CHAPTER 6
ITERATIVE ALGORITHM FOR FUZZY MAPPINGS FOR VARIATIONAL INCLUSIONS
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CHAPTER 6

ITERATIVE ALGORITHM FOR FUZZY MAPPINGS FOR
VARIATIONAL INCLUSIONS

The purpose of Chapter 6 is to prove some new concepts on different classes of variational inclusions for fuzzy mappings.

Section 6.1 deals with the iterative algorithm and convergence theorem for the general set-valued variational inclusions for fuzzy mappings. Section 6.2 contains fuzzy multivalued quasi-variational inclusion its algorithm and convergence analysis. Our results improve some known results, which are in crisp mappings.

The basic information about fuzzy mappings:

Let $H$ be a real Hilbert space with norm $\| \cdot \|$ and inner product $\langle \cdot , \cdot \rangle$, respectively. Let $\mathcal{F}(H)$ be collection of all fuzzy sets over $H$. A mapping $T$ from $H$ into $\mathcal{F}(H)$ is called a fuzzy mapping on $H$. If $T$ is a fuzzy mapping on $H$, then $T(x)$ (denote it by $T_x$ in the sequel) is a fuzzy set on $H$ and $T_x(y)$ is the membership function of $y$ in $T_x$. Let $B \in T_x(H)$, $\alpha \in [0,1]$, then the set $(B)_\alpha = \{ x \in H : B(x) \geq \alpha \}$ is called a $\alpha$-cut set of $B$.

Further, let $T:H \to \mathcal{F}(H)$ be fuzzy mapping such that there exist a real
number \( a \in [0, 1] \) such that for all \( x \in H \), the set \( (T_x)_a \) belongs to \( \text{CB}(H) \), where \( \text{CB}(H) \) denotes the family of all nonempty closed bounded subset of \( H \).

6.1. General set-valued variational inclusion for fuzzy mappings

Let \( S, T, G: H \rightarrow \mathcal{F}(H) \) be three set-valued fuzzy mappings with function \( a, b, c: H \rightarrow [0, 1] \), respectively and \( M: H \times H \rightarrow 2^H \) a maximal monotone mapping. For a nonlinear mapping \( N(\cdot, \cdot): H \times H \rightarrow H \), \( M(\cdot, t) \) is maximal \( \eta \)-monotone with \( \text{range}(G) \cup \text{dom}(M(\cdot, t)) \neq \emptyset \) for each \( t \in H \).

We consider the problem of finding \( u, x, y, z \in H \) such that

\[
S_u(x) \geq a(u), \quad T_u(y) \geq b(u), \quad G_u(z) \geq c(u),
\]

\[
0 \in N(x, y) + M(z, u) \tag{6.1.1}
\]

An inequality (6.1.1) is called the general set-valued variational inclusion for fuzzy mappings.

Now, we recall some definitions and notions.

**Definition 6.1.1.** Let \( S, T, G: H \rightarrow \mathcal{F}(H) \) be three fuzzy mapping with functions \( a, b, c: H \rightarrow [0, 1] \), respectively and \( N(\cdot, \cdot): H \times H \rightarrow H \) is said to be:

(i) Lipschitz continuous with respect to fuzzy mapping \( T \) if, for any

\[
x_1, x_2 \in H \text{ and } w_i \in (T_{x_i})_{a(x_i)}, \quad w_2 \in (T_{x_2})_{a(x_2)},
\]
Similarly, we can define the Lipschitz continuity of \( N(.,.) \) in the second argument.

(ii) strongly monotone if for all \( x, y \in H \) and \( u, v \in H \) with \( x \in \{ S_u \} \) and \( y \in \{ T_u \} \) constant \( r \in (0, 1) \) such that

\[
\langle N(x, .) - N(y, .), u - v \rangle \geq r \| u - v \|^2.
\]

**Definition 6.1.2.** A multivalued mapping \( T: H \to 2^H \) is said to be D-Lipschitz continuous if there exist a constant \( \xi \in (0, 1) \) such that

\[
D(Au, Av) \leq \xi \| u - v \| \quad \text{for all } u, v \in H. \quad (6.1.2)
\]

We now deal with lemma for fixed point, algorithm and its convergence theorem.

