CHAPTER III

CSMC FOR THE MEAN AND STANDARD DEVIATION OF NON-NORMAL POPULATION
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3.1 Introduction

Takagi (1978) employed the Burr distribution and Gram-Charlier series for developing acceptance sampling plans for life testing data. Gosh (1980) constructed Cumulative Sum Control Charts (CCCC) for the mean and standard deviation using type I censoring, assuming that the production process follows a normal distribution. In this chapter we have made an attempt to develop CCCC for the mean and standard deviation of the non-normal population expressed by the first four terms of an Edgeworth series under type I censoring. Here, by type I censoring, we mean that a sample of n units is put on test for a prefixed time, say \( t_0 \), and the number of failures are recorded. It may be stressed that acceptance sampling plans under this type of censoring have been worked out for the distributions like normal (lognormal), exponential, gamma distributions by Gupta (1962), Balasubramanian (1972) and Gupta and Groll (1961).

The procedure of construction of cumulative sum control chart is the same as in the previous chapter, using the pair of Sequential Probability Ratio (SPRT) test. The Johnson's (1961) technique is utilized to determine the
parameters (the lead distance \(d\) and the angle \(\theta\) between the arm of the mask and the horizontal line) of the mask. The expression for the Average Run Length (ARL) has also been given.

3.2 CUSUM for the mean under type I censoring.

Suppose the underlying distribution of the product life \(x\) has the mean \(\mu\) and standard deviation \(\sigma\). Let \(\lambda_3(=\mu_4), \lambda_4(=\mu_4 - 3)\) be the standardized third and fourth cumulants respectively, then the density function of \(x\) can be approximated by first four terms of an Edgeworth series as follows:

\[
f(x, \mu, \sigma) = \frac{1}{\sigma} \left[ \phi \left( \frac{x - \mu}{\sigma} \right) - \frac{\lambda_3}{6} \phi^{(3)} \left( \frac{x - \mu}{\sigma} \right) + \frac{\lambda_4}{24} \phi^{(4)} \left( \frac{x - \mu}{\sigma} \right) \\
+ \frac{\lambda_3}{72} \phi^{(6)} \left( \frac{x - \mu}{\sigma} \right) \right]
\]

(3.2.1)

where \(\phi(t) = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)\)

and

\[
\phi^{(n)}(t) = \frac{d^n}{dt^n} \phi(t)
\]

To develop the cumulative sum control chart for the mean of the distribution (3.2.1), we assume that \(\mu, \sigma, \lambda_3\) and \(\lambda_4\) are known.

Suppose \(n\) items are put on test for a preassigned truncation time \(t\) and let the number of failures occurring on or before the truncation time \(t\) be \(x\). Now, define a random variable \(y\) such that
\[ y = 0, \text{if an item survives beyond time } t. \]

\[ y = 1, \text{if it fails on or before the time } t. \]  

Thus the total number of failures during the truncation time \( t \) is given by

\[ F = \sum_{i=1}^{n} Y_i \]  

The probability of occurrence of each of these failures on or before the prefixed time \( t \) is

\[ P = \int_{-\infty}^{t} f(x, \mu, \sigma) \, dx = \Phi\left(\frac{t - \mu}{\sigma}\right) - \Phi\left(\frac{t - \mu}{\sigma}\right) \left[ \frac{\lambda}{2} H_2 \left(\frac{t - \mu}{\sigma}\right) + \frac{\lambda^2}{72} H_4 \left(\frac{t - \mu}{\sigma}\right) \right] \]

(3.2.4)

where \( f(x, \mu, \sigma) \) is given by (3.2.1) and \( H_y(x) \) is the well known Hermite polynomial of degree \( y \) in \( x \) given by

\[ \phi^y(x) = (-1)^y \phi(x) H_y(x) \]

(3.2.5)

In constructing \( U(M) \), we shall treat \( p \) as a function of \( \mu \) assuming the other parameters to be known. It is then easily seen that \( p = \psi\left(\frac{t - \mu}{\sigma}\right) \) (say) is a monotonically decreasing function of \( \mu \) and hence we have

\[ \psi\left(\frac{t - \mu}{\sigma}\right) \leq \psi\left(\frac{t - \mu_0}{\sigma}\right) \implies \mu \geq \mu_0 \]

or

\[ p \leq p_0 \implies \mu \geq \mu_0 \]

(3.2.6)
where \( p_0 \) is given by (3.2.4) for any value \( \mu_0 (\geq \mu) \) of \( \mu \). The basic idea behind the construction of UCCG using the number of failures occurring on or before a prefixed time \( t \), is that any hypothesis regarding \( \mu \) can uniquely be expressed in terms of \( p_0 \).

