CHAPTER 4
ITERATIVE ALGORITHMS FOR SOLVING
GENERALIZED NONLINEAR MIXED
VARIATIONAL-LIKE INEQUALITIES
62 - 80
CHAPTER 4

ITERATIVE ALGORITHMS FOR SOLVING GENERALIZED NONLINEAR MIXED VARIATIONAL-LIKE INEQUALITIES

In this chapter, we suggest predictor-corrector iterative scheme for solving a class of generalized mixed variational-like inequalities by using the concept of $(g,\eta)$-partially relaxed strong monotonicity set-valued mappings. The convergence analysis of algorithm only requires that the underlying mappings are continuous and $(g,\eta)$-partially relaxed strongly monotone. Our result generalizes the corresponding results of (see [25], [29], [30], [53], [132], [134]).

In Section 4.1, we introduced the concept of $(g,\eta)$-partially relaxed strongly monotone and other basic definitions are given. Section 4.2 contains the algorithm and the convergence theorem.

4.1. $(g,\eta)$-partially relaxed strongly monotone mapping

Let $H$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let $C(H)$ be the families of all nonempty compact subsets of $H$. Let $T, A: H \to C(H)$ be two set-valued mappings; $N, \eta: H \times H \to H$ be single-valued mappings, $g: H \to H$ be a single-valued invertible ma-
ping and \( \varphi : H \rightarrow (-\infty, \infty) \) be a real function.

We consider the following generalized nonlinear mixed variational-like inequalities problem:

To find \( x \in H, u \in T(x), v \in A(x) \) such that

\[
\langle N(u, v), \eta(g(y), g(x)) \rangle + \varphi(g(y)) - \varphi(g(x)) \geq 0 \quad \text{for all } g(y) \in H. \quad (4.1.1)
\]

**Special cases:**

1) If \( g = I \), the identity mapping, then (4.1.1) reduces to the following generalized mixed variational-like inequality problem given by Ding [29]:

To find \( x \in H, u \in T(x) \) and \( v \in A(x) \) such that

\[
\langle N(u, v), \eta(y, x) \rangle + \varphi(y) - \varphi(x) \geq 0 \quad \text{for all } y \in H. \quad (4.1.2)
\]

2) If \( \eta(x, y) = g(x) - g(y) \) for all \( x, y \in H \) where \( g : H \rightarrow H \) be single-valued mapping, then (4.1.2) reduces to the generalized mixed variational inequality problem given by Ding [25]:

To find \( x \in H, u \in T(x) \) and \( v \in A(x) \) such that

\[
\langle N(u, v), g(y) - g(x) \rangle + \varphi(y) - \varphi(x) \geq 0 \quad \text{for all } y \in H. \quad (4.1.3)
\]

It can be readily seen that (4.1.1) includes a number of extensions and generalizations of variational and variational-like inequalities studied in [25], [29], [30], [79], [132], [134] and the references therein.

Now here we give some definitions:
Definition 4.1.1. Let $T, A : H \to C(H)$ be two set-valued mappings, $N, N : H \times H \to H$ be two single-valued mappings and $g : H \to H$ be a single-valued one-one onto mapping. Then

1) $N(.,.)$ is said to be partially relaxed $(g, \eta)$-strongly monotone in first argument with respect to $T$ if there exists a constant $\alpha > 0$ such that

$$\langle N(u, .) - N(u_2, .), \eta(g(z), g(y)) \rangle \geq -\alpha \| g(x) - g(z) \|^2$$

for all $x, y, z \in H, u_1 \in T(x), u_2 \in T(y)$.

Similarly, we can define the partially relaxed $(g, \eta)$-strong monotonicity of $N(.,.)$ in second argument with respect to $A$.

2) $N(.,.)$ is said to be $(g, \eta)$-strongly monotone in first argument with respect to $T$ if there exists a constant $\alpha > 0$ such that

$$\langle N(u, .) - N(u_2, .), \eta(g(x), g(y)) \rangle \geq \alpha \| g(x) - g(y) \|^2$$

for all $x, y \in H, u_1 \in T(x), u_2 \in T(y)$.

3) $T$ is said to be $D$-continuous on $H$ if $\{x_n\} \subset H$ and $x_n \to x^*$, then $T(x_n) \to T(x^*)$ under the Hausdorff metric $D$ on $C(H)$.

