Interaction of two level atom and electromagnetic field in Kerr medium

4.1 Introduction

In this chapter we investigate the evolution of population inversion and von Neumann entropy in a system consisting of a two level atom in Kerr medium interacting with photon field, which is initially prepared in a quadrature squeezed state. The frequency of the field is set to be varying sinusoidally with time and also has a phase difference from the initial frequency. By following steps that used in the previous chapter we derived an analytical expressions for the probability amplitudes for constant frequency case. Again the time dependent field frequency case has to be investigated using the numerical techniques. Population inversion is examined for various nonlinear strengths for both time dependent and time independent frequencies. The effects of frequency modulation on the evolution of the system for phase zero and phase non-zero cases have been analysed. It is observed that in Kerr medium also frequency fluctuation modifies the interaction between squeezed field and two level atom. The presence of phase factor in the frequency fluctuation enhances the modifications. The von Neumann entropy of the system which is a direct measure of the entanglement between the two subsystems; the atom and the field is
also analysed for different damping and susceptibility. The effects of frequency fluctuation on the time evolution of von Neumann entropy and the corresponding entanglement is examined. It is also noticed that some interesting behaviour of the evolution of a two level atom in nonlinear medium can also be induced in linear medium itself by adding a phase factor in the field frequency modulations.

4.2 Model and Hamiltonian

Consider the system consists of a single two level atom interacting with single mode electromagnetic field in an infinite Q-cavity containing Kerr medium. Let $\omega_a$ be the frequency equivalent corresponding to the energy difference between the ground and exited states of the two level atom and $\omega_f$ is the single mode photon frequency. Kerr non-linearity of the medium can be modelled by an anharmonic oscillator with frequency $\omega_k$. Let $\hat{b}(\hat{b}^\dagger)$ be the annihilation(creation) operator corresponding to the medium and $\hat{a}(\hat{a}^\dagger)$ be that of the photon. The Hamiltonian of the system is derived using the Jayness Cumming model where the rotating wave approximation is applied. The total Hamiltonian of the system can be written as $(\hbar = 1)$:

$$\hat{H} = \omega_f \hat{a}^\dagger \hat{a} + \omega_\sigma \sigma_z + g (\hat{a}^\dagger \hat{\sigma}_- + \hat{a} \hat{\sigma}_+) + \omega_k \hat{b}^\dagger \hat{b} + q \hat{b}^\dagger \hat{b} + p (\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a})$$

(4.1)

Here $\sigma$'s are the atomic operators satisfying $[\sigma_+, \sigma_-] = i\sigma_z$, $q$ is the anharmonicity parameter, $p$ is the field-medium coupling strength and $g$ is the atom field coupling strength. If the response time of the nonlinear medium is so short that the medium follows the field in an adiabatic manner, the total Hamiltonian can be transformed to an effective Hamiltonian involving only the photon and the atomic operators. In the adiabatic limit the field frequency and anharmonic
frequency are assumed to be very far from each other (i.e., $\omega_k \ll \omega_f$). In such cases one can introduce the third order susceptibility factor in the Hamiltonian [71, 72] and it can be rewritten as,

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + \omega_a \hat{\sigma}_z + g \left( \hat{a}^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \right) + \chi \hat{a}^\dagger \hat{a}^2 \quad (4.2)$$

where the new frequency $\omega$ and the coupling constant $\chi$ are related to $q$ and $p$ by, $\chi = qp^4/\delta^4$, $\omega = \omega_f - p^2/\delta$, $\delta = \omega_f - \omega_k$. The coupling constant $\chi$ is the dispersive part of the third-order nonlinearity of the Kerr-like medium.

We now consider the case where the field frequency depends on time, such that

$$\omega(t) = \omega_0 + \lambda \sin(\beta t + \phi), \quad (4.3)$$

where the amplitude of fluctuation $\lambda \ll \omega_0$ and $\phi$ is the phase of the field frequency modulation. The modifications in the atom field coupling strength $g$ is obtained from Eq. (3.6) of section 3.2 and is given by

$$g = g_0 \left[ 1 + \lambda \sin(\beta t + \phi) \right]. \quad (4.4)$$

The susceptibility factor $\chi$ also changes due to the frequency fluctuation and takes the form

$$\chi = \chi_0 + \epsilon \lambda \sin(\beta t + \phi), \quad (4.5)$$

where $\epsilon \ll \chi_0$ such that the changes in $\chi$ due to the frequency fluctuation is always small. Now the total Hamiltonian of the system in the frequency fluctuating case becomes:

