CHAPTER 4

\( \tau^* \)-GENERALIZED CONTINUOUS FUNCTION IN TOPOLOGICAL SPACES

4.1 INTRODUCTION

Levine (1963) introduced semi continuous functions using semi open sets. Balachandran et al (1991) introduced and studied the concept of generalized continuous functions and proved that the class of generalized continuous functions includes the class of continuous functions and studied several properties related to it.

The purpose of this chapter is to introduce and study \( \tau^* \)-generalized continuous function, \( \tau^* \)-gc-irresolute function, strongly \( \tau^* \)-g-continuous function and perfectly \( \tau^* \)-g-continuous function in topological space.

4.2 \( \tau^* \)-GENERALIZED CONTINUOUS FUNCTION

The purpose of this section is to introduce and study the concepts of a new class of function, namely \( \tau^* \)-generalized continuous function. Moreover, a new operator \( \text{cl}_{\tau^*} \), known as \( \tau^* \)-generalized closure operator is also introduced.

**Definition 4.2.1:** A function \( f: X \rightarrow Y \) from a topological space \((X, \tau^*)\) into a topological space \((Y, \sigma^*)\) is called \( \tau^* \)-generalized continuous function (briefly
\( \tau^*\text{-g-continuous) if the inverse image of every g-closed (or g-open) set in Y is } \tau^*\text{-g-closed (or } \tau^*\text{-g-open) in X.}

**Theorem 4.2.2:** If a function \( f: X \to Y \) from a topological space \((X, \tau)\) into a topological space \((Y, \sigma)\) is continuous then it is \( \tau^*\text{-g-continuous, provided Y is a } T_{1/2} \text{ space.} \)

**Proof:** Let \( f: X \to Y \) be a continuous function. Suppose \( F \) is a g-closed set in \( Y. \) Since \( Y \) is a \( T_{1/2} \) space, \( F \) is closed in \( Y. \) By the definition of continuous function, \( f^{-1}(F) \) is closed in \( X. \) Also by Theorem 3.2.10, \( f^{-1}(F) \) is \( \tau^*\text{-g-closed in X. Hence } f \text{ is } \tau^*\text{-g-continuous.} \)

**Remark 4.2.3:** The following example shows that the above theorem need not be true if \( Y \) is not a \( T_{1/2} \) space.

**Example 4.2.4:** Let \( X=\{a, b, c\}, \tau= \{X, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}\} \) and \( \sigma= \{Y, \emptyset, \{c\}, \{a, b\}\}. \) Let \( f: X \to Y \) be an identity function. Then \( f \) is continuous. However, it is not \( \tau^*\text{-g-continuous, since for the g-closed set } V=\{b\} \text{ in } Y, \text{ the inverse image } f^{-1}(V) = \{b\} \text{ is not } \tau^*\text{-g-closed in } X. \)

**Theorem 4.2.5:** If a function \( f: X \to Y \) from a topological space \((X, \tau)\) into a topological space \((Y, \sigma)\) is g-continuous, then it is \( \tau^*\text{-g-continuous, provided Y is a } T_{1/2} \text{ space.} \)

**Proof:** Let \( f: X \to Y \) be a g-continuous function. Suppose \( F \) is a g-closed set in \( Y. \) Since \( Y \) is a \( T_{1/2} \) space, \( F \) is closed in \( Y. \) By the assumption, \( f^{-1}(F) \) is g-closed in \( X. \) Also by Theorem 3.2.13, \( f^{-1}(F) \) is \( \tau^*\text{-g-closed. Therefore } f \text{ is } \tau^*\text{-g-continuous.} \)

**Remark 4.2.6:** The following example shows that the above theorem need not be true if \( Y \) is not a \( T_{1/2} \) space.
Example 4.2.7: Let $X=Y=\{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}, \{c\}\}$. Let $f:X \to Y$ be an identity function. Then $f$ is $g$-continuous. But it is not $\tau^*$-g-continuous, since for the $g$-closed set $V=\{a, b\}$ in $Y$, the inverse image $f^{-1}(V)=\{a, b\}$ is not $\tau^*$-g-closed in $X$.

Theorem 4.2.8: If a function $f:X \to Y$ from a topological space $(X, \tau)$ into a topological space $(Y, \sigma)$ is strongly $g$-continuous then it is $\tau^*$-g-continuous.

Proof: Let $f:X \to Y$ be a strongly $g$-continuous function. Suppose $F$ is a $g$-closed set in $Y$. By the assumption, $f^{-1}(F)$ is closed in $X$. By Theorem 3.2.10, $f^{-1}(F)$ is $\tau^*$-g-closed. Hence $f$ is $\tau^*$-g-continuous.

Remark 4.2.9: The converse of the above theorem need not be true as seen from the following example.

Example 4.2.10: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{c\}, \{a, c\}, \{b, c\}\}$. Let $f:X \to Y$ be an identity function. Then $f$ is $\tau^*$-g-continuous. On the other hand, it is not strongly $g$-continuous, since for the $g$-closed set $\{a\}$ in $Y$, the inverse image of $\{a\}$ is not closed in $X$.

Definition 4.2.11: The $\tau^*$-generalized closure operator $cl_{\tau^*}$ for a subset $A$ of a topological space $(X, \tau^*)$ is defined by the intersection of all $\tau^*$-g-closed sets containing $A$. That is $cl_{\tau^*}(A) = \bigcap \{G: A \subseteq G$ and $G$ is $\tau^*$-g-closed\}$

Theorem 4.2.12: Let $f:X \to Y$ be a function from a topological space $(X, \tau^*)$ into a topological space $(Y, \sigma^*)$.

(i) The following statements are equivalent:

(a) $f$ is $\tau^*$-g-continuous.
(b) The inverse image of each $g$-open set in $Y$ is $\tau^*$-$g$-open in $X$.

(ii) If $f : X \to Y$ is $\tau^*$-$g$-continuous, then $f (\text{cl}_X^*(A)) \subseteq \text{cl} (f (A))$ for every subset $A$ of $X$.

(iii) The following statements are equivalent.

(a) For each point $x \in X$ and each $g$-open set $V$ containing $f (x)$, there exists a $\tau^*$-$g$-open set $U$ containing $x$ such that $f (U) \subseteq V$.

(b) For every subset $A$ of $X$, $f (\text{cl}_X^*(A)) \subseteq \text{cl} (f (A))$ holds.

(c) The function $f : X \to Y$ from a topological space $(X, \tau^*)$ into $(Y, \sigma^*)$ is $\tau^*$-$g$-continuous.