Definition 4.1.2. Hausdorff metric: Let $H$ be a real Hilbert space. Let $B(H)$ denotes the family of all nonempty closed bounded subset of $H$. Let $G : H \to CB(H), \varepsilon > 0$ be any real number, then for every $u_1, u_2 \in H$ and $v_1 \in G(u_1)$, there exist $v_2 \in G(u_2)$ such that

$$\|v_1 - v_2\| \leq D(G(u_1), G(u_2)) + \varepsilon \|u_1 - u_2\|,$$
here $D(.,.)$ is the Hausdorff metric defined on $CB(H)$ by

$$D(B,C) = \max_{v \in B} \{ \sup_{u \in C} d(v, B), \sup_{u \in C} d(u, C) \},$$

for $B, C \in CB(H)$ and $d(v, B) = \min_{u \in B} d(v, u)$.

We note that if $G:H \rightarrow C(H)$, where $C(H)$ denotes the family of all nonempty compact subsets of $H$, then it is true for $e = 0$.

### 1.2. Iterative algorithm and its convergence analysis

In this section, we suggest and analyze some new predictor-corrector iterative algorithm for solving (4.1.1) by using the auxiliary variational inequality technique.

For given $x \in H$, $u \in T(x)$ and $v \in A(x)$, we consider the following auxiliary variational inequality problem:

Find $\hat{x} \in H$ such that

$$\langle g(\hat{x}) - g(x), g(y) - g(\hat{x}) \rangle + \langle pN(u,v), \eta(g(y), g(\hat{x})) \rangle + \rho \varphi(g(y)) - \rho \varphi(g(\hat{x})) \geq 0 \quad \text{for all } g(y) \in H,$$

(4.2.1)

where $\rho > 0$ is a constant. Observe that if $g(\hat{x}) = g(x)$, $\hat{u} \in T(\hat{x})$ and $\hat{v} \in A(\hat{x})$, then $(\hat{x}, \hat{u}, \hat{v})$ is a solution of (4.1.1). Thus, one can suggest the following predictor-corrector algorithm for solving the (4.1.1).

**Algorithm 4.2.1.** For given $x_0 \in H$, $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, compute an approximate solution $(x_n, u_n, v_n)$ of generalized nonlinear mixed var-
ational-like inequalities problem by the following iterative scheme:

\[
\langle g(y_n) - g(x_n), g(y) - g(y_n) \rangle + \langle \mu N(u_n, v_n), \eta(g(y), g(y_n)) \rangle \\
+ \mu \varphi(g(y)) - \mu \varphi(g(y_n)) \geq 0 \quad \text{for all } g(y) \in H, \quad (4.2.2)
\]

\[
\langle g(z_n) - g(y_n), g(y) - g(z_n) \rangle + \langle \beta N(c_n, d_n), \eta(g(y), g(z_n)) \rangle \\
+ \beta \varphi(g(y)) - \beta \varphi(g(z_n)) \geq 0 \quad \text{for all } g(y) \in H, \quad (4.2.3)
\]

\[
\langle g(x_{n+1}) - g(z_n), g(y) - g(x_{n+1}) \rangle + \langle \rho N(c_n, f_n), \eta(g(y), g(x_{n+1})) \rangle \\
+ \rho \varphi(g(y)) - \rho \varphi(g(x_{n+1})) \geq 0 \quad \text{for all } g(y) \in H, \quad (4.2.4)
\]

\[
\begin{align*}
&u_n \in T(x_n), \quad \|u_{n+1} - u_n\| \leq D(T(x_{n+1}), T(x_n)), \\
v_n \in A(x_n), \quad \|v_{n+1} - v_n\| \leq D(A(x_{n+1}), A(x_n)), \\
c_n \in T(y_n), \quad \|c_{n+1} - c_n\| \leq D(T(y_{n+1}), T(y_n)), \\
d_n \in A(y_n), \quad \|d_{n+1} - d_n\| \leq D(A(y_{n+1}), A(y_n)), \\
e_n \in T(z_n), \quad \|e_{n+1} - e_n\| \leq D(T(z_{n+1}), T(z_n)), \\
f_n \in A(z_n), \quad \|f_{n+1} - f_n\| \leq D(A(z_{n+1}), A(z_n)),
\end{align*}
\]

here \( \mu > 0, \beta > 0, \rho > 0 \) are constants, and \( D \) is the Hausdorff metric on \( C(H) \).

If \( g = I \), then Algorithm 4.2.1 reduces to the following Predictor-Corrector iterative algorithm for solving the generalized mixed variational-like inequality problem (4.1.2).

