CHAPTER 5

τ*-GENERALIZED HOMEOMORPHISM IN TOPOLOGICAL SPACES

5.1 INTRODUCTION


5.2 τ*-GENERALIZED OPEN MAPS AND τ*-GENERALIZED CLOSED MAPS IN TOPOLOGICAL SPACES

In this section, the notion of τ*-generalized open maps and τ*-generalized closed maps in topological spaces are introduced and some of their properties are investigated. Relationships of these maps with some existing maps are examined.

Definition 5.2.1: A map $f : X \rightarrow Y$ is said to be τ*-generalized open map (respectively τ*-generalized closed map) if for each g-open set (respectively g-closed set) $U$ in $X$, $f(U)$ is τ*-g-open set (respectively τ*-g-closed set) in $Y$.

Theorem 5.2.2: For any bijection $f : X \rightarrow Y$, the following statements are equivalent:
(a) The inverse function \( f^{-1} : Y \rightarrow X \) is \( \tau^* \)-g-continuous.

(b) \( f \) is a \( \tau^* \)-g-open map.

(c) \( f \) is a \( \tau^* \)-g-closed map.

Proof:

(a) \( \Rightarrow \) (b). Let \( G \) be any \( g \)-open set in \( X \). Since \( f^{-1} \) is \( \tau^* \)-g-continuous, the inverse image of \( G \) under \( f^{-1} \) is \( \tau^* \)-g-open in \( Y \). That is \( (f^{-1})^{-1}(G) = f(G) \) is \( \tau^* \)-g-open in \( Y \) and so \( f \) is a \( \tau^* \)-g-open map. Hence (a) \( \Rightarrow \) (b).

(b) \( \Rightarrow \) (c) Let \( F \) be any \( g \)-closed set in \( X \). Then \( F^c \) is \( g \)-open in \( X \). Since \( f \) is a \( \tau^* \)-g-open map, \( f(F^c) \) is \( \tau^* \)-g-open in \( Y \). But \( f(F^c) = Y - f(F) \). Therefore \( Y - f(F) \) is \( \tau^* \)-g-open in \( Y \) and so \( f(F) \) is \( \tau^* \)-g-closed in \( Y \). Hence, \( f \) is a \( \tau^* \)-g-closed map. Thus, (b) \( \Rightarrow \) (c).

(c) \( \Rightarrow \) (a) Let \( F \) be any \( g \)-closed set in \( X \). Since \( f \) is a \( \tau^* \)-g-closed map, \( f(F) \) is \( \tau^* \)-g-closed in \( Y \). But \( f(F) = (f^{-1})^{-1}(F) \). Therefore the inverse map \( f^{-1} \) is \( \tau^* \)-g-continuous. Thus (c) \( \Rightarrow \) (a). Hence (a), (b) and (c) are equivalent.

**Theorem 5.2.3:** A map \( f : X \rightarrow Y \) is \( \tau^* \)-g-closed if and only if for each subset \( S \) of \( Y \) and for each \( g \)-open set \( U \) containing \( f^l(S) \), there is a \( \tau^* \)-g-open set \( V \) of \( Y \) such that \( S \subseteq V \) and \( f^l(V) \subseteq U \).

**Proof:** Suppose \( f \) is a \( \tau^* \)-g-closed map. Let \( S \) be a subset of \( Y \) and \( U \) be a \( g \)-open set of \( X \) such that \( f^l(S) \subseteq U \). Then \( V = Y - f(X - U) \) is a \( \tau^* \)-g-open set containing \( S \) such that \( f^l(V) \subseteq U \).
Conversely, suppose $F$ is a g-closed set in $X$. Then $f^{-1}(Y-f(F)) = X - F$ and $X - F$ is g-open. By hypothesis, there is a $\tau^*$-g-open set $V$ of $Y$ such that $Y-f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore $F \subseteq X - f^{-1}(V)$. Hence $Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$, which implies $f(F) = Y - V$. Since $Y - V$ is $\tau^*$-g-closed, $f(F)$ is $\tau^*$-g-closed and thus $f$ is a $\tau^*$-g-closed map.