**Lemma 6.1.1.** Let \( H \) be real Hilbert space. Let \((u, x, y, z)\), where \( u \in H \), \((S_u)(x) \geq a(u), (T_u)(y) \geq b(u) \) and \((G_u)(z) \geq c(u)\) is a solution of (6.1.1) if and only if

\[
z = J_p^{M(u)} (z - \rho N(x, y)), \quad (6.1.3)
\]

where \( J_p^{M(u)} = (I + \rho M(.,u))^{-1} \) and \( \rho > 0 \) is a constant.

**Proof.** By definition of \( J_p^{M(u)} \), we have

\[
z = (I + \rho M(.,u))^{-1} (z - \rho N(x, y))
\]
\[(I + \rho M(\cdot, u)) z = z - \rho N(x, y)\]
\[(z + \rho M(z, u) = z - \rho N(x, y)\]
\[0 \in N(x, y) + M(z, u),\]

with \(S_u(x) \geq a(u), T_u(y) \geq b(u), G_u(z) \geq c(u).\)

Invoking Lemma 6.1.1, we suggest the following algorithm for solving the general set-valued variational inclusion for fuzzy mappings.

**Algorithm 6.1.1.** For any given \(u \in H, x \in (S_u)a(u), y \in (T_u)b(u), z \in (G_u)c(u).\)

Define the iterative sequence \(\{u_n\}, \{x_n\}, \{y_n\} \) and \(\{z_n\}\) as follows:

\[
\begin{align*}
 u_{n+1} & = u_n - z_n + J_{\rho M(\cdot, u)}(z_n - \rho N(x_n, y_n)) \\
x_n & \in (S_u)a(u) \\
\| x_n - x_{n-1} \| \leq (1 + \frac{1}{1 + n})D((S_u)a(u), (S_u)a(u)) \\
y_n & \in (T_u)b(u) \\
\| y_n - y_{n-1} \| \leq (1 + \frac{1}{1 + n})D((T_u)b(u), (T_u)b(u)) \\
z_n & \in (G_u)c(u) \\
\| z_n - z_{n-1} \| \leq (1 + \frac{1}{1 + n})D((G_u)c(u), (G_u)c(u))
\end{align*}
\]

for all \(n \geq 0.\)

Now we present our main theorem of this section.

**Theorem 6.1.1.** Let \(S, T, G : H \to F(H)\) be three closed fuzzy mappings.
such that there exist real numbers $a \in [0,1]$, $b \in [0,1]$, $c \in [0,1]$ and for all $x, y, z \in H$, $(Sx)a$, $(Ty)b$, $(Gz)c$ belongs to $\mathcal{CB}(H)$. $S$, $T$, $G : H \to \mathcal{F}(H)$ be $D$-Lipschitz continuous fuzzy mappings with constant $\sigma$, $k$, and $\xi$, respectively, and $G$ is strongly monotone with constant $\gamma$ where $0 < \gamma < 1$. Let $\eta : H \times H \to H$ be strongly monotone and Lipschitz continuous with constant $\delta$ and $\tau$. $N : H \times H \to H$ be $\alpha$-Lipschitz continuous with respect to fuzzy mapping $S$ for any $y \in H$, and $\beta$-Lipschitz continuous with respect to fuzzy mapping $T$ for any given $x \in H$. Let $M : H \times H \to 2^H$ be such that for each $t \in H$, $M(.,t)$ is maximal $\eta$-monotone. Suppose there exist constants $\rho > 0$ and $\lambda > 0$ such that for each $x, y, z \in H$

$$\| J^{M,x}_\rho(z) - J^{M,y}_\rho(z) \| \leq \lambda \| x - y \|, \quad (6.1.5)$$

if the following condition holds:

$$0 = (1 + \frac{\tau}{\rho})\sqrt{1 - 2\gamma + \frac{\rho}{\xi}} + \frac{\tau}{\rho} \sqrt{1 - 2\rho \tau + \rho^2 \alpha^2 \sigma^2} + \beta k \frac{\tau}{\delta} + \lambda < 1, \quad (6.1.6)$$

then there exists $u \in H$, $x \in (S_u)_a(u)$, $y \in (T_u)_b(u)$, $z \in (G_u)_c(u)$ satisfying equation (6.1.3), and so $(u, x, y, z)$ is a solution of general set-valued variational inclusion for fuzzy mapping (6.1.1) and the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 6.1.2 converge strongly to $u, x, y, z$ in $H$, respectively.
\textbf{Proof.} From (6.1.4), we have