Algorithm 4.2.2. For given \( x_0 \in H, u_0 \in T(x_0) \) and \( v_0 \in A(x_0) \), compute the approximate solution \( (x_n, u_n, v_n) \) by the following iterative scheme:
\[
\langle y_n - x_n, y - y_n \rangle + (\mu N(u_n, v_n), \eta(y, y_n))
\]
\[+ \mu \varphi(y) - \mu \varphi(y_n) \geq 0 \text{ for all } y \in H, \quad (4.2.6)\]
\[
\langle z_n - y_n, y - z_n \rangle + (\beta N(c_n, d_n), \eta(y, z_n))
\]
\[+ \beta \varphi(y) - \beta \varphi(z_n) \geq 0 \text{ for all } y \in H, \quad (4.2.7)\]
\[
\langle x_{n+1} - z_n, y - x_{n+1} \rangle + (\rho N(c_n, f_n), \eta(y, x_{n+1}))
\]
\[+ \rho \varphi(y) - \rho \varphi(x_{n+1}) \geq 0 \text{ for all } y \in H, \quad (4.2.8)\]

\[
\begin{cases}
  u_n \in T(x_n), & \| u_{n+1} - u_n \| \leq D(T(x_{n+1}), T(x_n)), \\
  v_n \in A(x_n), & \| v_{n+1} - v_n \| \leq D(A(x_{n+1}), A(x_n)), \\
  c_n \in T(y_n), & \| c_{n+1} - c_n \| \leq D(T(y_{n+1}), T(y_n)), \\
  d_n \in A(y_n), & \| d_{n+1} - d_n \| \leq D(A(y_{n+1}), A(y_n)), \\
  e_n \in T(z_n), & \| e_{n+1} - e_n \| \leq D(T(z_{n+1}), T(z_n)), \\
  f_n \in A(z_n), & \| f_{n+1} - f_n \| \leq D(A(z_{n+1}), A(z_n)), \\
\end{cases}, \quad n = 0, 1, 2, \ldots, (4.2.9)
\]

Here \( \mu > 0, \beta > 0, \rho > 0 \) are constants, and \( D \) is the Hausdorff metric in \( C(H) \).

If \( \eta(x, y) = g(x) - g(y) \) for all \( x, y \in H \), where \( g : H \rightarrow H \) be single-valued mapping, then Algorithm 4.2.2 reduces to the following predictor-corrector algorithm for solving the generalized mixed variational inequality problem (4.1.3).

**Algorithm 4.2.3.** For given \( x_0 \in H, u_0 \in T(x_0) \) and \( v_0 \in A(x_0) \), compute the approximate solution \( (x_n, u_n, v_n) \) by the following iterative schemes
\begin{align}
\langle y_n - x_n, y - y_n \rangle &+ \langle \mu N(u_n, v_n), g(y) - g(y_n) \rangle \\
&+ \mu \eta(y) - \mu \eta(y_n) \geq 0 \text{ for all } y \in H, ~(4.2.10) \\
\langle z_n - y_n, y - z_n \rangle &+ \langle \beta N(c_n, d_n), g(y) - g(z_n) \rangle \\
&+ \beta \eta(y) - \beta \eta(z_n) \geq 0 \text{ for all } y \in H, ~(4.2.11) \\
\langle x_{n+1} - z_n, y - x_{n+1} \rangle &+ \langle \rho N(e_n, f_n), g(y) - g(x_{n+1}) \rangle \\
&+ \rho \eta(y) - \rho \eta(x_{n+1}) \geq 0 \text{ for all } y \in H, ~(4.2.12)
\end{align}

\begin{align*}
\begin{cases}
 u_n \in T(x_n), & \|u_{n+1} - u_n\| \leq D(T(x_{n+1}), T(x_n)), \\
v_n \in A(x_n), & \|v_{n+1} - v_n\| \leq D(A(x_{n+1}), A(x_n)), \\
c_n \in T(y_n), & \|c_{n+1} - c_n\| \leq D(T(y_{n+1}), T(y_n)), \\
d_n \in A(y_n), & \|d_{n+1} - d_n\| \leq D(A(y_{n+1}), A(y_n)), \\
e_n \in T(z_n), & \|e_{n+1} - e_n\| \leq D(T(z_{n+1}), T(z_n)), \\
f_n \in A(z_n), & \|f_{n+1} - f_n\| \leq D(A(z_{n+1}), A(z_n)),
\end{cases} \quad n = 0, 1, 2, \ldots, (4.2.13)
\end{align*}