From the above discussion it is clear that a detection of increase in the process mean \( \mu \) is equivalent to the detection of a decrease in the parameter \( p_0 \). Following Johnson (1961), for constructing the UCCG for mean we have to consider the sequential probability ratio (SPR) test for the following hypotheses:

\[
H_0: \mu = \mu_0 \text{ and } H_1: \mu = \mu_1 (\mu_1 > \mu_0) \\
\text{or} \\
H_0: p = p_0 \text{ and } H_1: p = p_1 (p_1 < p_0) \tag{3.2.7}
\]

For the given truncation time \( t \), the probability of occurrence of \( r \) failures out of \( n \) items placed on test, will be

\[
P(r) = \binom{n}{r} p^r (1 - p)^{n-r} \tag{3.2.8}
\]

where \( p \) is given by (3.2.4). The log likelihood function for discriminating between the hypotheses \( H_0 \) and \( H_1 \) is given as

\[
\gamma \log \frac{p_0 (1-p_1)}{p_1 (1-p_0)} - n \log \frac{1-p_1}{1-p_0} \tag{3.2.9}
\]
Obviously, the SNP test discriminating between \( H_0 \) and \( H_1 \) has its continuation region

\[
\log \frac{a_1}{1-a_0} + n \log \frac{1-p_1}{1-p_0} < \frac{1-a_1}{\log \frac{a_0}{p_0 (1-p_1)}} + \frac{1-p_1}{\log \frac{p_1 (1-p_0)}{p_0 (1-p_1)}}
\]

(3.2.10)

where \( a_1 = P \{ \text{accept } H_{1-1} / H_1 \} \), \( i = 0,1 \).

Now considering the C.KC as a reversed SNP test with a very small value of \( a_1 (= 0) \) and using only the right hand side of the inequality in (3.2.10), we have

\[
r > \frac{1-p_1}{\log \frac{a_0}{p_0 (1-p_1)}} \frac{1-p_0}{\log \frac{p_1 (1-p_0)}{p_0 (1-p_1)}}
\]

(3.2.11)

as the inequality for constructing the C.KC for the detection of an increase in the process mean \( \mu \) from \( \mu_0 \) to \( \mu_1 \).
The control limit $P_0$, as shown in Fig. 3(1.2), is obtained by plotting the points with co-ordinates $(p, r)$ in a two dimensional space. Any plotted point lying above the line $P_0$ is an evidence of lack of control i.e. an indication of decrease in $p$ or equivalently an increase in the process mean $\mu$. The lead distance $OP$ where $O$ is the last plotted point and $P$ is to the left of $O$ and the slope of the line $P_0$ with the horizontal line are respectively given by

$$OP = \frac{-\log s_0}{\log \frac{1-p_1}{1-p_0}} \quad (3.2.12)$$

and

$$\tan \angle OPQ = \log \left[ \frac{1-p_1}{1-p_0} \right] \div \log \left[ \frac{P_0 (1-P_1)}{P_1 (1-P_0)} \right] \quad (3.2.13)$$

Similarly for detecting a decrease in $\mu$ from $\mu_0$ to $\mu_2$, we consider the following set of hypotheses:

$$H_0 : \mu = \mu_0 \quad \text{and} \quad H_2 : \mu = \mu_2 \quad (\mu_2 < \mu_0)$$

or

$$H_0 : p = p_0 \quad \text{and} \quad H_2 : p = p_2 \quad (p_2 > p_0) \quad (3.2.14)$$

The inequality to detect a change in the parameter is given by
The control limits \( P'Q' \) as shown in Fig. 3 (1.5) can be obtained by plotting the points with co-ordinates \((n, r)\). The parameters of the control chart are given by

\[
0 P' = \left( - \log a_0 \right) / \log \frac{1 - p_0}{1 - p_2}
\]

and

\[
\tan \angle O P' Q' = \log \frac{1 - p_0}{1 - p_2} / \log \frac{P_2 (1 - p_0)}{P_0 (1 - p_2)}
\]

Any plotted point lying below the line \( P'Q' \) will indicate a lack of control in the process mean.