\begin{equation}
\| u_{n+1} - u_n \| = \| u_n - u_{n-1} - (z_n - z_{n-1}) + J_{\theta}^{\text{M}, u_n} (z_n - \rho N(x_n, y_n)) \\
- J_{\theta}^{\text{M}, u_{n-1}} (z_{n-1} - \rho N(x_{n-1}, y_{n-1})) \| \\
\leq \| u_n - u_{n-1} - (z_n - z_{n-1}) \| \\
+ \left\| J_{\theta}^{\text{M}, u_n} (z_n - \rho N(x_n, y_n)) - J_{\theta}^{\text{M}, u_{n-1}} (z_{n-1} - \rho N(x_{n-1}, y_{n-1})) \right\| \\
+ \left\| J_{\theta}^{\text{M}, u_{n-1}} (z_{n-1} - \rho N(x_{n-1}, y_{n-1})) - J_{\theta}^{\text{M}, u_{n-2}} (z_{n-2} - \rho N(x_{n-2}, y_{n-2})) \right\| \\
\leq \| u_n - u_{n-1} - (z_n - z_{n-1}) \| \\
+ \frac{\tau}{\delta} \| z_n - z_{n-1} - \rho (N(x_n, y_n) - N(x_{n-1}, y_{n-1})) \| + \lambda \| u_n - u_{n-1} \| \\
\leq \| u_n - u_{n-1} - (z_n - z_{n-1}) \| + \lambda \| u_n - u_{n-1} \| \\
+ \frac{\tau}{\delta} \| z_n - z_{n-1} - \rho (N(x_n, y_n) - N(x_{n-1}, y_{n-1})) \| \\
+ \rho \frac{\tau}{\delta} \| N(x_{n-1}, y_n) - N(x_{n-1}, y_{n-1}) \|. \tag{6.1.7}
\end{equation}

Since $G$ is strongly monotone and D-Lipschitz continuous, we obtain

\begin{equation}
\| u_n - u_{n-1} - (z_n - z_{n-1}) \|^2 \\
= \| u_n - u_{n-1} \|^2 - 2 \langle z_n - z_{n-1}, u_n - u_{n-1} \rangle + \| z_n - z_{n-1} \|^2 \\
\leq (1 - 2\gamma) \| u_n - u_{n-1} \|^2 + \left(1 + \frac{1}{1 + n}\right)^2 D^2 \langle (G u_{n-1})_{\text{stat}}, (G u_n)_{\text{stat}} \rangle \\
\leq (1 - 2\gamma + \left(1 + \frac{1}{1 + n}\right)^2 \xi^2) \| u_n - u_{n-1} \|^2. \tag{6.1.8}
\end{equation}

Further, from the assumption, we have
\[N(x, y) = N(x, y) - p \cdot \tilde{v}_n \cdot \nabla \tilde{u}_n \]

\[= \beta k (1 - \frac{1}{1 + \eta}) \tilde{u}_n \cdot \nabla u_n \cdot \nabla \tilde{u}_n \]

and

\[\|u_n \cdot \nabla u_n \cdot \nabla (N(x, y_n) \cdot N(x, y_n)) \|^2 \]

\[= \|u_n \cdot \nabla u_n \|^2 \cdot \|N(x, y_n) \cdot N(x, y_n), u_n, u_{n+1} \| \]

\[\gamma \|N(x, y_n) \cdot N(x, y_n)\|^2 \]

\[\leq \left[1 - 2 \rho r + \rho^2 \alpha^2 \sigma^2 (1 + \frac{1}{1 + n}) \right] \|u_n - u_{n+1}\|^2 \]

(6.1.10)

It follows from (6.1.7) - (6.1.10) that

\[\|u_{n+1} - u_n\| \leq (1 + \frac{\tau}{\delta}) \left[1 - 2 \rho r + (1 + \frac{1}{1 + n}) \right] \|u_n - u_{n+1}\| \]

\[+ \frac{\tau}{\delta} \beta k (1 + \frac{1}{1 + n}) \|u_n - u_{n+1}\| \]

\[+ \rho \frac{\tau}{\delta} \beta k (1 + \frac{1}{1 + n}) \|u_n - u_{n+1}\| \]

\[\leq \left[\beta + \frac{\tau}{\delta} \right] \left[1 - 2 \rho r + (1 + \frac{1}{1 + n}) \right] \|u_n - u_{n+1}\| \]

\[+ \frac{\tau}{\delta} \left(1 - 2 \rho r + \rho^2 \alpha^2 \sigma^2 (1 + \frac{1}{1 + n}) \right) \|u_n - u_{n+1}\| \]

\[+ \rho \frac{\tau}{\delta} \beta k (1 + \frac{1}{1 + n}) \|u_n - u_{n+1}\| \]