**Theorem 5.2.4:** If $f : X \rightarrow Y$ is $\tau^*$-g-continuous and $\tau^*$-g-closed and $A$ is a $\tau^*$-g-closed set of $X$, then $f(A)$ is a $\tau^*$-g-closed set in $Y$.

**Proof:** Let $f(A) \subseteq O$ where $O$ is a g-open set of $Y$. Since $f$ is $\tau^*$-g-continuous, $f^{-1}(O)$ is a $\tau^*$-g-open set containing $A$. Hence $\text{cl}^*(A) \subseteq f^{-1}(O)$ as $A$ is a $\tau^*$-g-closed set. Since $f$ is $\tau^*$-g-closed, $f(\text{cl}^*(A))$ is a $\tau^*$-g-closed set contained in the g-open set $O$, which implies that $\text{cl}^*(f(\text{cl}^*(A))) \subseteq O$ and hence $\text{cl}^*(f(A)) \subseteq O$. So $f(A)$ is a $\tau^*$-g-closed set in $Y$.

**Theorem 5.2.5:** If $f : X \rightarrow Y$ is $\tau^*$-g-closed and $A$ is g-closed set in $X$ then $f_A : A \rightarrow Y$ is $\tau^*$-g-closed.

**Proof:** Let $V$ be a g-closed set in $A$. Then $V$ is g-closed in $X$. By Theorem 3.2.13, $V$ is a $\tau^*$-g-closed set in $X$. By Theorem 5.2.4, $f(V)$ is $\tau^*$-g-closed in $Y$. But $f_A(V) = f(V)$. Therefore $f(V)$ is $\tau^*$-g-closed in $Y$. Therefore $f_A : A \rightarrow Y$ is a $\tau^*$-g-closed map.

**Theorem 5.2.6:** If $f : X \rightarrow Y$ is both continuous map and $\tau^*$-g-closed map from a normal space $X$ onto a space $Y$, then $Y$ is normal.

**Proof:** Let $A$ and $B$ be disjoint closed sets of $Y$. Since $f$ is g-continuous, $f^{-1}(A), f^{-1}(B)$ are disjoint closed sets of $X$. Since $X$ is normal, there are disjoint open sets $U, V$ in $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Since every open set
is g-open, \( f^{-1}(A) \) and \( f^{-1}(B) \) are g-open sets. By Theorem 5.2.3 and since \( f \) is \( \tau^* \)-g-closed, there are \( \tau^* \)-g-closed sets \( G, H \) in \( Y \) such that \( A \subseteq G, B \subseteq H \) and \( f^{-1}(G) \subseteq U \) and \( f^{-1}(H) \subseteq V \). Since \( U, V \) are disjoint, \( \text{int} (G) \) and \( \text{int} (H) \) are disjoint open sets. Since \( G \) is \( \tau^* \)-g-open, \( A \) is closed and \( A \subseteq G \Rightarrow A \subseteq \text{int} (G) \). Similarly \( B \subseteq \text{int} (H) \). Hence \( Y \) is normal.

**Theorem 5.2.7:** If \( f : X \rightarrow Y \) is an open map then it is \( \tau^* \)-g-open map provided \( X \) is a \( T_{1/2} \) space.

**Proof:** Let \( f : X \rightarrow Y \) be an open map. Suppose \( U \) is a g-open set in \( X \). Since \( X \) is a \( T_{1/2} \) space, \( U \) is open in \( X \). Then by the assumption, \( f(U) \) is open in \( Y \). Also by Theorem 3.2.10, \( f(U) \) is \( \tau^* \)-g-open in \( Y \). Hence \( f \) is a \( \tau^* \)-g-open map.

**Remark 5.2.8:** The following example shows that above theorem need not be true if \( X \) is not a \( T_{1/2} \) space.