**Proof:** (i) Assume that $f : X \to Y$ is $\tau^*$-$g$-continuous. Suppose $G$ is a $g$-open set in $Y$. Then $G^C$ is $g$-closed in $Y$. By assumption, $f^{-1}(G^C)$ is $\tau^*$-$g$-closed in $X$. But $f^{-1}(G^C) = X - f^{-1}(G)$. Thus $X - f^{-1}(G)$ is $\tau^*$-$g$-closed in $X$ and so $f^{-1}(G)$ is $\tau^*$-$g$-open in $X$. Therefore (a) $\Rightarrow$ (b).

Conversely assume that the inverse image of each $g$-open set in $Y$ is $\tau^*$-$g$-open in $X$. Suppose $F$ is a $g$-closed set in $Y$. Then $F^C$ is $g$-open in $Y$. By assumption, $f^{-1}(F^C)$ is $\tau^*$-$g$-open in $X$. But $f^{-1}(F^C) = X - f^{-1}(F)$. Thus $X - f^{-1}(F)$ is $\tau^*$-$g$-open in $X$ and so $f^{-1}(F)$ is $\tau^*$-$g$-closed in $X$. Therefore $f$ is $\tau^*$-$g$-continuous. Hence (b) $\Rightarrow$ (a). Thus (a) and (b) are equivalent.

(ii) Assume that $f$ is $\tau^*$-$g$-continuous. Suppose $A$ is a subset of $X$. Then $\text{cl} (f (A))$ is $g$-closed set in $Y$. By the assumption of $f$, $f^{-1}(\text{cl} (f (A)))$ is $\tau^*$-$g$-closed in $X$ and it contains $A$. But
\( \text{cl}^{\tau^*} (A) \) is the intersection of all \( \tau^* \)-g-closed set containing \( A \). Therefore \( \text{cl}^{\tau^*} (A) \subset f^{-1}(\text{cl} (f(A))) \) and so \( f(\text{cl}^{\tau^*} (A)) \subset \text{cl} (f(A)) \).

(iii) (a) \( \Rightarrow \) (b). Let \( y \in f(\text{cl}^{\tau^*} (A)) \) and let \( V \) be any \( \tau^* \)-open neighborhood of \( y \). Then there exists a point \( x \in X \) and a \( \tau^* \)-g-open set \( U \) such that \( f(x) = y \), \( x \in U \), \( x \in \text{cl}^{\tau^*} (A) \) and \( f(U) \subset V \). Since \( x \in \text{cl}^{\tau^*}(A) \), \( U \cap A \neq \emptyset \). Therefore \( y = f(x) \in \text{cl} (f(A)) \).

(b) \( \Rightarrow \) (a). Let \( x \in X \) and let \( V \) be any \( \tau^* \)-open set containing \( f(x) \). Suppose \( A = f^{-1}(V^C) \), then \( x \in A \). Now, \( \text{cl}^{\tau^*} (A) \subset f^{-1} (f(\text{cl}^{\tau^*} (A))) \subset f^{-1}(V^C) = A \). That is, \( \text{cl}^{\tau^*} (A) \subset A \). But \( A \subset \text{cl}^{\tau^*} (A) \). Therefore \( A = \text{cl}^{\tau^*} (A) \). Then, since \( x \notin \text{cl}^{\tau^*} (A) \), there exists a \( \tau^* \)-g-open set \( U \) containing \( x \) such that \( U \cap A = \emptyset \) and hence \( f(U) \subset f(A^C) \subset V \).

(b) \( \Rightarrow \) (c) By assumption \( f(\text{cl}^{\tau^*} (A)) \subset \text{cl} (f(A)) \). Therefore \( f \) is \( \tau^* \)-g-continuous.

(c) \( \Rightarrow \) (b) Let \( A \) be any subset of \( X \). Then \( \text{cl} (f(A)) \) is a \( g \)-closed set in \( Y \). Since \( f \) is \( \tau^* \)-g-continuous, \( f^{-1}(\text{cl} (f(A))) \) is \( \tau^* \)-g-closed in \( X \). Now \( A \subset f^{-1} (\text{cl} (f(A))) \Rightarrow \text{cl}^{\tau^*} (A) \subset \text{cl}^{\tau^*}(f^{-1}(\text{cl}(f(A)))) = f^{-1}(\text{cl}(f(A))) \) since \( f^{-1} (\text{cl}(f(A))) \) is \( \tau^* \)-closed in \( X \). This implies \( f(\text{cl}^{\tau^*} (A)) \subset \text{cl}(f(A)) \).

**Theorem 4.2.13:** Let \( X \) and \( Z \) be any topological spaces and \( Y \) be a \( \tau^* \)-T\(_g^* \)-space. Then the composition \( g \circ f: X \to Z \) of the \( \tau^* \)-g-continuous functions \( f:X \to Y \) and \( g:Y \to Z \) is also \( \tau^* \)-g-continuous.

**Proof:** Let \( F \) be a \( g \)-closed set in \( Z \). Since \( g \) is \( \tau^* \)-g-continuous, \( g^{-1}(F) \) is \( \tau^* \)-g-closed in \( Y \). But \( Y \) is \( \tau^* \)-T\(_g^* \) and so \( g^{-1}(F) \) is \( g \)-closed. Since \( f \) is
\( \tau^* \)-g-continuous, \( f^{-1}(g^{-1}(F)) \) is \( \tau^* \)-g-closed in \( X \). But \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \).
Therefore \( g \circ f \) is \( \tau^* \)-g-continuous.

**Remark 4.2.14:** The following example shows that the above theorem need not be true if \( Y \) is not a \( \tau^* \)-T_\( g \)-space.

**Example 4.2.15:** Let \( X = Y = Z = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\},\{b, c\}\} \) \( \sigma = \{Y, \phi, \{b\}\} \) and \( \eta = \{Z, \phi, \{b\}, \{c\}, \{b, c\}, \{a, b\}\} \). Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be identity functions. Then \( f \) and \( g \) are \( \tau^* \)-g-continuous. But \( g \circ f \) is not \( \tau^* \)-g-continuous, since for the g-closed set \( F = \{a\} \) in \( Z \), \( g^{-1}(F) = F \) and \( f^{-1}(g^{-1}(F)) = F \) is not \( \tau^* \)-g-closed in \( X \). Therefore \( g \circ f \) is not \( \tau^* \)-g-continuous.

**Theorem 4.2.16:** Let \( f : X \rightarrow Y \) be a \( \tau^* \)-g-continuous function from a topological space \( (X, \tau) \) into a topological space \( (Y, \sigma) \) and let \( H \) be a g-closed subset of \( X \). Then the restriction \( f / H : H \rightarrow Y \) is \( \tau^* \)-g-continuous where \( H \) is endowed with the relative topology.