(6.1.11)
\[ \| u_{n+1} - u_n \| \leq \| u_n - u_{n-1} \|, \quad (6.1.11) \]

where

\[ \theta_n = (1 + \frac{\gamma}{\delta}) \sqrt{1 - 2\gamma + (1 + \frac{1}{1+n})^2 \xi^2 + \frac{\tau}{\delta} \sqrt{1 - 2\rho + \rho^2 \alpha^2 \sigma^2 (1 + \frac{1}{1+n})}} \]

\[ + (1 + \frac{1}{1+n}) \beta k \frac{\tau}{\delta} + \lambda. \]

Letting

\[ 0 = (1 + \frac{\gamma}{\delta}) \sqrt{1 - 2\gamma + \xi^2 + \frac{\tau}{\delta} \sqrt{1 - 2\rho + \rho^2 \alpha^2 \sigma^2}} + \beta k \frac{\tau}{\delta} + \lambda. \]

We know that \( \theta_n \to \theta \) as \( n \to \infty \). It follows from (6.1.6) that \( 0 \leq \theta < 1 \).

Hence \( \theta_n < 1 \) for \( n \) sufficiently large. Therefore (6.1.11) implies that \( \{u_n\} \)

is a Cauchy sequence in \( H \). Let \( u_n \to u \) as \( n \to \infty \).

From (6.1.4), we get

\[ \| x_n - x_{n-1} \| \leq (1 + \frac{1}{1+n}) D \begin{pmatrix} S_{u_n} \end{pmatrix} (S_{u_{n-1}}) (S_{u_{n-1}}) \quad (6.1.12) \]

\[ \leq \sigma (1 + \frac{1}{1+n}) \| u_n - u_{n-1} \| \]

\[ \| y_n - y_{n-1} \| \leq (1 + \frac{1}{1+n}) D \begin{pmatrix} T_{u_n} \end{pmatrix} (T_{u_{n-1}}) (T_{u_{n-1}}) \]

\[ \leq k (1 + \frac{1}{1+n}) \| u_n - u_{n-1} \| \]

\[ \| z_n - z_{n-1} \| \leq (1 + \frac{1}{1+n}) D \begin{pmatrix} G_{u_n} \end{pmatrix} (G_{u_{n-1}}) (G_{u_{n-1}}) \]

\[ \leq \xi (1 + \frac{1}{1+n}) \| u_n - u_{n-1} \|. \]
isfies the condition

\[ \| J_{A(u)}w - J_{A(u)}v \| \leq \nu \| u - v \|, \]

where \( \nu > 0 \) is a constant.

Related to the fuzzy multivalued quasi-variational inclusion (6.2.1), we consider the following problem:

find \( x, u, w, y \in \mathcal{H} \) such that

\[
T_u(w) \succeq a(u), (V_u)(y) \succeq b(u),
\]

\[
N(w, y) + p^{-1}R_{A(u)}z = 0, \tag{6.2.2}
\]

where \( p > 0 \) is a constant and \( R_{A(u)} = I - J_{A(u)} \), where \( I \) is the identity operator and \( J_{A(u)} = (I + pA(u))^{-1} \) is the resolvent operator.

The following lemma is essential to find an algorithm and establish its convergence theorem.

**Lemma 6.2.1.** (\( u, w, y \) where \( u \in \mathcal{H} \), \( (T_u)(w) \succeq a(u) \), \( (V_u)(y) \succeq b(u) \) is solution of (6.2.1) if and only if \( (u, w, y) \) satisfies the relation

\[
g(u) = J_{A(u)}[g(u) - pN(w, y)], \tag{6.2.3}
\]

where \( p > 0 \) is a constant and \( J_{A(u)} = (I + pA(u))^{-1} \) is the resolvent operator.

**Proof.** Let \( u \in \mathcal{H} \), \( (T_u)(w) \succeq a(u) \), \( (V_u)(y) \succeq b(u) \) be a solution of (6.2.1).
Then for a constant $\rho > 0$

\[ 0 \in \rho N(w, y) + \rho A(g(u), u) \]

\[ 0 \in (g(u) - \rho N(w, y)) + (1 + \rho A(u)) g(u) \]

\[ g(u) = J_{\lambda_0}[g(u) - \rho N(w, y)] \]

This is the required result.