**Example 5.2.9:** Let \( X = Y = \{ a, b, c \}, \tau = \{ X, \phi, \{ a \} \} \) and \( \sigma = \{ Y, \phi, \{ a \}, \{ c \}, \{ a, c \}, \{ b, c \} \} \). Let \( f : X \rightarrow Y \) be an identity function. Then \( f \) is an open map. On the other hand, it is not a \( \tau^* \)-g-open map, since for the g-open set \( \{ b \} \) in \( X \), the image of \( \{ b \} \) under \( f \) is not \( \tau^* \)-g-open in \( Y \).

**Theorem 5.2.10:** If \( f : X \rightarrow Y \) is a g-open map then it is \( \tau^* \)-g-open provided \( X \) is a \( T_{1/2} \) space.

**Proof:** Let \( f : X \rightarrow Y \) be a g-open map. Suppose \( U \) is a g-open set in \( X \). Since \( X \) is a \( T_{1/2} \) space, \( U \) is open in \( X \). By the assumption, \( f(U) \) is g-open in \( Y \). Again, by the Theorem 3.2.13, \( f(U) \) is \( \tau^* \)-g-open in \( Y \). Therefore \( f \) is a \( \tau^* \)-g-open map.
Remark 5.2.11: The following example shows that above theorem need not be true if X is not a $T_{1/2}$ space.

Example 5.2.12: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{c\}, \{b, c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$. Let $f : X \to Y$ be an identity function. Then $f$ is g-open. However, it is not $\tau^*$-g-open, since for the g-open set $\{b\}$ in X, the image of $\{b\}$ under $f$ is not $\tau^*$-g-open in Y.

Theorem 5.2.13: If $f : X \to Y$ is strongly g-open map then it is $\tau^*$-g-open provided X is a $T_{1/2}$ space.

Proof: Let $f : X \to Y$ be a strongly g-open map. Suppose U is a g-open set in X. Since X is a $T_{1/2}$ space, U is open in X. By the assumption, $f(U)$ is strongly g-open in Y. Again, by the Theorem 3.2.16, $f(U)$ $\tau^*$-g-open in Y. Hence $f$ is a $\tau^*$-g-open map.

Remark 5.2.14: The following example shows that above theorem need not be true if X is not a $T_{1/2}$ space.

Example 5.2.15: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Let $f : X \to Y$ be an identity function. Then $f$ is strongly g-open map. However, it is not $\tau^*$-g-open, since for the g-open set $\{c\}$ in X, the image of $\{c\}$ under $f$ is not $\tau^*$-g-open in Y.

Remark 5.2.16: The following examples show that $\tau^*$-g-open map is independent from the strongly $\alpha$-open map.

Example 5.2.17: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{a, b\}\}$ and $\sigma = \{Y, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$. Then
$f$ is strongly $\alpha$-open. However, it is not $\tau^*$-g-open, since for the g-open set \{a\} in $X$, the image of \{a\} under $f$ is not $\tau^*$-g-open in $Y$.

**Example 5.2.18:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, c\}\}$. Then $f$ is a $\tau^*$-g-open map. On the other hand, it is not a strongly $\alpha$-open map, since for the $\alpha$-open set \{a\} in $X$, the image of \{a\} under $f$ is not $\alpha$-open in $Y$.

**Remark 5.2.19:** The following examples show that $\tau^*$-g-open map is independent from the strongly semi-open map.

**Example 5.2.20:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Then $f$ is a strongly semi-open map. But it is not a $\tau^*$-g-open map, since for the g-open set \{b\} in $X$, the image of \{b\} under $f$ is not $\tau^*$-g-open in $Y$.

**Example 5.2.21:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, c\}\}$. Then $f$ is a $\tau^*$-g-open map. However, it is not a strongly semi-open map, since for the semi-open set \{a, b\} in $X$, the image of \{a, b\} under $f$ is not semi-open in $Y$.