**Proof:** Let \( F \) be a g-closed subset of \( Y \). Since \( f \) is \( \tau^* \)-g-continuous, \( f^{-1}(F) \) is \( \tau^* \)-g-closed in \( X \). But by Remark 3.2.46, intersection of \( \tau^* \)-g-closed set and a g-closed set is \( \tau^* \)-g-closed set. Thus, if \( f^{-1}(F) \cap H = H_1 \), then \( H_1 \) is a \( \tau^* \)-g-closed in \( X \). Since \( (f / H)^{-1}(F) = H_1 \), it is sufficient to show that \( H_1 \) is \( \tau^* \)-g-closed in \( H \). Let \( G_1 \) be any g-open set of \( H \) such that \( G_1 \supseteq H_1 \). Let \( G_1 = G \cap H \), where \( G \) is g-open in \( X \). Now, \( H_1 \subseteq G \cap H \subseteq G \). Since \( H_1 \) is \( \tau^* \)-g-closed in \( X \), \( \text{cl}(H_1) \subseteq G \). Now \( \text{cl}_{H_1}(H_1) = \text{cl}(H_1) \cap H \subseteq G \cap H = G_1 \), where \( \text{cl}_{H_1}(A) \) is the closure of a subset \( A \) in a subspace \( H \) of \( X \). Therefore \( f / H \) is \( \tau^* \)-g-continuous.

**Theorem 4.2.17:** Let \( X = A \cup B \) be a topological space with topology \( \tau^* \) and \( Y \) be a topological space with topology \( \sigma^* \). Let \( f : (A, \tau^*) / A \rightarrow (Y, \sigma^*) \) and
$g : (B, \tau^* / B) \rightarrow (Y, \sigma^*)$ be $\tau^*$-g-continuous functions such that $f(x) = g(x)$ for every $x \in A \cap B$. Suppose that $A$ and $B$ are $\tau^*$-g-closed sets in $X$. Then $\alpha : (X, \tau^*) \rightarrow (Y, \sigma^*)$ is $\tau^*$-g-continuous.

**Proof:** Let $F$ be $\tau^*$-closed set in $Y$. Clearly $\alpha^{-1}(F) = f^{-1}(F) \cup g^{-1}(F) = C \cup D$ where $C = f^{-1}(F)$ and $D = g^{-1}(F)$. But $C$ is $\tau^*$-g-closed in $A$ and $A$ is $\tau^*$-g-closed in $X$ and so $C$ is $\tau^*$-g-closed in $X$, since if $B \subseteq A \subseteq X$, $B$ is $\tau^*$-g-closed in $A$ and $A$ $\tau^*$-g-closed in $X$ then $B$ is $\tau^*$-g-closed in $X$. Also $C \cup D$ is $\tau^*$-g-closed in $X$. Therefore, $\alpha^{-1}(F)$ is $\tau^*$-g-closed in $X$. Hence $\alpha$ is $\tau^*$-g-continuous.

**Remark 4.2.18:** The following examples show that $\tau^*$-g-continuous function is independent from the $\alpha$-continuous function.

**Example 4.2.19:** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, b\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is $\tau^*$-g-continuous. But it is not $\alpha$-continuous, since for the open set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not $\alpha$-open in $X$.

**Example 4.2.20:** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{c\}, \{c, a\}, \{c, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{c\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is $\alpha$-continuous. However, it is not $\tau^*$-g-continuous, since for the g-closed set $\{a, c\}$ in $Y$, the inverse image of $\{a, c\}$ is not $\tau^*$-g-closed in $X$.

**Remark 4.2.21:** The following examples show that $\tau^*$-g-continuous function is independent from the sp-continuous function.

**Example 4.2.22:** Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, b\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is
\(\tau^*-g\)-continuous. On the other hand, it is not sp-continuous, since for the open set \{b\} in \(Y\), the inverse image of \{b\} is not sp-open in \(X\).

**Example 4.2.23:** Let \(X = Y = \{a, b, c\}\), \(\tau = \{X, \phi, \{a\}, \{a, c\}\}\) and \(\sigma = \{Y, \phi, \{a\}, \{a, b\}\}\). Let \(f : X \to Y\) be an identity function. Then \(f\) is sp-continuous. However, it is not \(\tau^*-g\)-continuous, since for the \(g\)-closed set \{c\} in \(Y\), the inverse image of \{c\} is not \(\tau^*-g\)-closed in \(X\).

**Remark 4.2.24:** The following examples show that \(\tau^*-g\)-continuous function is independent from the sg-continuous function.

**Example 4.2.25:** Let \(X = Y = \{a, b, c\}\), \(\tau = \{X, \phi, \{a\}, \{a, b\}\}\) and \(\sigma = \{Y, \phi, \{b\}, \{a, b\}\}\). Let \(f : X \to Y\) be an identity function. Then \(f\) is \(\tau^*-g\)-continuous. On the other hand, it is not sg-continuous, since for the closed set \{a, c\} in \(Y\), the inverse image of \{a, c\} is not sg-closed in \(X\).

**Example 4.2.26:** Let \(X = Y = \{a, b, c\}\), \(\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}\) and \(\sigma = \{Y, \phi, \{a\}, \{a, b\}\}\). Let \(f : X \to Y\) be an identity function. Then \(f\) is sg-continuous. However, it is not \(\tau^*-g\)-continuous, since for the \(g\)-closed set \{a, c\} in \(Y\), the inverse image of \{a, c\} is not \(\tau^*-g\)-closed in \(X\).

**Remark 4.2.27:** The following examples show that \(\tau^*-g\)-continuous function is independent from the semi-continuous function.

**Example 4.2.28:** Let \(X = Y = \{a, b, c\}\), \(\tau = \{X, \phi, \{a\}, \{a, b\}\}\) and \(\sigma = \{Y, \phi, \{b\}, \{a, b\}\}\). Let \(f : X \to Y\) be an identity function. Then \(f\) is \(\tau^*-g\)-continuous. On the contrary, it is not semi-continuous, since for the open set \{b\} in \(Y\), the inverse image of \{b\} is not semi-open in \(X\).
Example 4.2.29: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b, c\}\}$. Let $f : X \to Y$ be an identity function. Then $f$ is semi-continuous. However, it is not $\tau^\ast$-g-continuous, since for the g-closed set $\{a, b\}$ in $Y$ the inverse image of $\{a, b\}$ is not $\tau^\ast$-g-closed in $X$.

Remark 4.2.30: The following examples show that $\tau^\ast$-g-continuous function is independent from the pre-continuous function.

Example 4.2.31: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, b\}\}$. Let $f : X \to Y$ be an identity function. Then $f$ is $\tau^\ast$-g-continuous. On the other hand, it is not pre-continuous, since for the open set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not pre-open in $X$.

