INTRODUCTION

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CHAPTER 1

INTRODUCTION

Stampacchia [136] in 1963-64 pioneered the theory of variational inequality. Variational inequality theory is very effective and powerful tool for studying a wide class of linear and nonlinear problems arising in many diverse fields of pure and applied sciences. These inequalities are an efficient tool in studying the existence of solutions of constrained problems arising in mechanics, optimization and control, operation research, engineering sciences etc. Variational inequalities can be used to study the problems of fluid flow through porous media (Bensousan [9]), contact problems in elasticity (Kikuchi and Oden [49]), transportation problems ( Berteskas [10], Harker [36]), assignment problems (Marcotte [61]), game theory (Border [11]), free and moving boundary value problem (Crank [16]) and economic equilibrium (Dafermos [18]). Ideas explaining these formulations led to the development of new and powerful techniques to solve a wide class of problems.

A branch of mathematics concerned with invariant points of operators is called fixed point theory. The fixed point formulation of any variational inequality problem is not only useful for existence of solution of the variational inequality problem, but it also provides the
facility to develop algorithms for approximation of solution of variational inequality problem. With this point of view the fixed point theory plays an important role in the theory of variational inequality.

1.1. Let \( H \) be a real Hilbert space whose inner product and norm are denoted by \( \langle ., . \rangle \) and \( \| . \| \), respectively. Let \( C \) be a closed convex subset of \( H \).

Let \( T : H \to H \) is a single-valued operator. We consider the problem to find \( u \in C \) such that

\[
\langle Tu, v - u \rangle \geq 0 \quad \text{for all} \quad v \in C,
\]

is known as the variational inequality (Stampacchia [136]).

Lions and Stampacchia [55] introduced projection method and studied convergence analysis of this method using fixed point formulation:

\[
u = P_C(u - \rho Tu),
\]

where \( P_C \) is projection of \( H \) onto \( C \) and \( \rho > 0 \) is a constant.

This alternative formulation has played a significant part in developing various projection-type methods for solving variational inequalities (Noor [70, 72, 109]).

For approximation of solution of (1.1.1) projection type method requires that the operator \( T \) must be strongly monotone and Lipschitz
continuous. Due to these strict conditions most of the applications based on projection methods are difficult to solve, hence modification in the methods became essential. One can modify the fixed point formulation (1.1.2) by the updating solution $u$ in another fixed point formulation:

$$u = P_c[u - \rho TP_c(u - \rho Tu)].$$  \hspace{1cm} (1.1.3)

This fixed point formulation is also used to develop algorithm for approximation of the solution of variational inequalities. This method is called the extragradient method, which was proposed by Korpelevich [50]. This method has been extensively studied by Sun [137], Khobotov [48], Marcotte [62] and many others.

The convergence of extragradient-type methods requires that a solution should exist and the monotone operator should Lipschitz continuous. If the operator involving in variational inequalities is not Lipschitz continuous, then the extragradient method and its different forms require extensive computational procedures. To overcome this weakness, several modified projection and extragradient-type methods were suggested and developed for solving the variational inequalities by Noor [82, 95, 109] and Tseng [140].

Motivated by the following fixed point formulation:

$$u = P_c(I - \rho T)P_c(I - \rho T)P_c(I - \rho T)u,$$ \hspace{1cm} (1.1.4)
Noor [87] introduced a splitting type method for solving variational inequalities, which is much similar to the splitting method of Peaceman and Rachford [120]. Noor's splitting method is also similar to the so-called splitting method of Glowinski and Le Tallec [34]. It has been shown in [34] that three-step schemes are numerically efficient for the application of the splitting method in partial differential equation (Ames [6]).

1.2. Noor [84] introduced and studied a new class of variational inequalities which is as follows:

Let $T, g: H \rightarrow H$ is a single-valued operator. We consider the problem to find $u \in C$ such that

$$\langle Tu, g(\nu) - g(u) \rangle \geq 0 \text{ for all } \nu \in C.$$  \hspace{1cm} (1.1.5)

The inequality of type (1.1.5) is known as general variational inequality.

The general variational inequalities have been used in solving the free, unilateral and moving boundary value problems arising in pure and applied sciences (see [16, 54, 123]).