\text{Here } \mu > 0, \beta > 0, \rho > 0 \text{ are constants.}

If } \eta(x, y) = x - y \text{ for all } x, y \in H, \text{ then Algorithm 4.2.2 reduces to the following predictor-corrector algorithm for solving the mixed variational inequality problem.}

\textbf{Algorithm 4.2.4.} \text{ For given } x_0 \in H, u_0 \in T(x_0) \text{ and } v_0 \in A(x_0), \text{ compute approximate solution } (x_n, u_n, v_n) \text{ of the generalized mixed variational inequality by the following iterative schemes:}

\begin{align}
\langle y_n - x_n, y - y_n \rangle &+ \langle \mu N(u_n, v_n), y - y_n \rangle
\end{align}
If \( \varphi \) is a proper convex and lower semicontinuous function on \( H \), then Algorithm 4.2.4 can be rewritten as follows.

**Algorithm 4.2.5.** For given \( x_0 \in H, u_0 \in T(x_0) \) and \( v_0 \in A(x_0) \), compute an approximate solution \( (x_n, u_n, v_n) \), by the following iterative schemes:

\[
\begin{align*}
y_n &= J^{\mu \varphi}_{\mu \varphi}(x_n - \mu N[u_n, v_n]), \\
z_n &= J^{\beta \varphi}_{\beta \varphi}(y_n - \beta N[c_n, d_n]), \\
x_{n+1} &= J^{\rho \varphi}_{\rho \varphi}(z_n - \rho N[e_n, f_n]),
\end{align*}
\]

(4.2.18)
\[ \begin{align*}
|u_n \in T(x_n), \quad & \|u_{n+1} - u_n\| \leq D(T(x_{n+1}), T(x_n)), \\
v_n \in A(x_n), \quad & \|v_{n+1} - v_n\| \leq D(A(x_{n+1}), A(x_n)), \\
c_n \in T(y_n), \quad & \|c_{n+1} - c_n\| \leq D(T(y_{n+1}), T(y_n)), \\
d_n \in A(y_n), \quad & \|d_{n+1} - d_n\| \leq D(A(y_{n+1}), A(y_n)), \\
e_n \in T(z_n), \quad & \|e_{n+1} - e_n\| \leq D(T(z_{n+1}), T(z_n)), \\
f_n \in A(z_n), \quad & \|f_{n+1} - f_n\| \leq D(A(z_{n+1}), A(z_n)),
\end{align*} \]

(4.2.19)

Here, \( J_\rho^{(m)} = (I + \rho \partial \varphi(\cdot))^{-1} \) is the resolvent operator associated with the associated differential \( \partial \varphi(\cdot) \), and \( \mu > 0, \beta > 0, \rho > 0 \) are constants.

Algorithm 4.2.5 is a three-step forward-backward splitting algorithm for solving the generalized mixed variational inequality problem.

Before presenting main result of this chapter, we prove the following:

**Lemma 4.2.1.** Let \((x, u, v)\) be an exact solution of (4.1.1) and \(\{x_n\}, \{u_n\}, \{v_n\}\) be the sequence of approximate solution of (4.1.1) generated by Algorithm 4.2.1. Suppose \( \eta(g(x), g(y)) = -\eta(g(y), g(x)) \) for all \( x, y \in H \). If \( N(\cdot, \cdot) \) is partially relaxed \((g, \eta)\)-strongly monotone in the first and second arguments with respect to \( T \) and \( A \) with constants \( \alpha > 0 \) and \( \gamma > 0 \), respectively. Then

\[ \begin{align*}
\|x_{n+1} - g(x)\|^2 & \leq \|g(x_{n+1}) - g(x)\|^2 - (1 - 2\rho(\alpha + \gamma))\|g(x_{n+1}) - g(z_n)\|^2, \\
\|y_n - g(y)\|^2 & \leq \|g(y) - g(y_n)\|^2 - (1 - 2\rho(\alpha + \gamma))\|g(y_n) - g(y)\|^2,
\end{align*} \]

(4.2.20) (4.2.21)
\[ (y_n) - g(x) \| ^2 \leq \| g(y_n) - g(x) \| ^2 - (1 - 2\mu(\alpha + \gamma)) \| g(y_n) - g(x) \| ^2. \]  
(4.2.22)