**Remark 5.2.22:** The following examples show that $\tau^*$-g-open map is independent from the strongly pre-open map.

**Example 5.2.23:** Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$. Then $f$ is a strongly pre-open map. But it is not a $\tau^*$-g-open map, since for the g-open set $V = \{b\}$ in $X$, the image of \{b\} under $f$ is not $\tau^*$-g-open in $Y.$
Example 5.2.24: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{a\}\}$ and $\sigma = \{Y, \emptyset, \{b\}, \{a, c\}\}$. Then $f$ is a $\tau^*$-g-open map. On the other hand, it is not a strongly pre-open map, since for the pre-open set $\{a\}$ in $X$, the image of $\{a\}$ under $f$ is not pre-open in $Y$.

Remark 5.2.25: The following examples show that $\tau^*$-g-open map is independent from the quasi $\alpha$-open map.

Example 5.2.26: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{a, b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then $f$ is a quasi $\alpha$-open map. However, it is not a $\tau^*$-g-open map, since for the g-open set $\{c\}$ in $X$, the image of $\{c\}$ under $f$ is not $\tau^*$-g-open in $Y$.

Example 5.2.27: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{b\}\}$ and $\sigma = \{Y, \emptyset, \{c\}, \{a, b\}\}$. Then $f$ is a $\tau^*$-g-open map. But it is not a quasi $\alpha$-open map, since for the $\alpha$-open set $\{b\}$ in $X$, the image of $\{b\}$ under $f$ is not open in $Y$.

Remark 5.2.28: The following examples show that $\tau^*$-g-open map is independent from the quasi semi-open map.

Example 5.2.29: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{b\}, \{a, c\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then $f$ is a quasi semi-open map. On the other hand, it is not a $\tau^*$-g-open map, since for the g-open set $\{c\}$ in $X$, the image of $\{c\}$ under $f$ is not $\tau^*$-g-open in $Y$.

Example 5.2.30: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \emptyset, \{c\}, \{a, c\}, \{b, c\}\}$ and $\sigma = \{Y, \emptyset, \{c\}\}$. Then $f$ is a $\tau^*$-g-open map. However, it is not a quasi semi-open map, since for the semi-open set $\{b, c\}$ in $X$, the image of $\{b, c\}$ under $f$ is not open in $Y$. 
Remark 5.2.31: The following examples show that $\tau^*$-g-open map is independent from the quasi pre-open map.

Example 5.2.32: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{c\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{a, b\}\}$. Then $f$ is quasi pre-open. On the other hand, it is not a $\tau^*$-g-open map, since for the g-open set $\{a, c\}$ in $X$, the image of $\{a, c\}$ under $f$ is not $\tau^*$-g-open in $Y$.

Example 5.2.33: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{c\}\}$. Then $f$ is a $\tau^*$-g-open map. But it is not a quasi pre-open map, since for the pre-open set $\{a, c\}$ in $X$, the image of $\{a, c\}$ under $f$ is not open in $Y$.

Remark 5.2.34: The following examples show that $\tau^*$-g-open map is independent from the pre-open map.

Example 5.2.35: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}, \{a, c\}\}$. Then $f$ is a pre-open map. However, it is not a $\tau^*$-g-open map, since for the g-open set $\{b\}$ in $X$, the image of $\{b\}$ under $f$ is not $\tau^*$-g-open in $Y$.

Example 5.2.36: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{c\}\}$. Then $f$ is a $\tau^*$-g-open map. On the other hand, it is not a pre-open map, since for the open set $\{a, b\}$ in $X$, the image of $\{a, b\}$ under $f$ is not pre-open in $Y$.

Remark 5.2.37: The following examples show that $\tau^*$-g-open map is independent from the semi-open map.