Example 4.2.32: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Let $f : X \to Y$ be an identity function. Then $f$ is pre-continuous. However, it is not $\tau^\ast$-g-continuous, since for the g-closed set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not $\tau^\ast$-g-closed in $X$.

Remark 4.2.33: The following examples show that $\tau^\ast$-g-continuous function is independent from the gs$^\ast$-continuous function.

Example 4.2.34: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, b\}\}$. Let $f : X \to Y$ be an identity function. Then $f$ is $\tau^\ast$-g-continuous. But it is not gs$^\ast$-continuous, since for the semi-open set $\{b, c\}$ in $Y$, the inverse image of $\{b, c\}$ is not gs-open in $X$.

Example 4.2.35: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Let $f : X \to Y$ be an identity function. Then $f$ is gs$^\ast$-continuous. On the other hand, it is not $\tau^\ast$-g-continuous, since for the g-closed set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not $\tau^\ast$-g-closed in $X$. 
Remark 4.2.36: The following examples show that $\tau^*-g$-continuous function is independent from the $sg^*$-continuous function.

Example 4.2.37: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, b\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is $\tau^*-g$-continuous. However, it is not $sg^*$-continuous, since for the semi-open set $\{b, c\}$ in $Y$, the inverse image of $\{b, c\}$ is not $sg^*$-open in $X$.

Example 4.2.38: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, b\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is $sg^*$-continuous. But it is not $\tau^*-g$-continuous, since for the $g$-closed set $\{a\}$ in $Y$, the inverse image of $\{a\}$ is not $\tau^*-g$-closed in $X$.

Remark 4.2.39: The following examples show that $\tau^*-g$-continuous function is independent from the strongly $sg^*$-continuous function.

Example 4.2.40: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, b\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is $\tau^*-g$-continuous. On the other hand, it is not strongly $sg^*$-continuous, since for the $sg^*$-open set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not open in $X$.

Example 4.2.41: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, c\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is strongly $sg^*$-continuous. However, it is not $\tau^*-g$-continuous, since for the $g$-closed set $\{a, b\}$ in $Y$, the inverse image of $\{a, b\}$ is not $\tau^*-g$-closed in $X$.

Remark 4.2.42: The following examples show that $\tau^*-g$-continuous function is independent from the weakly $sg^*$-continuous function.
Example 4.2.43: Let \( X = Y = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{b\}, \{a, b\}\} \). Let \( f : X \to Y \) be an identity function. Then \( f \) is \( \tau^* - g \)-continuous. On the contrary, it is not weakly sg-continuous, since for the sg-open set \( \{b, c\} \) in \( Y \), the inverse image of \( \{b, c\} \) is not semi-open in \( X \).

Example 4.2.44: Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\} \) and \( \sigma = \{Y, \phi, \{a\}\} \). Let \( f : X \to Y \) be an identity function. Then \( f \) is weakly sg-continuous. But it is not \( \tau^* - g \)-continuous, since for the g-closed set \( \{a, b\} \) in \( Y \), the inverse image of \( \{a, b\} \) is not \( \tau^* - g \)-closed in \( X \).

Remark 4.2.45: The following examples show that \( \tau^* - g \)-continuous function is independent from the strongly gs-continuous function.

Example 4.2.46: Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{a, b\}\} \) and \( \sigma = \{Y, \phi, \{b\}, \{a, b\}\} \). Let \( f : X \to Y \) be an identity function. Then \( f \) is \( \tau^* - g \)-continuous. On the other hand, it is not strongly gs-continuous, since for the gs-open set \( \{a\} \) in \( Y \), the inverse image of \( \{a\} \) is not open in \( X \).

Example 4.2.47: Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{c\}, \{c, a\}, \{c, b\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}\} \). Let \( f : X \to Y \) be an identity function. Then \( f \) is strongly gs-continuous. However, it is not \( \tau^* - g \)-continuous, since for the g-closed set \( \{b, c\} \) in \( Y \), the inverse image of \( \{b, c\} \) is not \( \tau^* - g \)-closed in \( X \).

Remark 4.2.48: The following examples show that \( \tau^* - g \)-continuous function is independent from the gsp-continuous function.

Example 4.2.49: Let \( X = Y = \{a, b, c\} \), \( \tau = \{X, \phi, \{c\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\} \). Let \( f : X \to Y \) be an identity function. Then \( f \) is \( \tau^* - g \)-continuous. But it is not gsp-continuous, since for the closed set \( \{c\} \) in \( Y \), the inverse image of \( \{c\} \) is not gsp-closed in \( X \).
Example 4.2.50: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{c\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{c\}, \{c, a\}, \{c, b\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is gsp-continuous. On the other hand, it is not $\tau^* - \sigma$-continuous, since for the $\sigma$-closed set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not $\tau^* - \sigma$-closed in $X$.

Remark 4.2.51: The following examples show that $\tau^* - \sigma$-continuous function is independent from the gsp-continuous function.

Example 4.2.52: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is $\tau^* - \sigma$-continuous. On the contrary, it is not gsp-continuous, since for the closed set $\{c\}$ in $Y$, the inverse image of $\{c\}$ is not gsp-closed in $X$.

Example 4.2.53: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{c\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{b\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is gsp-continuous. On the other hand, it is not $\tau^* - \sigma$-continuous, since for the $\sigma$-closed set $\{b, c\}$ in $Y$, the inverse image of $\{b, c\}$ is not $\tau^* - \sigma$-closed in $X$.

Remark 4.2.54: The following examples show that $\tau^* - \sigma$-continuous function is independent from the $\alpha \sigma$-continuous function.

Example 4.2.55: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is $\tau^* - \sigma$-continuous. However, it is not $\alpha \sigma$-continuous, since for the closed set $\{a\}$ in $Y$, the inverse image of $\{a\}$ is not $\alpha \sigma$-closed in $X$.

Example 4.2.56: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \phi, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Let $f : X \rightarrow Y$ be an identity function. Then $f$ is $\alpha \sigma$-continuous. But it
is not $\tau^*\text{-}g$-continuous, since for the $g$-closed set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not $\tau^*\text{-}g$-closed in $X$.

**Remark 4.2.57:** Pictorial representation of the above discussion is shown in Figure 4.1.

![Figure 4.1 Isolation of $\tau^*\text{-}g$–continuous function](image-url)

### 4.3 $\tau^*$- GC– IRRESOLUTE FUNCTIONS IN TOPOLOGICAL SPACES

In this section, a new class of function called $\tau^*$-gc-irresolute function is introduced which is included in the class of $\tau^*$-g-continuous function. Also, two forms of functions in topological spaces namely strongly $\tau^*$-g-continuous function and perfectly $\tau^*$-g-continuous function are introduced.