$$\hat{H} = \left[ \omega_0 + \lambda \sin(\beta t + \phi) \right] \hat{a}^\dagger \hat{a} + \omega_a \hat{\sigma}_z + \left[ \chi_0 + \epsilon \lambda \sin(\beta t + \phi) \right] \hat{a}^\dagger \hat{a}^2$$

$$+ g_0 \left( 1 + \frac{\lambda \sin(\beta t + \phi)}{\nu_0} \right) \left( \hat{a}^\dagger \hat{\sigma}_- + \hat{\sigma}_+ \right) \quad (4.6)$$
The general state of the atom field system at any time, \( t \) can be taken as
\[
|\psi(t)\rangle = \sum_n C_n^e(t)|e\rangle|n\rangle + \sum_n C_{n+1}^g(t)|g\rangle|n+1\rangle. \tag{4.7}
\]
As we have done in the preceding sections substitute Eq. (4.6) and Eq. (4.7) in the time dependent Schrödinger equation and obtain
\[
\frac{d}{dt} C_n^e(t) = -ig\sqrt{n+1} \left( 1 + \frac{\lambda \sin(\beta t + \phi)}{\omega_f} \right) C_{n+1}^g(t) - i \left[ n\omega_0 - \frac{\omega_0}{2} + n(n-1)\chi_0 \right] C_n^e(t) - i \left[ \lambda \sin(\beta t + \phi) \right] [1 + n(n-1)] C_n^e(t) \tag{4.8}
\]
\[
\frac{d}{dt} C_{n+1}^g(t) = -ig\sqrt{n+1} \left( 1 + \frac{\lambda \sin(\beta t + \phi)}{\omega_f} \right) C_n^e(t) - i \left[ n\omega_0 - \frac{\omega_0}{2} + n(n+1)\chi_0 \right] C_{n+1}^g(t) - i \left[ \lambda \sin(\beta t + \phi) \right] [1 + n(n+1)] C_{n+1}^g(t). \tag{4.9}
\]
By solving Eqs. (4.8) and (4.9) the time evolution of the system can be obtained. Variation of population inversion and entanglement entropy are examined from the obtained solutions.

### 4.3 Evolution of probabilities

If the field frequency does not have any time dependence i.e., \( \lambda = 0 \), we can solve the Eqs. (4.8) and (4.9) analytically by following the steps that we have used in the previous chapter. When \( \lambda = 0 \) Eqs.
(4.8) and (4.9) becomes
\[
\frac{d}{dt} C_{n}^{e}(t) = -ig\sqrt{n+1} C_{n+1}^{g}(t) - i(n\omega + \omega_a/2) C_{n}^{e}(t) - i n(n-1)\chi_0 C_{n}^{e}(t)
\]
\[
\frac{d}{dt} C_{n+1}^{g}(t) = -ig\sqrt{n+1} C_{n}^{e}(t) - i(n\omega - \omega_a/2) C_{n+1}^{g}(t) - i n(n+1)\chi_0 C_{n+1}^{g}(t).
\]

(4.10)

Now define \( H_{e,n} \) and \( H_{g,n+1} \) such that
\[
C_{e,n}(t) = \exp\left[-i\left(n\omega + \frac{1}{2}\omega_a + \chi_0 n(n+1)\right)H_{e,n}(t)\right]
\]
\[
C_{g,n+1}(t) = \exp\left[-i\left((n+1)\omega - \frac{1}{2}\omega_a + \chi_0 (n+1)n\right)H_{g,n+1}(t)\right].
\]

(4.13)

It is useful to note that
\[
|C_{e,n}(t)|^2 = |H_{e,n}(t)|^2,
\]
\[
|C_{g,n+1}(t)|^2 = |H_{g,n+1}(t)|^2.
\]

(4.14)

(4.15)

Substituting Eqs. (4.12) and (4.13) in Eqs. (4.10) and (4.11) we get the following equations
\[
i\frac{d}{dt} H_{e,n} = \lambda\sqrt{n+1} H_{g,n+1}(t) e^{-ik_n t},
\]
\[
i\frac{d}{dt} H_{g,n+1} = \lambda\sqrt{n+1} H_{e,n}(t) e^{+ik_n t},
\]

(4.16)

(4.17)

where
\[
k_n = \omega - \omega_0 + 2\chi n
\]

(4.18)