Example 5.2.38: Let $X = Y = \{a, b, c\}$ and let $f : X \to Y$ be an identity function. Let $\tau = \{X, \phi, \{b\}\}$ and $\sigma = \{Y, \phi, \{b\}, \{c\}, \{b, c\}, \{a, b\}\}$. Then $f$ is a semi-open map. However, it is not a $\tau^*$-g-open map, since for the g-open set $\{a\}$ in $X$, the image of $\{a\}$ under $f$ is not $\tau^*$-g-open in $Y$. 
**Example 5.2.39:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a, c\}\}$. Then $f$ is a $\tau^*\text{-g-open}$ map. But it is not a semi-open map, since for the open set $\{a\}$ in $X$, the image of $\{a\}$ under $f$ is not semi-open in $Y$.

**Remark 5.2.40:** The following examples show that $\tau^*\text{-g-open}$ map is independent from the $\alpha\text{-open}$ map.

**Example 5.2.41:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a, b\}, \{a, c\}\}$. Then $f$ is an $\alpha\text{-open}$ map. On the other hand, it is not a $\tau^*\text{-g-open}$ map, since for the g-open set $\{c\}$ in $X$, the image of $\{c\}$ under $f$ is not $\tau^*\text{-g-open}$ in $Y$.

**Example 5.2.42:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. Then $f$ is a $\tau^*\text{-g-open}$ map. However, it is not an $\alpha\text{-open}$ map, since for the open set $\{a, b\}$ in $X$, the image of $\{a, b\}$ under $f$ is not $\alpha\text{-open}$ in $Y$.

**Remark 5.2.43:** The following examples show that $\tau^*\text{-g-open}$ map is independent from the semi pre-open map.

**Example 5.2.44:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. Then $f$ is a semi pre-open map. On the contrary, it is not a $\tau^*\text{-g-open}$ map, since for the g-open set $\{c\}$ in $X$, the image of $\{c\}$ under $f$ is not $\tau^*\text{-g-open}$ in $Y$.

**Example 5.2.45:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{b\}, \{a, b\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Then $f$ is a $\tau^*\text{-g-open}$ map. But it is not a semi pre-open map, since for the open set $\{b\}$ in $X$, the image of $\{b\}$ under $f$ is not sp-open in $Y$. 
**Remark 5.2.46:** The above discussion leads to the Figure 5.1.

![Diagram](https://via.placeholder.com/150)

**Figure 5.1** Independent relationship of $\tau^*$-g-open map from other open maps

### 5.3 $\tau^*$-GENERALIZED HOMEOMORPHISMS IN TOPOLOGICAL SPACES

In this section, we introduce and study the notion of $\tau^*$-g-homeomorphism and $\tau^*$-gc-homeomorphisms in topological spaces.

**Definition 5.3.1:** A bijection $f : (X, \tau^*) \to (Y, \sigma^*)$ is called $\tau^*$-generalized homeomorphism (briefly $\tau^*$-g-homeomorphism) if $f$ is both $\tau^*$-g-continuous function and $\tau^*$-g-open map.

**Theorem 5.3.2:** If $f : X \to Y$ is bijective and $\tau^*$-g-continuous, then the following statements are equivalent:

(a) $f$ is a $\tau^*$-g-open map.

(b) $f$ is a $\tau^*$-g-homeomorphism.

(c) $f$ is a $\tau^*$-g-closed map.
Proof:

(a) \(\Rightarrow\) (b). By assumption, \(f\) is bijective, \(\tau^*\)-g-continuous and \(\tau^*\)-g-open. Then by definition, \(f\) is \(\tau^*\)-g-homeomorphism. Hence (a) \(\Rightarrow\) (b).

(b) \(\Rightarrow\) (c). Since \(f\) is \(\tau^*\)-g-homeomorphism, it is bijective, \(\tau^*\)-g-open and \(\tau^*\)-g-continuous. Then by Theorem 5.2.2, \(f\) is a \(\tau^*\)-g-closed map. Hence (b) \(\Rightarrow\) (c).