**Definition 4.3.1:** A function $f : X \to Y$ from a topological space $(X, \tau^*)$ into a topological space $(Y, \sigma^*)$ is called $\tau^*$-ge-irresolute if the inverse image of every $\tau^*$-g-closed set in $Y$ is $\tau^*$-g-closed in $X$. 
**Theorem 4.3.2:** A function \( f : X \rightarrow Y \) is \( \tau^*\)-gc-irresolute if and only if the inverse image of every \( \tau^*\)-g-open set in \( Y \) is \( \tau^*\)-g-open in \( X \).

**Proof:** Assume that \( f \) is \( \tau^*\)-gc-irresolute. Suppose \( A \) is a \( \tau^*\)-g-open set in \( Y \). Then \( A^C \) is \( \tau^*\)-g-closed in \( Y \). By assumption, \( f^{-1}(A^C) \) is \( \tau^*\)-g-closed in \( X \). But \( f^{-1}(A^C) = X - f^{-1}(A) \) and so \( f^{-1}(A) \) is \( \tau^*\)-g-open in \( X \). Hence the inverse image of every \( \tau^*\)-g-open set in \( Y \) is \( \tau^*\)-g-open in \( X \).

Conversely assume that the inverse image of every \( \tau^*\)-g-open set in \( Y \) is \( \tau^*\)-g-open in \( X \). Suppose \( A \) is a \( \tau^*\)-g-closed in \( Y \), then \( A^C \) is \( \tau^*\)-g-open in \( Y \). By assumption, \( f^{-1}(A^C) \) is \( \tau^*\)-g-open in \( X \). But \( f^{-1}(A^C) = X - f^{-1}(A) \) and so \( f^{-1}(A) \) is \( \tau^*\)-g-closed in \( X \). Therefore \( f \) is \( \tau^*\)-gc-irresolute.

**Theorem 4.3.3:** Let \( X, Y \) and \( Z \) be any topological spaces. For a \( \tau^*\)-gc-irresolute function \( f : X \rightarrow Y \) and a \( \tau^*\)-g-continuous function \( g : Y \rightarrow Z \), the composition \( g \circ f : X \rightarrow Z \) is \( \tau^*\)-g-continuous.

**Proof:** Let \( F \) be any \( \tau^*\)-g-closed set in \( Z \). Since \( g \) is \( \tau^*\)-g-continuous, \( g^{-1}(F) \) is \( \tau^*\)-g-closed in \( Y \). Since \( f \) is \( \tau^*\)-gc-irresolute, \( f^{-1}(g^{-1}(F)) \) is \( \tau^*\)-g-closed in \( X \). But \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \). Therefore \( g \circ f \) is \( \tau^*\)-g-continuous.

**Theorem 4.3.4:** If a function \( f : X \rightarrow Y \) from a topological space \( (X, \tau^*) \) into a topological space \( (Y, \sigma^*) \) is bijective, \( \tau^*\)-g-open and \( \tau^*\)-g-continuous then \( f \) is \( \tau^*\)-gc-irresolute.

**Proof:** Let \( A \) be a \( \tau^*\)-g-closed set in \( Y \) and \( f^{-1}(A) \subseteq O \), where \( O \) is \( \tau^*\)-g-open in \( X \). Therefore \( A \subseteq f(O) \) holds. Since \( f(O) \) is \( \tau^*\)-g-open and \( A \) is \( \tau^*\)-g-closed in \( Y \), \( \text{cl}(A) \subseteq f(O) \) holds and hence \( f^{-1}(\text{cl}(A)) \subseteq O \). Since \( f \) is \( \tau^*\)-g-continuous and \( \text{cl}(A) \) is \( \tau^*\)-g-closed in \( Y \), \( \text{cl}(f^{-1}(\text{cl}(A))) \subseteq O \) and so \( \text{cl}(f^{-1}(A)) \subseteq O \). Therefore, \( f^{-1}(A) \) is \( \tau^*\)-g-closed in \( X \). Hence \( f \) is \( \tau^*\)-gc-irresolute.
**Theorem 4.3.5:** If a function $f : X \to Y$ is gc-irresolute then it is $\tau^*$-gc-irresolute provided $Y$ is a $\tau^*$-$T_g$ space.

**Proof:** Assume that $f : X \to Y$ is gc-irresolute. Suppose $V$ is a $\tau^*$-g-closed set in $Y$. Since $Y$ is a $\tau^*$-$T_g$ space, $V$ is g-closed in $Y$. By the assumption of $f$, $f^{-1}(V)$ is g-closed in $X$. But by Theorem 3.2.13, $f^{-1}(V)$ is $\tau^*$-g-closed in $X$. Therefore $f$ is $\tau^*$-gc-irresolute.

**Remark 4.3.6:** The above theorem need not be true as seen from the following example if $Y$ is not a $\tau^*$-$T_g$ space.

**Example 4.3.7:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Then $f$ is gc-irresolute. On the other hand, it is not gc-irresolute, since for the $\tau^*$-g-closed set $\{a\}$ in $Y$, the inverse image of $\{a\}$ is not $\tau^*$-g-closed in $X$.

**Remark 4.3.8:** The following examples show that $\tau^*$-gc-irresolute function is independent from the pre-irresolute function.

**Example 4.3.9:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$. Then $f$ is $\tau^*$-gc-irresolute. However, it is not pre-irresolute, since for the pre-open set $\{c\}$ in $Y$, the inverse image of $\{c\}$ is not pre-open in $X$.

**Example 4.3.10:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{b\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, b\}, \{b, c\}\}$. Then $f$ is pre-irresolute. On the other hand, it is not $\tau^*$-gc-irresolute, since for the $\tau^*$-g-closed set $\{c\}$ in $Y$, the inverse image of $\{c\}$ is not $\tau^*$-g-closed in $X$.

**Remark 4.3.11:** The following examples show that $\tau^*$-gc-irresolute function is independent from the irresolute function.
**Example 4.3.12:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$. Then $f$ is $\tau^*$-irresolute. But it is not irresolute, since for the semi-open set $\{a, b\}$ in $Y$, the inverse image of $\{a, b\}$ is not semi-open in $X$.

**Example 4.3.13:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $f$ is irresolute. On the contrary, it is not $\tau^*$-irresolute, since for the $\tau^*$-closed set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not $\tau^*$-closed in $X$.

**Remark 4.3.14:** The following examples show that $\tau^*$-irresolute function is independent from the $\alpha$-irresolute function.

**Example 4.3.15:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $f$ is $\tau^*$-irresolute. However, it is not $\alpha$-irresolute, since for the $\alpha$-open set $\{a, b\}$ in $Y$, the inverse image of $\{a, b\}$ is not $\alpha$-open in $X$.