**Proof.** Let \((x, u, v)\) be a solution of (4.1.1), then \(u \in T(x), v \in A(x)\) and

\[ \begin{align*}
N(u, v), \eta(g(y), g(x)) + \mu \varphi(g(y)) - \mu \varphi(g(x)) \geq 0 & \text{ for all } g(y) \in H, \\
N(u, v), \eta(g(y), g(x)) + \beta \varphi(g(y)) - \beta \varphi(g(x)) \geq 0 & \text{ for all } g(y) \in H, \\
N(u, v), \eta(g(y), g(x)) + \rho \varphi(g(y)) - \rho \varphi(g(x)) \geq 0 & \text{ for all } g(y) \in H,
\end{align*} \]

(4.2.23) (4.2.24) (4.2.25)

where \(\mu > 0, \beta > 0\) and \(\rho > 0\) are constants. Taking \(y = x_n\) in (4.2.25), we get

\[ \text{Id } y = x \text{ in (4.2.4), we get} \]

\[ \langle \rho N(u, v), \eta(g(x_{n+1}), g(x)) \rangle + \rho \varphi(g(x_{n+1})) - \rho \varphi(g(x)) \geq 0, \]
(4.2.26)

\[ \langle g(x_{n+1}) - g(z_n), g(x) - g(x_{n+1}) \rangle + \langle \rho N(e_n, f_n), \eta(g(x), g(x_{n+1})) \rangle + \rho \varphi(g(x)) - \rho \varphi(g(x_{n+1})) \geq 0. \]
(4.2.27)

that \(\eta(g(x), g(y)) = -\eta(g(y), g(x))\) for all \(x, y \in X\), adding (4.2.26) and (4.2.27), we get

\[ \begin{align*}
\langle x_{n+1} - g(z_n), g(x) - g(x_{n+1}) \rangle \geq & \rho \langle N(e_n, f_n) - N(u, v), \eta(g(x_{n+1}), g(x)) \rangle \\
& + \rho \langle N(u, v) - N(u, v), \eta(g(x_{n+1}), g(x)) \rangle \\
& + \rho \langle N(u, v) - N(u, v), \eta(g(x_{n+1}), g(x)) \rangle \\
& \geq - \rho(\alpha + \gamma) \| g(x_{n+1}) - g(z_n) \| ^2, \]
(4.2.28)

where we have used the assumption that \(N(\cdot, \cdot)\) is partially relaxed \(\eta\)-strongly monotone in first and second arguments with respect to \(\alpha\) and \(\beta\) with constants \(\alpha > 0\) and \(\gamma > 0\), respectively. Since
follows from (4.2.28) that
\[
\langle g(x_{n+1}) - g(z_n), g(x) - g(x_{n+1}) \rangle = \frac{1}{2} \left[ \| g(x) - g(z_n) \|^2 - \| g(x_{n+1}) - g(z_n) \|^2 - \| g(x_{n+1}) - g(x) \|^2 \right]
\]
\[
\geq - \rho (\alpha + \gamma) \| g(x_{n+1}) - g(z_n) \|^2.
\]
Therefore, we get that for \( \rho < 1/2 (\alpha + \gamma) \),
\[
\| x_{n+1} - g(x) \|^2 \leq \| g(z_n) - g(x) \|^2 - (1 - 2\rho (\alpha + \gamma)) \| g(x_{n+1}) - g(z_n) \|^2
\tag{4.2.29}
\]
Taking \( y = z_n \) in (4.2.24) and \( y = x \) in (4.2.3), we have
\[
\langle \beta N(u, v), \eta(g(z_n), g(x)) + \beta \phi(g(z_n)) - \beta \phi(g(x)) \rangle \geq 0, \tag{4.2.30}
\]
\[
\langle g(z_n) - g(y_n), g(x) - g(z_n) \rangle + \langle \beta N(c_n, d_n), \eta(g(x), g(z_n)) \rangle
\]
\[
+ \beta \phi(g(x)) - \beta \phi(g(z_n)) \geq 0. \tag{4.2.31}
\]
Taking \( \eta(g(x), g(y)) = -\eta(g(y), g(x)) \) for all \( x, y \in X \), adding (4.2.30)
\tag{4.2.31}, we get
\[
z_n - g(y_n), g(x) - g(z_n) \rangle \geq \beta \langle N(c_n, d_n) - N(u, v), \eta(g(z_n), g(x)) \rangle
\]
\[
\geq \beta \langle N(c_n, d_n) - N(u, d_n), \eta(g(z_n), g(x)) \rangle
\]
72
+ \beta \langle N(u,d_n) - N(u,v), \eta(g(z_n), g(x)) \rangle