From above Lemma 6.2.1, we conclude that fuzzy multivalued quasi-variational inclusion (6.2.1) is equivalent to the fuzzy implicit fixed point problem (6.2.3). This alternative formula is very useful to propose some iterative algorithm for solving fuzzy quasi-variational inclusion (6.2.1) and related problems.

**Algorithm 6.2.1.** For given $u \in H$, $(T_u)(w) \in a(u)$, $(V_u)(y) \in b(u)$, define the iterative sequences $\{u_n\}, \{w_n\}$ and $\{y_n\}$ such that

\[ u_{n+1} = (1 - \lambda) u_n + \lambda (u_n - g(u_n) + J_{\lambda_0}[g(u_n) - \rho N(w_n, y_n)]) \]

\[ w_n \in (T_{u_n})_{a(u_n)} : \| w_{n+1} - w_n \| \leq D((T_{u_n})_{a(u_n)}, (T_{u_n})_{a(w_n)}) \]

\[ y_n \in (V_{u_n})_{b(u_n)} : \| y_{n+1} - y_n \| \leq D((V_{u_n})_{b(u_n)}, (V_{u_n})_{b(y_n)}), \quad n = 0, 1, 2, \ldots \]

where $D(\cdot, \cdot)$ is the Hausdorff metric on $C(H)$.

Now we give the convergence of Algorithm 6.2.1.

**Theorem 6.2.1.** Let $T, V : H \to \mathcal{F}(H)$ be fuzzy mappings such that there exists real numbers $a, b : H \to [0, 1]$ such that for all $x, y \in H$,
\((T_\alpha, (V_\beta)_\alpha)\) belongs to \(CB(H)\). \(T\) and \(V\) are \(D\)-Lipschitz continuous with constants \(\mu\) and \(\xi\), respectively. Let \(N(.,.)\) be strongly monotone with constant \(\alpha > 0\) and Lipschitz continuous with constant \(\beta > 0\) with respect to fuzzy mapping \(T\) for any given \(y \in H\), and \(\mu > 0\) with respect to fuzzy mapping \(V\) for any given \(x \in H\). Let \(g: H \to H\) be strongly monotone with constant \(\sigma > 0\) and be Lipschitz continuous with constant \(\delta > 0\). If Assumption 6.2.1 holds and

\[
\left| \frac{\alpha - (1 - k) \eta \xi}{\beta^2 \mu^2 - \eta^2 \xi^2} \right| < \sqrt{\left( \frac{\alpha - (1 - k) \eta \xi}{\beta^2 \mu^2 - \eta^2 \xi^2} \right)^2 - k \left( \frac{\beta^2 \mu^2 - \eta^2 \xi^2}{2 - k} \right)},
\]

(6.2.4)

\[
\alpha > (1 - k) \eta \xi + \sqrt{\left( \frac{\beta^2 \mu^2 - \eta^2 \xi^2}{2 - k} \right)},
\]

(6.2.5)

\[
\rho \eta \xi < 1 - k,
\]

(6.2.6)

\[
k = 2\sqrt{1 - 2\sigma + \delta^2} + \nu.
\]

(6.2.7)

Then there exist \(u \in H\), \(w \in (T_\alpha)_u\), \(y \in (V_\beta)_w\) satisfying the fuzzy multivalued quasi-variational inclusion (6.2.1) and the iterative sequences \(\{u_n\}\), \(\{w_n\}\) and \(\{y_n\}\) generated by Algorithm 6.2.1 converges to \(u\), \(w\) and \(y\) in \(H\), respectively.

**Proof.** From Algorithm 6.2.1 and Assumption 6.2.1, we have

\[
\| u_{n+1} - u_n \| \leq (1 - \lambda) \| u_n - u_{n-1} \| + \lambda \| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \|
\]

\[
+ \lambda \| J_{\alpha(u_n)}(g(u_n) - p N(w_n, y_n)) - J_{\alpha(u_{n-1})}(g(u_{n-1}) - p N(w_{n-1}, y_{n-1})) \|
\]

\[
\leq (1 - \lambda) \| u_n - u_{n-1} \| + \lambda \| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \|
\]