**Example 4.3.16:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{c\}, \{a, b\}\}$. Then $f$ is $\alpha$-irresolute. On the other hand, it is not $\tau^*$-irresolute, since for the $\tau^*$-closed set $\{a\}$ in $Y$, the inverse image of $\{a\}$ is not $\tau^*$-closed in $X$.

**Remark 4.3.17:** The following examples show that $\tau^*$-irresolute function is independent from the sg-irresolute function.

**Example 4.3.18:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}\}$. Then $f$ is $\tau^*$-irresolute. But it is not sg-irresolute, since for the sg-closed set $\{c\}$ in $Y$, the inverse image of $\{c\}$ is not sg-closed in $X$. 
Example 4.3.19: Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{c\}, \{a, c\}, \{c, b\}\}$ and $\sigma = \{Y, \phi, \{c\}\}$. Then $f$ is sg-irresolute. However, it is not $\tau^*\text{-}gc$-irresolute, since for the $\tau^*\text{-}g$-closed set $\{b, c\}$ in $Y$, the inverse image of $\{b, c\}$ is not $\tau^*\text{-}g$-closed in $X$.

Remark 4.3.20: The following examples show that $\tau^*\text{-}gc$-irresolute function is independent from the gs-irresolute function.

Example 4.3.21: Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, c\}\}$. Then $f$ is $\tau^*\text{-}gc$-irresolute. On the other hand, it is not gs-irresolute, since for the gs-closed set $\{a\}$ in $Y$, the inverse image of $\{a\}$ is not gs-closed in $X$.

Example 4.3.22: Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then $f$ is gs-irresolute. On the contrary, it is not $\tau^*\text{-}gc$-irresolute, since for the $\tau^*\text{-}g$-closed set $\{c\}$ in $Y$, the inverse image of $\{c\}$ is not $\tau^*\text{-}g$-closed in $X$.

Remark 4.3.23: The following examples show that $\tau^*\text{-}gc$-irresolute function is independent from the gsp-irresolute function.

Example 4.3.24: Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, c\}\}$. Then $f$ is $\tau^*\text{-}gc$-irresolute. On the other hand, it is not gsp-irresolute, since for the gsp-closed set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not gsp-closed in $X$.

Example 4.3.25: Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}\}$. Then $f$ is gsp-irresolute. However, it is not $\tau^*\text{-}gc$-irresolute, since for the $\tau^*\text{-}g$-closed set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not $\tau^*\text{-}g$-closed in $X$. 
Remark 4.3.26: The following examples show that $\tau^*\text{-gc}$-irresolute function is independent from the $\alpha g$-irresolute function.

Example 4.3.27: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a, c\}\}$. Then $f$ is $\tau^*\text{-gc}$-irresolute. On the contrary, it is not $\alpha g$-irresolute, since for the $\alpha g$-closed set $\{b\}$ in $Y$, the inverse image of $\{b\}$ is not $\alpha g$-closed in $X$.

Example 4.3.28: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{a\}, \{a, c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Then $f$ is $\alpha g$-irresolute. However, it is not $\tau^*\text{-gc}$-irresolute, since for the $\tau^*\text{-g}$-closed set $\{a\}$ in $Y$, the inverse image of $\{a\}$ is not $\tau^*\text{-g}$-closed in $X$.

Remark 4.3.29: The following examples show that $\tau^*\text{-gc}$-irresolute function is independent from the $g^*\text{-irresolute}$ function.

Example 4.3.30: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then $f$ is $\tau^*\text{-gc}$-irresolute. On the contrary, it is not $g^*$-irresolute, since for the $g^*$-closed set $\{b, c\}$ in $Y$, the inverse image of $\{b, c\}$ is not $g^*$-closed in $X$.

Example 4.3.31: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \emptyset, \{c\}\}$. Then $f$ is $g^*$-irresolute. But it is not $\tau^*\text{-gc}$-irresolute, since for the $\tau^*\text{-g}$-closed set $\{a\}$ in $Y$, the inverse image of $\{a\}$ is not $\tau^*\text{-g}$-closed in $X$.

Remark 4.3.32: The following examples show that $\tau^*\text{-gc}$-irresolute function is independent from the $g\alpha\text{-irresolute}$ function.

Example 4.3.33: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{a\}\}$ and $\sigma = \{Y, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$. Then $f$ is
\( \tau^*\text{-gc- irresolute.} \) However, it is not \( g\alpha\text{-irresolute,} \) since for the \( g\alpha\text{-closed set} \) \( \{a\} \) in \( Y \), the inverse image of \( \{a\} \) is not \( g\alpha\text{-closed in} \) \( X \).

**Example 4.3.34:** Let \( X = Y = \{a, b, c\} \) and let \( f : X \to Y \) be an identity function. Let \( \tau = \{X, \emptyset, \{b, c\}\} \) and \( \sigma = \{Y, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}\} \). Then \( f \) is \( g\alpha\text{-irresolute.} \) On the other hand, it is not \( \tau^*\text{-gc- irresolute,} \) since for the \( \tau^*\text{-g-closed set} \{c\} \) in \( Y \), the inverse image of \( \{c\} \) is not \( \tau^*\text{-g-closed in} \) \( X \).

**Remark 4.3.35:** From the above discussion, we obtain the following Figure 4.2.

\[
\begin{array}{cccc}
\text{pre-irresolute} & \text{irresolute} & \text{gsp-irresolute} \\
\alpha g\text{- irresolute} & \tau^*\text{-gc-irresolute} & \alpha g\text{- irresolute} \\
\alpha\text{- irresolute} & \text{sg- irresolute} & \text{gs- irresolute} & \text{g}^*\text{- irresolute}
\end{array}
\]

**Figure 4.2** Independence of \( \tau^*\text{-gc-irresolute function from other irresolute functions} \)

**Definition 4.3.36:** A function \( f : X \to Y \) from a topological space \((X, \tau^*)\) into a topological space \((Y, \sigma^*)\) is said to be strongly \( \tau^*\text{-g-continuous} \) if the inverse image of every \( \tau^*\text{-g-open set} \) (or \( \tau^*\text{-g-closed set} \)) in \( Y \) is \( g\text{-open} \) (or \( g\text{-closed} \)) in \( X \).

**Theorem 4.3.37:** If a function \( f : X \to Y \) from a topological space \((X, \tau^*)\) into a topological space \((Y, \sigma^*)\) is strongly \( \tau^*\text{-g-continuous} \) then it is \( \tau^*\text{-g-continuous.} \)

**Proof:** Assume that \( f \) is strongly \( \tau^*\text{-g-continuous.} \) Suppose \( G \) is any \( g\text{-closed} \) set in \( Y \). By Theorem 3.2.13, \( G \) is \( \tau^*\text{-g-closed} \) in \( Y \). By assumption, \( f^{-1}(G) \) is \( g\text{-closed} \) in \( X \). Therefore \( f \) is \( \tau^*\text{-g-continuous.} \)
Remark 4.3.38: Converse of the above theorem need not be true as seen from the following example.