\geq - \beta (\alpha + \gamma) \| g(z_n) - g(y_n) \|^2, \hspace{1cm} (4.2.32)

due to the assumption that \( N(.,.) \) is partially relaxed \((g,\eta)\)
monotone in the first and second arguments with constant \( \alpha > 0 \)
and \( \gamma > 0 \), respectively. Since

\[ \| g(x) - g(y_n) \|^2 = \| g(x) - g(z_n) + g(z_n) - g(y_n) \|^2 \]

\[ = \| g(z_n) - g(x) \|^2 + \| g(z_n) - g(y_n) \|^2 \]

\[ + 2 \langle g(z_n) - g(y_n), g(x) - g(z_n) \rangle \]

on (4.2.32), we have

\[ \langle g(z_n) - g(y_n), g(x) - g(z_n) \rangle = \frac{1}{2} \left[ \| g(y_n) - g(x) \|^2 - \| g(z_n) - g(y_n) \|^2 \right] \]

\[ \geq - \beta (\alpha + \gamma) \| g(z_n) - g(y_n) \|^2. \]

Therefore, for \( \beta < 1/(\alpha + \gamma) \),

\[ g(z_n) - g(x) \|^2 \leq \| g(y_n) - g(x) \|^2 - (1 - 2\beta (\alpha + \gamma)) \| g(z_n) - g(y_n) \|^2 \]

\[ \leq \| g(y_n) - g(x) \|^2. \hspace{1cm} (4.2.33) \]

Using \( y = y_n \) in (4.2.23) and \( y = x \) in (4.2.2), we have

\[ \langle \mu N(u,v), \eta(g(y_n), g(x)) \rangle + \mu \varphi(g(y_n)) - \mu \varphi(g(x)) \geq 0, \hspace{1cm} (4.2.34) \]

\[ \langle g(y_n) - g(x_n), g(x) - g(y_n) \rangle + \mu N(u,v_n), \eta(g(x), g(y_n)) \]

\[ + \mu \varphi(g(x)) - \mu \varphi(g(y_n)) \geq 0. \hspace{1cm} (4.2.35) \]
\[ g(4.2.34) \text{ and } (4.2.35), \text{ and using } \eta(g(x), g(y)) = -\eta(y, g(x)), \text{ we have} \]

\[ \langle y_n - g(x_n), g(x) - g(y_n) \rangle \geq \mu \langle N(u, v_n) - N(u, v), \eta(g(y_n), g(x)) \rangle \]
\[ \geq \mu \langle N(u, v_n) - N(u, v), \eta(g(y_n), g(x)) \rangle \]
\[ + \mu \langle N(u, v_n) - N(u, v), \eta(g(y_n), g(x)) \rangle \]
\[ \geq -\mu (\alpha + \gamma) \| g(y_n) - g(x_n) \|^2, \quad (4.2.36) \]

where we have used the assumption that \( N(., .) \) is partially relaxed strongly monotone in the first and second arguments with \( \alpha \) to \( T \) and \( \gamma \) to \( A \) with constants \( \alpha > 0 \) and \( \gamma > 0 \), respectively. Since

\[ \| g(x) - g(x_n) \|^2 = \| g(x) - g(y_n) + g(y_n) - g(x_n) \|^2 \]
\[ = \| g(y_n) - g(x) \|^2 + \| g(y_n) - g(x_n) \|^2 + 2 \langle g(y_n) - g(x_n), g(x) - g(y_n) \rangle, \]

follows from (4.2.36) that

\[ \langle g(y_n) - g(x_n), g(x) - g(y_n) \rangle = \frac{1}{2} \left[ \| g(x_n) - g(x) \|^2 - \| g(y_n) - g(x_n) \|^2 \right] \]
\[ \geq -\mu (\alpha + \gamma) \| g(y_n) - g(x_n) \|^2. \]

Therefore, we get that for \( \mu < 1/2 (\alpha + \gamma) \),

\[ \| g(y_n) - g(x) \|^2 \leq \| g(x_n) - g(x) \|^2 - (1 - 2\mu (\alpha + \gamma)) \| g(y_n) - g(x_n) \|^2 \]
\[ \leq \| g(x_n) - g(x) \|^2. \quad (4.2.37) \]