It is assumed that initially the atom is in excited state, i.e.,
\[
C_{g,n+1}^{g}(0) = H_{g,n+1}^{g}(0) = 0
\]

(4.19)
Using this initial condition Eq. (4.19), Eqs. (4.16) and (4.17) are solved and the solutions are

\[
H_{g,n+1}(t) = \frac{\lambda}{\Omega_n} \sqrt{n+1} e^{(ik_n t/2)} 2i \sin (\Omega_n t/2), \quad (4.20)
\]

\[
H_{e,n}(t) = \frac{1}{2\Omega_n} e^{(-ik_n t/2)} \{ ik \sin (\Omega_n t/2) + \Omega_n \cos (\Omega_n t/2) \}, \quad (4.21)
\]

with \( \Omega_n^2 = k_n^2 + 4g^2(n+1) \). Using Eqs. (4.12) and (4.13) in Eqs. (4.20) and (4.21) we get

\[
C_{e,n}(t) = \frac{1}{\Omega_n} [ ik_n \sin (\Omega_n t/2) + \Omega_n \cos (\Omega_n t/2) ]
\times \exp \{ -i (n\omega + \omega_a/2 + \chi_0(n+1) + k_n t/2) \}, \quad (4.22)
\]

\[
C_{g,n+1}(t) = \left( \frac{2\lambda \sqrt{n+1}}{\Omega_n} \right) 2i \sin (\Omega_n t/2)
\times \exp \{ -i [(n+1)\omega - \omega_a/2 + \chi_0(n+1)n - k_n t/2] \}, \quad (4.23)
\]

From Eqs. (4.22) and (4.23) the population inversion of the system is

\[
W(t) = \frac{-1}{\Omega_n^2} \left[ -k_n^2 \sin^2 (\Omega_n t/2) + \Omega_n^2 \cos^2 (\Omega_n t/2) - k_n \Omega_n \sin (\Omega_n t/2) \cos (\Omega_n t/2) \right] - \frac{\lambda^2(n+1)}{\Omega_n^2} \sin^2 (\Omega_n t/2) \quad (4.24)
\]

In the case of time dependent field frequencies we use numerical techniques to solve the Eqs. (4.8) and (4.9) to find the population inversion.
4.4 Entanglement entropy

In this section we look at the quantum entropy properties of the atom field system. It is well known that the interaction between a two level atom and photon field leads to entanglement between the two sub systems, the atom and the field. Many methods are there to study entanglement dynamics between the atom and field and we calculated the von Neumann entropy which gives a direct measure of the entanglement. The reduced density operator of the atom($\rho_a$) in the bare basis is obtained by taking the partial trace over all the field states and is given by,

$$\rho_a(t) = \left[ \begin{array}{cc} \sum_{n=0}^{\infty} C_n^e(t)C_n^e(t) + C_{n+1}^g(t)C_{n+1}^e(t) \\ \sum_{n=0}^{\infty} C_{n+1}^g(t)C_{n+1}^g(t) + C_n^g(t)C_n^g(t) \end{array} \right]. \quad (4.25)$$

It can be shown that the reduced density operator for field($\rho_f$) is also the same. The components of Bloch vectors are written in terms of the elements in the reduced density matrix as,

$$s_1(t) = \sum_{n=0}^{\infty} \left[ C_n^e(t)C_{n+1}^g(t) + C_{n+1}^g(t)C_n^e(t) \right]$$

$$s_2(t) = \sum_{n=0}^{\infty} \left[ C_n^e(t)C_{n+1}^g(t) - C_{n+1}^g(t)C_n^e(t) \right]$$

$$s_3(t) = \sum_{n=0}^{\infty} \left[ |C_n^e(t)|^2 - |C_{n+1}^g(t)|^2 \right] \quad (4.26)$$

and the von Neumann entropy, $S(\rho_a)$, is

$$S(\rho_a) = -g_1(t) \ln \{g_1(t)\} - g_2 \ln \{g_2(t)\}, \quad (4.27)$$

where

$$g_1(t) = 1 + \sqrt{|s_1(t)|^2 + |s_2(t)|^2 + |s_3(t)|^2},$$

$$g_2(t) = 1 - \sqrt{|s_1(t)|^2 + |s_2(t)|^2 + |s_3(t)|^2}. \quad (4.28)$$
The effects of frequency modulation on the entropy evolution in a system of two level atom and field in a linear medium has been already studied by Jia Fei et al.[73]. According to their results, in the case of interaction between two level atom and coherent field, when the decay coefficient is small, the system is in the entangled state all the time except the initial time; when the decay coefficient increases, the entanglement between the atom and the field decays to zero as the time increases. Here we focus on the entropy evolution of the system in Kerr medium and also the role of phase shifted frequency fluctuation on the entropy evolution. The initial photon field is chosen to be quadrature squeezed.