Example 4.3.39: Let \( \tau = \{X, \phi, \{c\}\}, \sigma = \{Y, \phi, \{c\}, \{a, b\}\} \), where \( X = Y = \{a, b, c\} \). Let \( f : X \to Y \) be an identity function. Then \( f \) is \( \tau^*\)-g-continuous. But it is not strongly \( \tau^*\)-g-continuous, since for the \( \tau^*\)-g-closed set \( \{c\} \) in \( Y \), the inverse image of \( \{c\} \) is not \( g \) closed in \( X \).

Theorem 4.3.40: A function \( f : X \to Y \) from a topological space \((X, \tau^*)\) into a topological space \((Y, \sigma^*)\) is strongly \( \tau^*\)-g-continuous if and only if the inverse image of every \( \tau^*\)-g-closed set in \( Y \) is \( g \) closed in \( X \).

Proof: Assume that \( f \) is strongly \( \tau^*\)-g-continuous. Suppose \( F \) is a \( \tau^*\)-g-closed set in \( Y \). Then \( F^C \) is \( \tau^*\)-g-open set in \( Y \). By assumption, \( f^{-1}(F^C) \) is \( g \)-open in \( X \). But \( f^{-1}(F^C) = X - f^{-1}(F) \) and so \( f^{-1}(F) \) is \( g \)-closed in \( X \).

Conversely assume that the inverse image of every \( \tau^*\)-g-closed set in \( Y \) is \( g \)-closed in \( X \). Suppose \( G \) is a \( \tau^*\)-g-open set in \( Y \). Then \( G^C \) is \( \tau^*\)-g-closed set in \( Y \). By assumption, \( f^{-1}(G^C) \) is \( g \)-closed in \( X \). But \( f^{-1}(G^C) = X - f^{-1}(G) \) and so \( f^{-1}(G) \) is \( g \)-open in \( X \). Therefore \( f \) is strongly \( \tau^*\)-g-continuous.

Theorem 4.3.41: If a function \( f : X \to Y \) is strongly \( \tau^*\)-g-continuous and a function \( g : Y \to Z \) is \( \tau^*\)-g-continuous then the composition \( g \circ f : X \to Z \) is \( \tau^*\)-g-continuous.

Proof: Let \( G \) be any \( g \)-closed set in \( Z \). Since \( g \) is \( \tau^*\)-g-continuous, \( g^{-1}(G) \) is \( \tau^*\)-g-closed in \( Y \). Since \( f \) is strongly \( \tau^*\)-g-continuous, \( f^{-1}(g^{-1}(G)) \) is \( g \)-closed in \( X \). By Theorem 3.2.13, \( f^{-1}(g^{-1}(G)) \) is \( \tau^*\)-g-closed. But \( f^{-1}(g^{-1}(G)) = (g \circ f)^{-1}(G) \). Therefore \( g \circ f \) is \( \tau^*\)-g-continuous.
Theorem 4.3.42: If a function \( f : X \rightarrow Y \) from a topological space \((X, \tau)\) into a topological space \((Y, \sigma)\) is continuous, then it is strongly \( \tau^* \)-g-continuous provided \( Y \) is both \( \tau^* \)-T\(_g\) space and T\(_{1/2}\) space.

**Proof:** Let \( f : X \rightarrow Y \) be continuous. Suppose \( F \) is a \( \tau^* \)-g-closed set in \( Y \). Since \( Y \) is \( \tau^* \)-T\(_g\) space and T\(_{1/2}\) space, \( F \) is closed in \( Y \). Then by the assumption, \( f^{-1}(F) \) is closed in \( X \). Again, since every closed set is g-closed, \( f^{-1}(F) \) is g-closed. Hence \( f \) is strongly \( \tau^* \)-g-continuous.

Remark 4.3.43: The following example shows that the above theorem need not be true if \( Y \) is not \( \tau^* \)-T\(_g\) space and T\(_{1/2}\) space.

**Example 4.3.44:** Let \( X = Y = \{a, b, c\} \). Let \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\} \) and \( \sigma = \{Y, \phi, \{a\}\} \). Here \( Y \) is not a \( \tau^* \)-T\(_g\) space. Let \( f : X \rightarrow Y \) be an identity function. Then \( f \) is continuous. But it is not strongly \( \tau^* \)-g-continuous, since for the \( \tau^* \)-g-closed set \( V = \{b\} \) in \( Y \), \( f^{-1}(V) = \{b\} \) is not g-closed in \( X \).

**Example 4.3.45:** Let \( X = Y = \{a, b, c\} \). Let \( \tau = \{X, \phi, \{c\}, \{c, a\}, \{c, b\}\} \) and \( \sigma = \{Y, \phi, \{c\}, \{a, c\}\} \). Here \( Y \) is not a T\(_{1/2}\) space. Let \( f : X \rightarrow Y \) be an identity function. Then \( f \) is continuous. But it is not strongly \( \tau^* \)-g-continuous, since for the \( \tau^* \)-g-closed set \( V = \{b, c\} \) in \( Y \), \( f^{-1}(V) = \{b, c\} \) is not g-closed in \( X \).

Theorem 4.3.46: If a function \( f : X \rightarrow Y \) from a topological space \((X, \tau)\) into a topological space \((Y, \sigma)\) is g-continuous then it is strongly \( \tau^* \)-g-continuous provided \( Y \) is both \( \tau^* \)-T\(_g\)-space and T\(_{1/2}\) space.

**Proof:** Let \( f : X \rightarrow Y \) be a g-continuous function. Suppose \( F \) is a \( \tau^* \)-g-closed set \( Y \). Since \( Y \) is both \( \tau^* \)-T\(_g\)-space and T\(_{1/2}\) space, \( F \) is closed in \( Y \). By
definition of g-continuous function, \( f^{-1}(F) \) is g-closed in X. Thus for the \( \tau^* \)-g-closed set F in Y, the inverse image \( f^{-1}(F) \) is g-closed in X. Therefore \( f \) is strongly \( \tau^* \)-g-continuous.

**Remark 4.3.47:** The following example shows that the above theorem need not be true if Y is not \( \tau^* \)-T_g-space and T_{1/2} space.

**Example 4.3.48:** Let \( X = Y = \{a, b, c\} \). Let \( \tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\} \) and \( \sigma = \{Y, \phi, \{a\}, \{a, b\}\} \). Let \( f : X \to Y \) be an identity function. Here Y is not a T_{1/2} space. Clearly \( f \) is g-continuous. On the contrary, it is not strongly \( \tau^* \)-g-continuous, since for the \( \tau^* \)-g-closed set \( V = \{a, c\} \) in Y, \( f^{-1}(V) = \{a, c\} \) is not g-closed in X.