(c) \(\Rightarrow\) (a). By assumption, \(f\) is \(\tau^*\)-g-closed and bijective. Therefore by Theorem 5.2.2, \(f\) is \(\tau^*\)-g-open map. Hence (c) \(\Rightarrow\) (a). Thus (a), (b) and (c) are equivalent.

\textbf{Theorem 5.3.3:} Let \(X\) and \(Z\) be any two topological spaces and let \(Y\) be a \(\tau^*\)-T\(_g\)-space. If \(f : X \rightarrow Y\) and \(g : Y \rightarrow Z\) be \(\tau^*\)-g-homeomorphisms, then the composition \(g \circ f : X \rightarrow Z\) is also \(\tau^*\)-g-homeomorphisms.

\textbf{Proof:} Let \(U\) be a g-closed set in \(Z\). Since \(g : Y \rightarrow Z\) is \(\tau^*\)-g-continuous, \(g^{-1}(U)\) is \(\tau^*\)-g-closed in \(Y\). Since \(Y\) is a \(\tau^*\)-T\(_g\)-space, \(g^{-1}(U)\) is g-closed in \(Y\).

By the assumption of \(f\), \(f^{-1}[g^{-1}(U)]\) is a \(\tau^*\)-g-closed in \(X\). But \(f^{-1}[g^{-1}(U)] = (g \circ f)^{-1}(U)\). Hence \(g \circ f\) is \(\tau^*\)-g-continuous.

Again, let \(U\) be a g-open set in \(X\). Since \(f\) is a \(\tau^*\)-g-open map, \(f(U)\) is \(\tau^*\)-g-open in \(Y\). And since \(Y\) is a \(\tau^*\)-T\(_g\)-space, \(f(U)\) is g-open in \(Y\). Also by the assumption of \(g\), \(g[f(U)]\) is a \(\tau^*\)-g-open set in \(Z\). But \(g[f(U)] = (g \circ f)(U)\). Hence \((g \circ f)\) is a \(\tau^*\)-g-open map. Thus, \(g \circ f : X \rightarrow Z\) is both \(\tau^*\)-g-continuous and \(\tau^*\)-g-open map. Hence it is \(\tau^*\)-g-homeomorphisms.
Theorem 5.3.4: If \( f : X \rightarrow Y \) is g-homeomorphism then it is \( \tau^* \)-g-homeomorphism provided both X and Y are T\(_{1/2}\) spaces.

Proof: Let \( f : X \rightarrow Y \) be g-homeomorphism. Then \( f \) is both g-continuous function and g-open map. By Theorem 4.2.5, \( f \) is \( \tau^* \)-g-continuous. Also, by Theorem 5.2.10, \( f \) is a \( \tau^* \)-g-open map. Hence \( f \) is \( \tau^* \)-g-homeomorphism.

Remark 5.3.5: The following examples show that the above theorem need not be true if X and Y are not T\(_{1/2}\) spaces.

Example 5.3.6: Let \( X=\{a, b, c\} \), \( \tau = \{X, \emptyset, \{b\}\} \) and \( \sigma = \{Y, \emptyset, \{b\}, \{a, b\}, \{b, c\}\} \). Let \( f:X \rightarrow Y \) be an identity function. Here X is not a T\(_{1/2}\) space. Clearly \( f \) is g-homeomorphism. But it is not \( \tau^* \)-g-open map, since for the g-open set \( V=\{a\} \) in X, the image \( f(V) \) is not \( \tau^* \)-g-open in Y.

Example 5.3.7: Let \( X=\{a, b, c\} \), \( \tau = \{X, \emptyset, \{c\}\} \) and \( \sigma = \{Y, \emptyset, \{b\}, \{b, c\}\} \). Let \( f:X \rightarrow Y \) be an identity function. Here Y is not a T\(_{1/2}\) space. Then \( f \) is g-homeomorphism. However, it is not \( \tau^* \)-g-open map, since for the g-open set \( V=\{a, c\} \) in X, the image \( f(V) \) is not \( \tau^* \)-g-open in Y.