4.5 Evolution of population inversion

4.5.1 No field frequency fluctuations

We have the analytical solutions for this case given by Eqs. (4.22) and (4.23) and using it, the analytical expression for the population inversion is obtained in Eq. (4.24). In the Kerr medium population inversion oscillates with periodic collapses and revivals as shown in Fig. 4.1, similar to the case of linear medium. But the population inversion never collapses to zero, rather it settles to a positive value during the collapse region, which indicates that the probability of atom being in the initial excited state is large comparing with that of the atom being in ground state. When the value of nonlinear susceptibility $\chi_0$ is high, the probability of the atom being in the excited state itself is very near to 1 and the ground state probability is close to zero. One can say that the high nonlinearity of the medium suppresses the probability of deexcitation of the two level atom. It is also noted that the number of collapses and revivals in the popu-
lation inversion during a particular time interval increases with the nonlinearity ($\chi_0$) of the medium.

4.5.2 With field frequency fluctuations

For frequency fluctuating cases we use numerical techniques to solve the time evolution equation given in Eqs. (4.8) and (4.9). In our discussion the parameters of frequency variation such as amplitude and frequency are chosen to be $\lambda = 30g_0$ and $\beta = 1g_0$. In the phase shifted case we take $\phi = \pi/2$ for which the effects are maximum, which we have already seen in the previous chapter.
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Figure 4.1: Population inversion versus time for constant field frequency. The initial field is squeezed with parameters $\bar{n} = 25$, $r = 0.8$, $\theta = \pi$. Damping $\gamma = 0$. 

(a): Susceptibility $\chi_0 = 0.4g_0$

(b): Susceptibility $\chi_0 = 0.8g_0$
With field frequency fluctuations

Figure 4.2: Population inversion versus time for fluctuating field frequency with $c = 30g_0$ and $\beta = 1g_0$. Squeezed field parameters are $\bar{n} = 25$, $r = 0.8$, Damping $\gamma = 0$. Dotted curve shows the field frequency fluctuation.
Figure 4.3: Population inversion versus time for fluctuating field frequency. Initial squeezed field parameters are $\bar{n} = 25$, $r = 0.8$, $\theta = 0$. Susceptibility $\chi_0 = 0.5g_0$ and damping $\gamma = 0$. Dotted curve shows the field frequency fluctuation.
Case: 1  Phase $\phi = 0$.
As in the case of coherent field atom interaction in Kerr medium, discussed by L Wang et al. [70], for the case of initial squeezed field also, collapses and revivals in population inversion are unclear. Population inversion or the probabilities correspond to the atom in excited and ground states executes oscillations with time and these oscillations are very close to a sinusoidal wave, which is shown in Figs. 4.2 and 4.3. The period of oscillation of the population inversion is same as that of the field frequency modulation.

Case: 2  Phase $\phi \neq 0$.
When there is a phase factor in the frequency modulation the oscillations in population inversion are more sinusoidal compared to the zero phase case, which is clear from the Fig. 4.4. The oscillation in population inversion starts earlier than the zero phase case, depending on $\phi$. In the figures shown we set $\phi = \pi/2$, for which the effect of phase factor on the evolution is most visible. The same behaviour of the population inversion is observed in linear medium also when there is a phase factor in the frequency modulation. In the Fig. 4.5, where the population inversion is plotted for linear medium i.e., $\chi_0 = 0$ with $\phi = \pi/2$, the evolution of population inversion is very similar to the behaviour of population inversion when a two level atom interacts with frequency modulated field in Kerr medium plotted in Fig. 4.4. The dynamics of a two level atom-photon system in nonlinear medium is induced in a linear medium due to the phase factor in the field frequency fluctuations. It is suggested that the effect of non-linearity in the medium in a two level atom-time varying frequency field interaction can be induced in linear medium also by introducing a phase factor in the frequency modulation.
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Figure 4.4: Population inversion versus time where the field has a phase shifted frequency fluctuation. Initial squeezed field parameters are $\bar{n} = 25$, $r = 0.8$ and $\theta = 0$. Susceptibility $\chi_0 = 0.5g_0$ and damping $\gamma = 0$. Dotted curve shows the field frequency fluctuation.
Figure 4.5: Population inversion versus time in linear medium where field has a phase shifted frequency fluctuation. Initial squeezed field parameters are $r = 0.8$, $\theta = \pi$, $\bar{n} = 25$, $\chi_0 = 0$, $\gamma = 0$ and $\phi = \pi/2$. Dotted curve shows the field frequency fluctuation.
4.6 Evolution of entanglement entropy

The von Neumann entropy of the system can be calculated by using the atom field evolution coefficients in Eqs. (4.25), (4.26) and (4.27). The evolution of entropy is plotted in Figs. 4.6 - 4.9, by varying various parameters.