In 2000, Noor [72] developed and studied a new approximation scheme for general variational inequalities. In 2003, Sahu [125] has introduced generalized Ishikawa iteration process of rank $r$ and studied the approximation of fixed points of nonexpansive mappings.
in Banach spaces. Due to the utility of splitting method we have used
the concept of both Noor [72] and Sahu [125].

The purpose of Chapter 2 is to introduce the system of general
variational inequalities and design propose the three-step iterative
algorithm for a system of general variational inequalities that appears
to be a new one. The convergence criterion is also discussed (see
Theorem 2.1.1). The convergence theorem generalizes the result of
Noor [72]. In Section 2.2 three-step perturbed iterative algorithm for a
system of general variational inequalities is introduced and its
convergence analysis is discussed (see Theorem 2.2.1).

Our results unify and improve several results known in the
theory of variational inequalities (see [45, 60, 72, 99, 111]).

1.3 In Section 1.1, we have already discussed that how iteration
processes are useful to approximate the solution of variational
inequalities. There are lot of research papers (see [2, 17, 21, 22, 25,
26, 28-30, 34, 37, 40, 42, 45, 47, 53, 57, 60, 61, 67-80, 82, 83, 88,
90, 92, 94, 95, 97-99, 104, 105, 108-111, 117, 118, 124, 125, 128,
134, 135, 140, 142]) which have been used to deal with the problem
of approximation of solution of various nonlinear problems by three
iteration processes which are Picard [121], Mann [60] and Ishikawa
[45].
The Picard iteration process is mostly used to find the approximate solution in variational inequalities. Note that the rate of convergence of the Picard iteration is faster than the Mann iteration for contraction mappings.

There is a drawback in the Picard iteration that it fails to calculate the fixed point if the mapping concerned is nonexpansive or more general than nonexpansive. To overcome this drawback of the Picard iteration in 2006, Agrawal, O'Regan and Sahu [3] introduced a new iteration process named as S-iteration process whose rate of convergence is similar to the Picard iteration process and faster than the Mann iteration process. They applied it to deal with the problem of approximation of fixed points of nearly asymptotically nonexpansive mappings.

Note that the S-iteration process is totally independent from the Mann and the Ishikawa iteration processes.

At this stage here arises a natural question:

Is it possible to approximate solutions of variational inequalities by the S-iteration process?

In Chapter 3, we answer the above question affirmatively by introducing some algorithms for solving variational inequalities. The results presented in this chapter extend several known results of
variational inequalities and variational inclusions in the context of \(S\)-iteration process.

1.4 We have already seen in Section 1.1 to 1.3 that the variational inequality (1.1.1) can be generalized in different ways. There is another generalization of variational inequality in which antisymmetric function is used. This type of inequality is called variational-like inequality. Variational-like inequalities are more general and include many known results in the field of variational inequalities. These inequalities have been studied by Lee, Ansari and Yao [53], Noor [79, 115], Dien [20] and Ding [24, 27, 29]. Since the nonlinear term is nondifferentiable, hence we cannot use the projection methods or resolvent type methods for solving the mixed type variational-like inequalities.

The variational inequality (1.1.1) can be formulated to another inequality as given below:

For given \(u \in C\), consider a problem of finding \(w \in C\) such that

\[
\langle \rho T u + w - u, v - w \rangle \geq 0, 
\]

which is known as \textit{auxiliary variational inequality} (Lions and Stampachia [55]). This shows that the solution of auxiliary problem (1.1.6) is the solution of variational inequality problem (1.1.1). Motivated by above fact Glowinski [34] has developed an auxiliary
principle technique to study the existence of the solution of variational inequalities. Ding [24, 27, 29], Dien [20] and Noor [79, 115] have modified and extended the auxiliary principle technique to study the existence of solution of variational-like inequalities and suggested the predictor-corrector type methods for solving variational-like inequalities.

In 2004, Ding [29] introduced the concept of partially relaxed \(\eta\)-strong monotonicity of set-valued mapping and suggested the predictor-corrector iterative algorithm for solving generalized mixed variational-like inequalities. In [25], Ding further introduced \(g\)-partially relaxed strong monotonicity and suggested iterative algorithm for solving generalized nonlinear mixed variational inequalities.