Theorem 5.3.8: If \( f : X \rightarrow Y \) is strongly g-homeomorphism then it is \( \tau^* \)-g-homeomorphism provided X is a T\(_{1/2}\) space.

Proof: Let \( f : X \rightarrow Y \) be strongly g-homeomorphism. Then by the definition, \( f \) is both strongly g-continuous function and strongly g-open map. By Theorem 4.2.8, \( f \) is \( \tau^* \)-g-continuous. Also, by Theorem 5.2.13, \( f \) is \( \tau^* \)-g-open. Therefore \( f \) is \( \tau^* \)-g-homeomorphism.

Remark 5.3.9: The following example shows that the above theorem need not be true if X is not a T\(_{1/2}\) space.
Example 5.3.10: Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a, b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Let $f : X \to Y$ be an identity function. Here $X$ is not a $T_{1/2}$ space. Clearly $f$ is strongly $g$-homeomorphism. But it is not $\tau^*$-g-open map, since for the g-open set $V = \{a\}$ in $X$, the image $f(V)$ is not $\tau^*$-g-open in $Y$.

Definition 5.3.11: A bijection $f : X \to Y$ is said to be $\tau^*$-gc-homeomorphisms if $f$ is $\tau^*$-gc-irresolute and its inverse $f^{-1}$ is also $\tau^*$-gc-irresolute. The space $(X, \tau^*)$ and $(Y, \sigma^*)$ are said to be $\tau^*$-gc-homeomorphic if there exists a $\tau^*$-gc-homeomorphism from $(X, \tau^*)$ to $(Y, \sigma^*)$.

Notations: The family of all $\tau^*$-gc-homeomorphisms [respectively $\tau^*$-g-homeomorphisms, homeomorphisms, $g$-homeomorphism, gc-homeomorphism] from a topological space $X$ onto itself is denoted by $\tau^*$-gch(X) [respectively $\tau^*$-gh(X), h(X), gh(X), gch(X)].

Theorem 5.3.12: Let $X$ be a topological space. Then (i) the set $\tau^*$-gch(X) is a group under the composition of maps and (ii) gh(X) is a subgroup of the group $\tau^*$-gch(X).

Proof:

(i) Let $f, g \in \tau^*$-gch(X). Then $g \circ f \in \tau^*$-gch(X) and so $\tau^*$-gch(X) is closed under the composition of functions. Composition of maps is always associative. The identity function $i : X \to X$ is a $\tau^*$-gc-homeomorphisms and so $i \in \tau^*$-gch(X). Also, $f \circ i = i \circ f = f$ for every $f \in \tau^*$-gch(X). If $f \in \tau^*$-gch(X), then $f^{-1} \in \tau^*$-gch(X) and $f \circ f^{-1} = f^{-1} \circ f = i$. 
Hence $\tau^*-\text{gch}(X)$ is a group under the composition of functions.

(ii) Let $f : X \rightarrow X$ be g-homeomorphism. Then by Theorem 4.3.4, both $f$ and $f^{-1}$ are $\tau^*-\text{gc}$-irresolute and so $f$ is $\tau^*-\text{gc}$-homeomorphism. Therefore every g-homeomorphism is a $\tau^*-\text{gc}$-homeomorphism and so $\text{gh}(X)$ is a subset of $\tau^*-\text{gch}(X)$. Also $\text{gh}(X)$ is a group under the composition of maps. Therefore $\text{gh}(X)$ is a subgroup of the group $\tau^*-\text{gch}(X)$.

**Remark 5.3.13:** Semi-homeomorphism is independent from $\tau^*-\text{g}$-homeomorphism as seen from the following examples.

**Example 5.3.14:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, c\}\}$. Then $f$ is both irresolute and pre semi-open. So $f$ is semi-homeomorphism. However, it is not $\tau^*-\text{g}$-homeomorphism, since for the g-open set $\{b\}$ in $X$, the image of $\{b\}$ under $f$ is not $\tau^*-\text{g}$-open in $Y$.