**Case: 1 No frequency fluctuation**

Evolution of entropy for no field frequency fluctuation ($\lambda = 0$) is shown in Fig. 4.6.
Figure 4.6: Evolution of entropy with time in Kerr medium with initial squeezed field with $\bar{n} = 25$, $\tau = 0.8$ and $\theta = 0$. 

(a): Susceptibility $\chi = 0.4g_0$ 

(b): Susceptibility $\chi = 0.8g_0$
It is clear that the entropy remains a constant value with frequent periodic drops to some minimum value, where few of the minimum points are very close to zero. Interesting thing is that these drops occurs at the same time when revivals occurs in the population inversion. The number of falls to minimum increases with increase in the susceptibility. Thus the average entropy is small for higher susceptibility values. We can infer that during each revival period the atom and field becomes minimally entangled and the nonlinearity reduces the entanglement between atom and field. If the damping is present, entropy increases to a maximum and then it decreases and damp out to zero.

**Case: 2 With frequency fluctuation**

Now consider the effect of frequency fluctuations with a zero phase, shown in Figs. 4.7 and 4.8. Here the entropy oscillates in the shape of a quasi sine wave, which is in sync with the applied field frequency fluctuation. We can say that the frequency fluctuation make the entropy more ordered and controllable. The additional phase factor does not make any noticeable modifications in the entropy evolution. As seen from Fig. 4.9, where the entropy evolution is plotted for linear medium with a phase shifted($\phi = \pi/2$) field frequency, quasi sinusoidal oscillation in entropy is present in this case also. Thus for the entropy evolution also the effects of non linearity can be induced by the phase shifted frequency modulation in linear medium itself. One can understand this by comparing Fig. 4.9 with Fig.4.7.
Figure 4.7: Evolution of entropy with time in Kerr medium for initial squeezed field with $\bar{n} = 25, r = 0.8$ and $\theta = 0$. Field frequency is sinusoidally fluctuating, damping $\gamma = 0$. Dotted curve shows the field frequency fluctuation.
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Figure 4.8: Evolution of entropy in Kerr medium with phase shifted field frequency variation. Initial squeezed field parameters are $\bar{n}$, $r = 0.8$ and $\theta = \pi$. Damping $\gamma = 0$. Dotted curve shows the field frequency fluctuation.
Figure 4.9: Evolution of entropy in linear medium with phase shifted frequency fluctuation. Initial squeezed field parameters are $r = 0.8$, $\theta = \pi$, $\bar{n} = 25$. Susceptibility $\chi_0 = 0$, damping $\gamma = 0$ and phase $\phi = \pi/2$. Dotted curve shows the field frequency fluctuation.
4.7 Conclusion

In this work we have considered the system consisting of a two level atom in Kerr medium interacting with quadrature squeezed photon field in the adiabatic limit. The frequency of the field is set to be fluctuating and phase shifted. Evolution of population inversion and entanglement entropy of the system is analysed by varying parameters. It is observed that in the nonlinear medium also sinusoidal frequency fluctuation modifies the time evolution of population inversion. These modifications are enhanced in the presence of a phase in the frequency fluctuation. The entanglement entropy of the system also has a close dependence on the field frequency fluctuations. It becomes more ordered and controllable when the frequency is sinusoidally fluctuating. We noticed that the entanglement between the atom and field can be controlled by varying the period of the field frequency fluctuations. As we have discussed in the previous chapter, many interesting behaviour in the evolution of a two level atom in Kerr medium can also be produced in linear medium by including phase factor in the frequency modulation. State evolution during the interaction of an isolated atom-photon system is an interesting area of research nowadays, which contribute much to the developments of controllable quantum computation and QIP. The understanding of the entanglement between the atom and field can be applied in generating optimal methods for the qubit operations. Thus the results produced in this work may help to progress the research in QIP and speed up the realisation of a quantum computer.