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\[ + \lambda \left\| J_{\lambda(u_n)}[g(u_n) - \rho N(w_n, y_n)] - J_{\lambda(u_{n-1})}[g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})] \right\| \]

\[ + \lambda \left\| J_{\lambda(u_{n-1})}[g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})] - J_{\lambda(u_{n-1})}[g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})] \right\| \]

\[ \leq (1 - \lambda) \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \left\| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \right\| + \lambda \nu \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \left\| g(u_n) - \rho N(w_n, y_n) - (g(u_{n-1}) - \rho N(w_{n-1}, y_{n-1})) \right\| \]

\[ \leq (1 - \lambda) \left\| u_n - u_{n-1} \right\| + 2 \lambda \left\| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \right\| \]

\[ + \lambda \nu \left\| u_n - u_{n-1} \right\| + \lambda \left\| u_n - u_{n-1} - \rho (N(w_n, y_n) - N(w_{n-1}, y_{n-1})) \right\| \]

\[ \leq (1 - \lambda) \left\| u_n - u_{n-1} \right\| + 2 \lambda \left\| u_n - u_{n-1} - (g(u_n) - g(u_{n-1})) \right\| \]

\[ + \lambda \nu \left\| u_n - u_{n-1} \right\| + \lambda \left\| u_n - u_{n-1} - \rho (N(w_n, y_n) - N(w_{n-1}, y_{n-1})) \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

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\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

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\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]

\[ + \lambda \nu \left\| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \right\| \left\| u_n - u_{n-1} \right\| \]
respect to fuzzy mapping $T$, it follows that:

$$
\|u_n - u_{n-1} - \rho [N(w_n, y_n) - N(w_{n-1}, y_n)]\|^2
= \|u_n - u_{n-1}\|^2 - 2\rho \langle N(w_n, y_n) - N(w_{n-1}, y_n), u_n - u_{n-1} \rangle
+ \rho^2 \|N(w_n, y_n) - N(w_{n-1}, y_n)\|^2
\leq (1 - 2\rho \alpha + \rho^2 \beta^2 \eta^2) \|u_n - u_{n-1}\|^2.
$$

From (6.2.8) - (6.2.11), we have

$$
\|N(w_{n-1}, y_{n-1}) - N(w_{n-1}, y_{n-1})\| \leq \|y_n - y_{n-1}\|
\leq \eta D(Vu_{n-1}, Vu_{n-1})
\leq \eta \xi \|u_n - u_{n-1}\|.,
$$

where

$$
v = k + \rho \eta \xi + t(\rho),
$$

$$
t(\rho) = \sqrt{1 - 2\rho \alpha + \rho^2 \beta^2 \mu^2},
$$

$$
h = 1 - \lambda(1 - \theta),
$$

Using the Lipschitz continuity of operator $N(.,.)$ with respect to fuzzy mapping $V$ and the D-Lipschitz continuity of $V$, we have

$$
\|N(w_{n-1}, y_{n-1}) - N(w_{n-1}, y_{n-1})\| \leq \|y_n - y_{n-1}\|
\leq \eta D(Vu_{n-1}, Vu_{n-1})
\leq \eta \xi \|u_n - u_{n-1}\|.,
$$

From (6.2.8) - (6.2.11), we have

$$
\|u_{n+1} - u_n\| \leq \lambda \left\{ 2\sqrt{(1 - 2\rho \alpha + \delta^2)} + v + \rho \eta \xi \right\}
+ \sqrt{1 - 2\rho \alpha + \delta^2 \eta^2} \|u_n - u_{n-1}\| + (1 - \lambda) \|u_n - u_{n-1}\|
\leq \left\{ (1 - \lambda) + \lambda (k + \rho \eta \xi + t(\rho)) \right\} \|u_n - u_{n-1}\|
\leq \left\{ 1 - \lambda (1 - \theta) \right\} \|u_n - u_{n-1}\| = h \|u_n - u_{n-1}\|.,
$$

where

$$
v = k + \rho \eta \xi + t(\rho),
$$

$$
t(\rho) = \sqrt{1 - 2\rho \alpha + \rho^2 \beta^2 \mu^2},
$$

$$
h = 1 - \lambda(1 - \theta),
$$

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Now from (6.2.4) - (6.2.7), we have $0 < q < 1$. Consequently, it follows that $h < 1$. From (6.2.12), the sequence $\{u_n\}$ is a Cauchy sequence in $H$, i.e., there exists $u \in H$ with $u_{n+1} \to u$ as $n \to \infty$. Similarly $y_n \to y$ and $w_n \to w$.