74
Combining (4.2.29), (4.2.33) and (4.2.37), it is easy to see that conclusions (4.2.20), (4.2.21) and (4.2.22) hold.

Now, we denote the solution set $M^*$ of (4.1.1) as follows:

$$(x, u, v) \in H \times H \times H: u \in T(x), v \in A(x) \text{ and } \langle N(u, v), \eta(g(y), g(x)) \rangle + \varphi(g(y)) - \varphi(g(x)) \leq 0 \text{ for all } g(y) \in H.$$  

Now we are in a position to establish our main result:

**Theorem 4.2.1.** Let $H$ be a finite-dimensional Hilbert space, $T$, $A$: $H \rightarrow H$ be D-continuous set-valued mapping and $N, \eta: H \times H \rightarrow H$ and $\eta: H \rightarrow \mathbb{R}$ are continuous single-valued mappings such that $g$ is invertible and $\eta(g(x), g(y)) = -\eta(g(y), g(x))$ for all $x, y \in H$.

Let $\varphi: H \rightarrow (-\infty, \infty)$ be a lower semicontinuous. Suppose that $\eta$ is partially relaxed $(g, \eta)$-strongly monotone in first and second arguments with respect to $T$ and $A$ with constants $\alpha > 0$ and $\gamma > 0$, respectively, and the solution set $M^*$ of generalized nonlinear mixed variational-like inequalities problem (4.1.1) is nonempty. Then for any $x_0, u_0 \in T(x_0)$ and $v_0 \in A(x_0)$ the iterative sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ defined by Algorithm 4.2.1, with $0 < \rho$, $\beta$, $\mu < 1/2 (\alpha + \gamma)$ converges strongly to a solution $(\hat{x}, \hat{u}, \hat{v})$ of (4.1.1).

**Proof.** For any $(x, u, v) \in M^*$, from (4.2.20)-(4.2.22) in Lemma 4.2.1 it
allows that the sequences \( \| g(x_{n+1}) - g(x) \|, \| g(z_n) - g(x) \| \) and \( \| g(y_n) - g(x) \| \) are non-increasing and hence \( \{x_n\}, \{z_n\} \) and \( \{y_n\} \) are bounded. Furthermore, we have

\[
\sum_{n=0}^{\infty} (1 - 2p(a + \gamma)) \| g(x_{n+1}) - g(z_n) \|^2 \leq \| g(x_0) - g(x) \|^2,
\]

\[
\sum_{n=0}^{\infty} (1 - 2q(a + \gamma)) \| g(z_n) - g(y_n) \|^2 \leq \| g(z_0) - g(x) \|^2,
\]

\[
\sum_{n=0}^{\infty} (1 - 2u(a + \gamma)) \| g(y_n) - g(x_n) \|^2 \leq \| g(y_0) - g(x) \|^2.
\]

These inequalities imply \( \| g(x_{n+1}) - g(z_n) \| \to 0, \| g(z_n) - g(y_n) \| \to 0 \) and \( g(y_n) - g(x_n) \| \to 0 \) as \( n \to \infty \).

Therefore, we have

\[
\| g(x_{n+1}) - g(x_n) \| \leq \| g(x_{n+1}) - g(z_n) \| + \| g(z_n) - g(y_n) \| + \| g(y_n) - g(x_n) \| \to 0
\]

as \( n \to \infty \).

Since \( \{x_n\} \) is bounded, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) such that \( x_{n_k} \to \bar{x} \) and hence we have \( g(x_{n_k}) \to g(\bar{x}) \) and \( g(y_{n_k}) \to g(\bar{x}) \).

Since \( T \) and \( A \) are \( D \)-continuous on \( H \), by Proposition 1.5.2 of Aubin and Cellina [4.9, p.66], \( T \) and \( A \) are both upper semicontinuous on \( H \). Since \( u_n \in T(x_n) \) and \( v_n \in A(x_n) \) for all \( n = 0, 1, 2, \ldots \), it follows from Proposition 11.11 of Border [4.10, p.57] that there exist subsequence \( u_{n_k} \) of \( \{u_n\} \) and subsequence \( v_{n_k} \) of \( \{v_n\} \) such that \( u_{n_k} \to \hat{u}, v_{n_k} \to \hat{v} \)
\[ \hat{u} \in T(\hat{x}) \text{ and } \hat{v} \in A(\hat{x}), \text{ respectively. By (4.2.2), we have} \]
\[ \langle g(y_n) - g(x_n), g(y) - g(y_n) \rangle + \left( \mu N(u_n, v_n), \eta(g(y), g(y_n)) \right) \]
\[ + \mu \varphi(g(y)) - \mu \varphi(g(y_n)) \geq 0 \text{ for all } g(y) \in H. \quad (4.2.38) \]

