric distribution of the force over the plate, acting in the positive (downward) direction. For various values of $\phi$ we shall have a number of force distributions of different intensities.

(iii) Finally, when $\phi = \alpha = \beta = \sigma = 0$, then

$$\Phi(x) = K_0$$

(7.7.5)

Which denotes a uniform force distribution of magnitude $K_0$ over the plate area. This particular type of loading with rectangular pulse shape (7.6.1) has been considered by Sneddon (2) in one of his investigations.

It is clear from the above analysis that the single function $\Phi(x)$ represents several types of loadings and therefore the result obtained in the present work is capable of unifying scores of hitherto scattered results in the concerned literature.

As for example, in the first series of problems, the load shape parameters $\alpha', \beta, n$, and $\sigma$ may be held fixed and the results may be obtained for various pulse shapes $\psi(t)$, such as given by equations (7.1.3) - (7.1.6) with $\psi_{\text{max}}$, being varied. A number of combinations of the parameters may be used for the purpose.

Secondly, the pulses shape should be held fixed and load shape may be altered by varying $\alpha', \beta, n$, and $\sigma$. Thus a series of problems regarding a fixed pulse shape may be studied for various load shapes.

It may also be pointed out at this stage that wherever the product of load shape and the pulse shape gives a negative value, the force governing the vibrations of the plate should be
treated as acting in the negative (upward) direction. Since the pulse shape may change its sign a large number of times in a finite interval, as for instance in the case of harmonic pulse, the driving force will also fluctuate a number of times. This justifies the statement that the exciting force should be characterised by the pulse shape.
REFERENCES