A two sided CSCE can be obtained by simultaneous operation of the pair of SRR tests given by (3.2.11) and (3.2.15). The control chart diagram is shown in Fig. (3.2). Any plotted point with co-ordinate \((n, r)\), lying outside the two limits \(PQ\) and \(P'Q'\) will indicate a lack of control. The relevant parameters for the two sided CSCE are still given by the equations (3.2.12).
(3.2.13), (3.2.16) and (3.2.17) with first kind of error being twice of $\alpha_0$ and second kind of error $\alpha_1 (= \alpha)$. 

3.2.1 Average Run Length (ARL)

Following Johnson (1961), the approximate formula for the average run length (ARL), when $H_0$ is true is given by

$$\text{ARL} = ( - \log \alpha_0 ) e^k \quad (3.2.18)$$

where

$$E = \mathbb{E} \left[ \log \frac{f (y / p = p_1)}{f (y / p = p_0)} \bigg/ H_0 \right] \quad (3.2.19)$$

and $\mathbb{E} ( \cdot / H_0 )$ is the expectation under the condition that $H_0 (i = 1, 2)$ is true. Using the equation (3.2.18) and (3.2.19), the approximate ARL can be obtained as

$$\text{ARL} = \frac{- \log \alpha_0}{E}$$

$$H_1 = \frac{p_1 \log (p_1 / p_0) + (1-p_1) \log [(1-p_1)/(1-p_0)]}{(3.2.20)$$

when $H_1$ is true $(i = 1, 2)$.

For two-sided CUSC, the approximate ARL can be obtained by

$$\text{ARL} = (\text{ARL})^{-1}_{H_1} + (\text{ARL})^{-1}_{H_2} \quad (3.2.21)$$
where \((H_0)\) is the null when \(H_0 (i = 1, 2)\) is true.

3.3 G-CC for the standard deviation under type I censoring.

In this section we shall again assume that \(n\) items are put on test for a truncation time \(t\) and number of failures \(x\) occurring on or before this time is noted, like in previous section, to construct G-CC for standard deviation, using the number of failures \(x\), we would like to connect \(\sigma\) and \(x\). This can be had again by the equation \((3.2.4)\). They are differing only in interpretation. In the previous section we assumed that fraction defective \(p\) is a function of \(\mu\). Here, we assume that \(p\) is a function of \(\sigma\) and the other parameters are known.

In the light of above discussion, for constructing G-CC for variability we have to consider the following hypotheses:

\[
H : \sigma = \sigma_0 \quad \text{and} \quad H_1 : \sigma = \sigma_1 (\sigma_1 > \sigma_0) \quad (3.3.1)
\]

It is easily seen that \(p\) is monotonically decreasing function of \(\sigma\) and therefore, we can have

\[
P \leq P_0 \iff \sigma \geq \sigma_0 \quad (3.3.2)
\]

where \(p\) is still given by \((3.2.4)\) for any value \(\sigma_0 (\leq \sigma)\) of \(\sigma\).
Thus the construction of C3CC for detecting a change in $c$ can be carried out in the same way as for the mean by using the control charts as shown in Fig. 3(1.a), 3(1.b) and (3.2). The only difference is that we have to consider $p$ as a function of $c$ alone in place of $\mu$. The relevant parameters and the expressions for ARL can be obtained as given in the section (3.2).
Fig. 3(1.a): CCCC for detecting the increase in the process mean.

Fig. 3(1.b): CCCC for detecting the decrease in the process mean.

Fig. 3.2: Two sided CCCC for detecting the mean of a Normal distribution.