By the continuity of \( N(\cdot, \cdot), \eta(\cdot, \cdot) \) and \( g \) and the lower semicontinuity of \( \varphi \), letting \( j \to \infty \) in (4.2.38), we obtain
\[ \langle N(\hat{u}, \hat{v}), \eta(g(y), g(\hat{x})) \rangle + \varphi(g(y)) - \varphi(g(\hat{x})) \geq 0 \text{ for all } g(y) \in H, \]
i.e., \( (\hat{x}, \hat{u}, \hat{v}) \) is a solution of (4.1.1). Since (4.2.20) in Lemma 4.2.2 holds for any \( (x, u, v) \in M^* \). Hence
\[ \| g(x_n) - g(\hat{x}) \| \leq \| g(x_n) - g(\hat{x}) \| \text{ for all } n = 0, 1, 2, 3..., \]
which implies that \( g(x_n) \to g(\hat{x}) \) as \( n \to \infty \). Since \( g \) is invertible, we have \( x_n \to \hat{x} \) as \( n \to \infty \). Since \( T \) and \( A \) are \( D \)-continuous on \( H \), by (4.2.5), we have
\[ \| u_n - u_n' \| \leq D(T(x_n), T(x_n')) \to 0, \text{ as } n \to \infty. \]
It follows that for any \( n > 0 \), we have
\[ \| u_n - \hat{u} \| \leq \| u_n - u_n' \| + \| u_n' - u_n'' \| + \cdots + \| u_n'' - \cdots - u_n - \hat{u} \| \to 0, \]
as \( n \to \infty \), i.e., \( u_n \to \hat{u} \) as \( n \to \infty \). Similarly, we can prove that \( v_n \to \hat{v} \) as \( n \to \infty \). This completes the proof. \( \blacksquare \)

**Corollary 4.2.1.** (Theorem 3.1, Ding [29]): Let \( H \) be a finite-dimensional Hilbert space, \( T, A : H \to C(H) \) be \( D \)-continuous set-valued
mapping and \( N, \eta : H \times H \to H \) are continuous single-valued mappings such that \( \eta(x, y) = -\eta(y, x) \) for all \( x, y \in H \). Let \( \varphi : H \to (-\infty, \infty) \) be a lower semicontinuous. Suppose that \( N(.,.) \) is partially relaxed \( \eta \)-strongly monotone in first and second arguments with respect to \( T \) and \( A \) with constants \( \alpha > 0 \) and \( \gamma > 0 \), respectively, and the solution set \( \text{Sol}(2.1) \) of \( \text{GMVLIP}(2.1) \) is nonempty. Then for any \( x_0 \in H, u_0 \in T(x_0) \) and \( v_0 \in A(x_0) \) the iterative sequences \( \{x_n\}, \{u_n\} \) and \( \{v_n\} \) defined by Algorithm 3.1, with \( 0 < \rho, \beta, \mu < \frac{1}{2(\alpha + \gamma)} \) converges strongly to a solution \((\hat{x}, \hat{u}, \hat{v})\) of \( \text{GMVLIP}(2.1) \).

**Corollary 4.2.2.** (Theorem 3.1, Ding [25]): Let \( H \) be a finite-dimensional Hilbert space, \( T, A : H \to C(H) \) be \( \bar{H} \)-continuous set-valued mapping and \( N : H \times H \to H \) and \( g : H \to H \) are continuous single-valued mappings such that \( g \) is invertible.

Let \( \varphi : H \to (-\infty, \infty) \) be a lower semicontinuous. Suppose that \( N(.,.) \) is \( g \)-partially relaxed strongly monotone in first and second arguments with respect to \( T \) and \( A \) with constants \( \alpha > 0 \) and \( \gamma > 0 \), respectively, and the solution set \( \text{Sol}(2.1) \) of \( \text{GNMVIP}(2.1) \) is nonempty. Then for any given \( x_0 \in H, u_0 \in T(x_0) \) and \( v_0 \in A(x_0) \) the iterative sequences \( \{x_n\}, \{u_n\} \) and \( \{v_n\} \) defined by Algorithm 3.1, with \( 0 < \rho, \beta, \mu < \frac{1}{2(\alpha + \gamma)} \) converges strongly to a solution \((\hat{x}, \hat{u}, \hat{v})\) of \( \text{GNMVIP}(2.1) \).
REFERENCES


* * * * *