**Example 5.3.15:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{b\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. Then $f$ is $\tau^*-\text{g}$-homeomorphisms. On the other hand, it is not pre semi-open, since for the semi-open set $\{a, c\}$ in $X$, the image of $\{a, c\}$ under $f$ is not a semi-open set in $Y$.

**Remark 5.3.16:** $\text{sg}$-homeomorphism is independent from $\tau^*-\text{g}$-homeomorphism as seen from the following examples.

**Example 5.3.17:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Then $f$ is
sg-homeomorphism. On the other hand, it is not $\tau^*-g$-homeomorphism, since for the g-closed set $\{a, b\}$ in $Y$, the inverse image of $\{a, b\}$ is not $\tau^*-g$-closed in $X$.

**Example 5.3.18:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{c\}, \{a, b\}\}$. Then $f$ is $\tau^*-g$-homeomorphisms. But it is not sg-continuous, since for the closed set $\{a, b\}$ in $Y$, the inverse image of $\{a, b\}$ is not sg-closed in $X$. Therefore, $f$ is not sg-homeomorphism.

**Remark 5.3.19:** sg-continuous is independent from $\tau^*-g$-homeomorphism as seen from the following examples.

**Example 5.3.20:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{a, b\}\}$. Then $f$ is sg-homeomorphism. On the contrary, it is not a $\tau^*-g$-open map, since for the $g$-open set $\{c\}$ in $X$, the image of $\{c\}$ under $f$ is not $\tau^*-g$-open in $Y$. Therefore $f$ is not $\tau^*-g$-homeomorphism.

**Example 5.3.21:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{c\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b, c\}\}$. Then $f$ is $\tau^*-g$-homeomorphisms. However, it is not sg-irresolute, since for the sg-closed set $\{b, c\}$ in $Y$, the inverse image of $\{b, c\}$ is not sg-closed in $X$. Therefore $f$ is not sg-homeomorphism.

**Remark 5.3.22:** sg-continuous is independent from $\tau^*-g$-homeomorphism as seen from the following examples.

**Example 5.3.23:** Let $X = Y = \{a, b, c\}$ and let $f : X \rightarrow Y$ be an identity function. Let $\tau = \{X, \phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ and $\sigma = \{Y, \phi, \{c\}, \{a, c\}, \{b, c\}\}$,
Then \( f \) is gsc-homeomorphism. On the other hand, it is not a \( \tau^* \)-g-open map, since for the g-open set \( \{a\} \) in \( X \), the image of \( \{a\} \) under \( f \) is not \( \tau^* \)-g-open in \( Y \). Therefore \( f \) is not \( \tau^* \)-g-homeomorphism.

**Example 5.3.24:** Let \( X = Y = \{a, b, c\} \) and let \( f : X \rightarrow Y \) be an identity function. Let \( \tau = \{X, \emptyset, \{a\}\} \) and \( \sigma = \{Y, \emptyset, \{c\}, \{a, b\}\} \). Then \( f \) is \( \tau^* \)-g-homeomorphisms. But \( f^{-1} : Y \rightarrow X \) is not gs-irresolute, since for the gs-closed set \( \{a\} \) in \( Y \), the inverse image of \( \{a\} \) is not gs-closed in \( X \). Therefore \( f \) is not gsc-homeomorphism.

**Remark 5.3.25:** Above arguments give the following Figure 5.2.

![Diagram](attachment:image.png)

**Figure 5.2 Independenty of \( \tau^* \)-g-homeomorphisms**

5.4 **CONCLUSION**

In this chapter, \( \tau^* \)-generalized open maps and \( \tau^* \)-generalized closed maps in topological spaces have been defined and some of their properties are studied. Relationship with some existing maps is also investigated. \( \tau^* \)-g-homeomorphism and \( \tau^* \)-gc-homeomorphisms have also been introduced and investigated. In future, the above defined notions can be studied on fuzzy topological spaces and supra topological spaces.