In chapter 4, we have introduced a concept of \((g,\eta)\)-partially relaxed strong monotonicity for set-valued mappings. Then by using auxiliary principle technique we have suggested an iterative algorithm for solving generalized nonlinear mixed variational-like inequalities. Our result generalizes the algorithm of Ding [25, 29, 30]. We have also discussed the convergence criterion.

In Section 4.1, we have introduced the \((g,\eta)\)-partially relaxed strong monotone operator and some more results which helps us to obtain our main result. In Section 4.2, we have developed the iterative
algorithm for \((g, \eta)\)-partially relaxed strong monotonicity for set-valued mappings (see Algorithm 4.2.1) and discussed its convergence criterion (see Theorem 4.2.1). Our result significantly improves and generalizes many previously known results (see [25, 29, 30, 53, 132, 134]).

1.5. Variational inclusions, which are an important and useful generalization of variational inequalities, have a wide range of applications to optimizations and control, economics and transportation equilibrium and engineering sciences. These variational inclusions have been studied by many authors including Huang [42], Noor [81, 103], Ding and Luo [21], Adly [2], Lee, Ansari and Yao [53], Huang and Fang [39] and others.

Projection method and auxiliary principle technique could not be extended and modified for solving variational inclusions. Thus, resolvent operator technique associated with the maximal monotone operator was used. Using this technique one can show that the variational inclusion problem is equivalent to the fixed point problem.

In 2000, Ding and Luo [21] introduced two new concepts of \(\eta\)-subdifferential and \(\eta\)-proximal mapping of a proper function in Hilbert space. Some related work, has also been studied by Ding [24] and Lee, Ansari and Yao [53]. In 2003, Huang and Fang [39] introduced a new
class of $\eta$-monotone mappings. They also studied a new class of general variational inclusions involving maximal $\eta$-monotone mappings and developed a new algorithm to solve this class of generalized variational inclusions by using the resolvent operator technique.

The purpose of Chapter 5 is to study generalized variational inclusion by introducing the concept of maximal $(g,\eta)$-monotone mapping and define the resolvent operator for maximal $(g,\eta)$-monotone mapping.

In Section 5.3, we introduce an algorithm (see Algorithm 5.3.1) for solving generalized variational inclusion and discuss its convergence criterion (see Theorem 5.3.1). Theorem 5.3.1 generalizes the Theorem 3.1 of Huang and Fang [39]. Section 5.4 deals with the perturbed iterative algorithm (see Algorithm 5.4.1) and its convergence theorem (see Theorem 5.4.1) for the generalized variational inclusion involving maximal $\eta$-monotone and $g$-monotone mapping.

1.6. Fuzzy set theory was introduced by Zadeh [148]. The application of the fuzzy set theory can be found in many branches of Physics, mathematical sciences, computer and engineering sciences (see Chang and Huang [15], Ding [23] and Noor [112, 113]). Chang and
Zhu [13] were first to introduce the concept of variational inequalities for fuzzy mappings. Since then several classes of variational inequalities for fuzzy mappings were considered by Chang and Huang [15], Noor [112], Noor and Al-Said [80], Ding [23] and Park and Jeong [119] in setting of Hilbert spaces. Noor [112] has used the projection methods to show variational inequalities for fuzzy mappings. This development leads to new and significant results in the area of mathematical sciences, engineering sciences, computer sciences, management sciences and operation research. In 1999, Ding [23] suggested generalized implicit quasi-variational inclusions for fuzzy set-valued mapping. In 2002, Ding and Park [26] studied a new class of generalized nonlinear implicit quasi-variational inclusions with fuzzy mappings.

The purpose of chapter 6 is to introduce some new results on different classes of variational inclusions for fuzzy mappings. In Section 6.1 we study an iterative approximation problem (see Algorithm 6.1.1) and discuss the convergence criterion (see Theorem 6.1.1) to obtain the solution to general set-valued variational inclusions involving $\eta$-maximal monotone mapping for fuzzy mappings.

In Section 6.2, we suggest an iterative algorithm (see Algorithm 6.2.1) for fuzzy multivalued quasi-variational inclusions and its
convergence criterion (see Theorem 6.2.1). In the same section we also prove that the fuzzy quasi-variational inclusions are equivalent to the fuzzy implicit resolvent equations and we suggest the algorithm (see Algorithm 6.2.2) and convergence criterion (see Theorem 6.2.2).