Finally we prove that $w \in (T_u, a(u))$, $y \in (V_u, b(u))$. Since $w_n \in (T_u, a(u))$, we have

$$d(w_n, (T_u, a(u))) \leq ||w - w_n|| + d(w_n, (T_u, a(u)))$$

$$\leq ||w - w_n|| + d(w_n, (T_u, a(u))) + D((T_u, a(u)), (T_u, a(u)))$$

$$\leq ||w - w_n|| + 0 + \mu ||u_n - u|| \to 0, \text{ (as } n \to \infty)$$

where $d(w_n, (T_u, a(u))) = \inf\{||w - z|| : z \in T(u)\}$. Since $\{w_n\}$ and $\{u_n\}$ are Cauchy sequences, hence $d(w_n, (T_u, a(u))) = 0$, and so $w \in (T_u, a(u))$. In a similar way we can show that $y \in (V_u, b(u))$. This implies that $(u, w, y)$ is a solution of fuzzy multivalued quasi-variational inclusion and consequently $u_n \to u$, $w_n \to w$ and $y_n \to y$ strongly in $H$.

This completes the proof.

Next, we prove that the fuzzy quasi-variational inclusions are equivalent to the fuzzy implicit resolvent equations. This equivalent is used to propose and analyze a number of iterative methods for solving the fuzzy variational inclusions and related problems.

**Lemma 6.2.2.** The fuzzy multivalued quasi-variational inclusion (6.2.1) has a solution $u \in H$, $w \in (T_u, a(u))$, $y \in (V_u, b(u))$ if and only if $z$, $u \in H$,
(T_u)(w) ≥ a(u), (V_u)(y) ≥ b(u) is a solution of the implicit resolvent equation (6.2.2), where

\[ z = g(u) - \rho N(w, y) \]  
(6.2.16)

\[ g(u) = J_{A_{\mu_0}} z \]  
(6.2.17)

and \( \rho > 0 \) is a constant.

**Proof.** Let \( u \in H, \ (T_u)(w) \geq a(u), \ (V_u)(y) \geq b(u), \) be a solution of (6.2.1). Then

\[ g(u) = \int A_{\mu_0}^1 u dz \]  
(6.2.18)

Using \( R_{A_{\mu_0}} = I - J_{A_{\mu_0}} \) and equation (6.2.18), we obtain

\[ R_{A_{\mu_0}} [g(u) - \rho N(w, y)] = g(u) - \rho N(w, y) - J_{A_{\mu_0}} [g(u) - \rho N(w, y)] \]
\[ = g(u) - \rho N(w, y) - g(u) \]
\[ = -\rho N(w, y) \]

\[ R_{A_{\mu_0}} [g(u) - \rho N(w, y)] = -\rho N(w, y), \]

which implies that

\[ N(w, y) + \rho^{-1} R_{A_{\mu_0}} z = 0 \] with \( z = g(u) - \rho N(w, y) \).

Conversely, let \( z, u \in H, \ w \in (T_u)a(u), \ y \in (V_u)b(u), \) be a solution of (6.2.2), then

\[ \rho N(w, y) = -R_{A_{\mu_0}} z = J_{A_{\mu_0}} z - z. \]  
(6.2.19)

Thus from (6.2.18) and (6.2.19), it follows that

\[ g(u) = J_{A_{\mu_0}} [g(u) - \rho N(w, y)] \]
where \( u = g^{-1}J_{\mathcal{A}_0}[g(u) - \rho N(w, y)] \),

is the solution of (6.2.1), which is the required result. ■

The above fixed point formulation suggest the following iterative method.

**Algorithm 6.2.2.** For a given \( z_0, u_0 \in H \), \( w_0 \in (T_{u_0} h_{u_0}) \), and \( y_0 \in (V_{u_0} h_{u_0}) \), compute the sequences \( \{z_n\} \), \( \{w_n\} \), \( \{y_n\} \) and \( \{u_n\} \) by the iterative scheme:

(i) \( w_n \in (T_{u_n} h_{u_n}) \),

\[
\| w_n - w_{n-1} \| \leq D((T_{u_n} h_{u_n})(T_{u_{n-1}} h_{u_{n-1}})),
\]  

(6.2.20)

(ii) \( y_n \in (V_{u_n} h_{u_n}) \),

\[
\| y_n - y_{n-1} \| \leq D((V_{u_n} h_{u_n})(V_{u_{n-1}} h_{u_{n-1}})),
\]  

(6.2.21)

(iii) \( z_{n+1} = g(u_n) - \rho N(w_n, y_n) \),

(6.2.22)

(iv) \( g(u_n) = J_{\mathcal{A}[u]}[z_n] \).