**Example 4.3.49:** Let \( X = Y = \{a, b, c\} \). Let \( \tau = \{X, \phi, \{b\}, \{c\}, \{b, c\}\} \) and \( \sigma = \{Y, \phi, \{c\}\} \). Let \( f : X \to Y \) be an identity function. Here Y is not a \( \tau^* \)-T_g space. Clearly \( f \) is g-continuous. On the contrary, it is not strongly \( \tau^* \)-g-continuous, since for the \( \tau^* \)-g-closed set \( V = \{b\} \) in Y, \( f^{-1}(V) = \{b\} \) is not g-closed in X.

**Definition 4.3.50:** A function \( f : X \to Y \) from a topological space \( (X, \tau^*) \) into a topological space \( (Y, \sigma^*) \) is said to be perfectly \( \tau^* \)-g-continuous if the inverse image of every \( \tau^* \)-g-closed set in Y is both g-open and g-closed in X.

**Theorem 4.3.51:** A function \( f : X \to Y \) from a topological space \( (X, \tau^*) \) into a topological space \( (Y, \sigma^*) \) is perfectly \( \tau^* \)-g-continuous if and only if the inverse image of every \( \tau^* \)-g-open set in Y is both g-open and g-closed in X.

**Proof:** Assume that \( f \) is a perfectly \( \tau^* \)-g-continuous function. Suppose \( F \) is a \( \tau^* \)-g-open set in Y. Then \( F^c \) is \( \tau^* \)-g-closed in Y. By assumption, \( f^{-1}(F^c) \) is both g-open and g-closed in X. But \( f^{-1}(F^c) = X - f^{-1}(F) \) and so \( f^{-1}(F) \) is both g-open and g-closed in X.
Conversely assume that the inverse image of every \(\tau^*-g\)-open set in \(Y\) is both \(g\)-open and \(g\)-closed in \(X\). Suppose \(G\) is a \(\tau^*-g\)-closed set in \(Y\). Then \(G^c\) is \(\tau^*-g\)-open in \(Y\). By assumption, \(f^{-1}(G^c)\) is both \(g\)-open and \(g\)-closed in \(X\). But \(f^{-1}(G^c) = X - f^{-1}(G)\) and so \(f^{-1}(G)\) is both \(g\)-open and \(g\)-closed in \(X\). Therefore \(f\) is perfectly \(\tau^*-g\)-continuous.

**Theorem 4.3.52:** If a function \(f : X \to Y\) from a topological space \((X, \tau^*)\) into a topological space \((Y, \sigma^*)\) is perfectly \(\tau^*-g\)-continuous then it is strongly \(\tau^*-g\)-continuous.

**Proof:** Assume that \(f\) is perfectly \(\tau^*-g\)-continuous. Suppose \(G\) is a \(\tau^*-g\)-closed set in \(Y\). Then \(f^{-1}(G)\) is \(g\)-closed in \(X\). Therefore \(f\) is strongly \(\tau^*-g\)-continuous.

**Remark 4.3.53:** Converse of the above theorem need not be true as seen from the following example.

**Example 4.3.54:** Let \(\tau = \{\emptyset, \{a\}\}\), \(\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}\). Let \(f : X \to Y\) be an identity function. Then \(f\) is strongly \(\tau^*-g\)-continuous. But it is not perfectly \(\tau^*-g\)-continuous, since for the \(\tau^*-g\)-closed set \(\{b, c\}\) in \(Y\), the inverse image of \(\{b, c\}\) is \(g\)-open but not \(g\)-closed in \(X\).

**Theorem 4.3.55:** If a function \(f : X \to Y\) from a topological space \((X, \tau^*)\) into a topological space \((Y, \sigma^*)\) is strongly \(\tau^*-g\)-continuous then it is \(\tau^*-gc\)-irresolute.

**Proof:** Let \(f : X \to Y\) be strongly \(\tau^*-g\)-continuous function. Suppose \(F\) is a \(\tau^*-g\)-closed set in \(Y\). By assumption, \(f^{-1}(F)\) is \(g\)-closed in \(X\). By Theorem 3.2.13, \(f^{-1}(F)\) is \(\tau^*-g\)-closed in \(X\). Hence \(f\) is \(\tau^*-gc\)-irresolute.

**Remark 4.3.56:** Converse of the above theorem need not be true as seen from the following example.
Example 4.3.57: Let \( \tau = \{X, \phi, \{c\}\} \), \( \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\} \). Let \( f : X \to Y \) be an identity function. Then \( f \) is \( \tau^* \)-gc-irresolute. However, it is not strongly \( \tau^* \)-g-continuous, since for the \( \tau^* \)-g-closed set \( \{c\} \) in \( Y \), the inverse image of \( \{c\} \) is not \( g \)-closed in \( X \).

Theorem 4.3.58: If a function \( f : X \to Y \) from a topological space \((X, \tau^*)\) into a topological space \((Y, \sigma^*)\) is perfectly \( \tau^* \)-g-continuous, then it is \( \tau^* \)-gc-irresolute.

Proof: Let \( f : X \to Y \) be a perfectly \( \tau^* \)-g-continuous function. Suppose \( F \) is a \( \tau^* \)-g-closed set in \( Y \). Then by assumption, \( f^{-1}(F) \) is both \( g \)-open and \( g \)-closed in \( X \). By Theorem 3.2.13, \( f^{-1}(F) \) is \( \tau^* \)-g-closed in \( X \). Hence \( f \) is \( \tau^* \)-gc-irresolute.

Remark 4.3.59: Converse of the above theorem need not be true as seen from the following example.

Example 4.3.60: Let \( \tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\} \), \( \sigma = \{Y, \phi, \{b\}, \{a, b\}, \{b, c\}\} \). Let \( f : X \to Y \) be a function defined by \( f(a) = b \), \( f(b) = c \), \( f(c) = a \). Then \( f \) is \( \tau^* \)-gc-irresolute. On the other hand, it is not perfectly \( \tau^* \)-g-continuous, since for the \( \tau^* \)-g-closed set \( \{c\} \) in \( Y \), the inverse image of \( \{c\} \) is \( g \)-open in \( X \) but not \( g \)-closed in \( X \).

4.4 CONCLUSION

In this chapter, continuous functions, namely \( \tau^* \)-generalized continuous function, strongly \( \tau^* \)-g-continuous function, perfectly \( \tau^* \)-g-continuous function and \( \tau^* \)-gc-irresolute function in topological spaces are introduced and their characteristics are investigated. These functions are compared with several other continuous functions. In future, these functions can be studied elaborately on various other topological spaces.