(6.2.23)

We now conclude the section by the following:

**Theorem 6.2.2.** Let \( T, V : H \to \mathcal{F}(H) \) below fuzzy mapping with function \( a, b : H \to [0,1] \), respectively. \( T \) and \( V \) are \( D \)-Lipschitz continuous with constant \( \mu > 0 \) and \( \xi > 0 \), \( g : H \to H \) be strongly monotone with constant \( \sigma \) where \( 0 < \sigma < 1 \) and Lipschitz continuous with constant...
\( \delta > 0 \). Let operator \( N(.,.) \) be strongly monotone with constant \( \alpha > 0 \) and Lipschitz continuous with \( \beta > 0 \) with respect to fuzzy mapping \( T \) for any \( y \in H \). If Assumption 6.2.1 and relations (6.2.4) to (6.2.7) hold, then there exist \( z, u \in H \), \( w \in (T_u)_{b(0)} \), \( y \in (V_u)_{b(0)} \) satisfying the implicit equation (6.2.2) and the sequences \( \{z_n\}, \{w_n\}, \{y_n\} \) and \( \{u_n\} \) generated by Algorithm 6.2.2 converge strongly to \( z, w, y \) and \( u \) in \( H \), respectively.

**Proof.** From Algorithm 6.2.2 and using (6.2.9) - (6.2.11), we have

\[
\| z_{n-1} - z_n \| \leq \| u_n - u_{n-1} - (g(u_n)) - (g(u_{n-1})) \|
+ \rho \| N(w_n, y_n) - N(w_{n-1}, y_{n-1}) \|
+ \| u_n - u_{n-1} - \rho \{ N(w_n, y_n) - N(w_{n-1}, y_{n}) \} \|
\leq \sqrt{1 - 2\sigma + \delta^2 + \rho \eta \xi + \sqrt{1 - 2\rho \alpha + \rho^2 \beta^2}} \| u_n - u_{n-1} \|
\leq \left\{ \frac{k - \nu}{2} + \rho \eta \xi + t(p) \right\} \| u_n - u_{n-1} \|.
\] (6.2.24)

From (6.2.9), (6.2.23) and Assumption 6.2.1, we have

\[
\| u_n - u_{n-1} \| \leq \| u_n - u_{n-1} - (g(u_n)) - (g(u_{n-1})) \| + \| J_{A(u)} z_n - J_{A(u)} z_{n-1} \|
+ \| J_{A(u)} z_{n-1} - J_{A(u)} z_{n-1} \|
\leq \frac{k - \nu}{2} \| u_n - u_{n-1} \| + \| z - z_{n-1} \| + \nu \| u_n - u_{n-1} \|,
\]

which implies that

\[
\| u_n - u_{n-1} \| \leq \left\{ \frac{1}{k + \nu} \right\} \| z_n - z_{n-1} \|. \quad (6.2.25)
\]
Combining (6.2.24) and (6.2.25), we get

\[ \| z_{n+1} - z_n \| \leq \{ k - \frac{\nu}{2} + \rho \eta \xi + t(p)\}\left(\frac{1}{k + \nu}\right) \| z_n - z_{n-1} \| \]

\[ \| z_{n+1} - z_n \| \leq 0 \| z_n - z_{n-1} \|. \]

From (6.2.4) – (6.2.7), 0 < 1. Consequently, \{z_n\} is a Cauchy sequence in \( H \), that is there exists \( z \in H \) such that \( z_n \to z \) as \( n \to \infty \).

From Theorem 6.2.1, we know that \{w_n\}, \{y_n\} and \{u_n\} are also Cauchy sequence. We can show that \( z, u \in H, w \in (Tu)_w \), \( y \in (Vu)_u \) is a solution of (6.2.2) and \( \{z_n\}, \{w_n\}, \{y_n\} \) and \( \{u_n\} \) converge strongly to \( z, w, y \) and \( u \) in \( H \), respectively. Thus, we obtain the required result.

**Remark 6.2.1.** Theorem 6.1.1 improve the Theorem 3.1 of Huang and Fang [39], Theorem 6.2.1 and Theorem 6.2.2 improves the Theorem 3.1 and Theorem 4.2, respectively of Noor [94] from crisp mappings to fuzzy mappings.

**REFERENCES**


