CHAPTER-4

A Special Class of Singular Singularly Perturbed Problems via B-Spline Method

4.1 Introduction

In this chapter, the following special class of singular singularly perturbed two-point boundary value problems (BVPs) is considered:

\[ \varepsilon u''(x) + \frac{p(x)}{x} u'(x) + \frac{q(x)}{x^2} u(x) = R(x), \quad x \in (0,1) \]  \hspace{1cm} (4.1.1)

\[ u(0) = \gamma_0, \quad u(1) = \delta_0, \quad \gamma_0, \delta_0 \in \mathbb{R} \]  \hspace{1cm} (4.1.2)

where \( 0 < \varepsilon \ll 1, \ p(x) \geq 1 \) and \( q(x) , R(x) \) are smooth to satisfy the existence, uniqueness of the solution given by Russell and Shampine (1975). The above problem is called singular because the coefficient of derivative term has regular singularity. The solution of these types of problems is very difficult to handle as compared to singularly perturbed problems and very little literature is available for these types of problems. Such types of problems frequently arise in optimal control theory, chemical reactor theory, aerodynamics, geophysics etc. Various numerical techniques [Gupta et al. (2011), Kumar et al. (2007b), Mishra et al. (2014), Miller et al.(1996), O’Malley (1979)] are developed to solve the such type of problems. Schmeiser and Weiss (1984, 1986) have given the asymptotic and numerical methods to solve these kinds of problems. Ascher (1985) has studied some families of symmetric difference schemes equivalent to certain collocation schemes based on Gauss and Labatto points to solve singular singularly perturbed boundary value problems. Zhu (1994) has given asymptotic solution constructed by modified Vasil-eva method. In addition, the existence and uniqueness of the exact solution and the uniform validity of the formal asymptotic solution for the boundary value problems have been proved. Lin (1996) has shown asymptotic expansion of the solution and its uniform
validity on a finite interval. Mohanty et al. (2005), Mohanty and Arora (2006) have given the numerical solution of singularly perturbed two-point singular boundary value problem using convergent tension spline method and non-uniform mesh tension spline method. Kadalbajoo et al. (2005a) have given B-spline method for solving a class of singular singularly perturbed boundary value problems. Chang et al. (2011) explained the comparison of B-spline method and finite difference method to solve the BVPs of linear ODEs. Rashidnia et al. (2007b) and Jian-li (2008) have given cubic spline and reproducing kernel space method for solving singular singularly perturbed boundary value problems respectively.

In this chapter, a direct method is presented which is based on fitted-mesh techniques and B-spline for solution of singular singularly perturbed two-point boundary value problems. In section 4.2, B-spline method is described for solving equations (4.1.1) with boundary conditions (4.1.2). In section 4.3, convergence analysis is given. In section 4.4, numerical examples are demonstrated and conclusion is discussed in the section 4.5.

### 4.2 Description of the Method

Let us consider the differential equation (4.1.1) in the following form:

\[ \varepsilon u''(x) + \frac{p(x)}{x} u'(x) + \frac{q(x)}{x^2} u(x) = R(x), \quad x \in (0,1) \]  

(4.2.1)

\[ u(0) = \gamma_0, \quad u(1) = \delta_0, \quad \gamma_0, \delta_0 \in \mathbb{R} \]  

(4.2.2)

Equation (4.2.1) has regular singularity at \( x = 0 \). Therefore, at \( x = 0 \), we modified equation (4.2.1) as:

\[ (\varepsilon + p(x) + q(x) / 2)u''(x) = R(x), \]  

(4.2.3)

Subdivide the interval \([0,1]\), and choose piecewise uniform mesh points represented by \( \pi = \{x_0, x_1, x_2, \ldots, x_N\} \) such that \( x_0 = 0 \) and \( x_N = 1 \) and \( \tilde{h} \) is the piecewise uniform spacing. We
define $L_2[0,1]$ as a vector space of all the square integrable functions on $[a,b]$. Let $X$ be the linear subspace of $L_2[0,1]$. Now, B-spline method is applied on $X$ as follows:

$$B_i(x) = \begin{cases} 
\frac{1}{h^3} \left( (x-x_{i-2})^3, \right. & \text{if } x \in [x_{i-2}, x_{i-1}] \\
\tilde{h}^3 + 3\tilde{h}^2 (x-x_{i-1}) + 3\tilde{h}(x-x_{i-1})^2 - 3(x-x_{i-1})^3, & \text{if } x \in [x_{i-1}, x_i] \\
\tilde{h}^3 + 3\tilde{h}^2 (x_{i+1} - x) + 3\tilde{h}(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & \text{if } x \in [x_i, x_{i+1}] \\
(x_{i+2} - x)^3, & \text{if } x \in [x_{i+1}, x_{i+2}] \\
0, & \text{otherwise.} 
\end{cases} \quad (4.2.4)$$

for $i = 0,1,2,\ldots,N$.

Four additional knots as $x_{-2} < x_{-1} < x_0$ and $x_{N+2} > x_{N+1} > x_N$ are introduced. From equation (4.2.4), it can be checked that each of the functions $B_i(x)$ is twice continuously differentiable on the entire real line. In addition,

$$B_i(x_j) = \begin{cases} 
4, & \text{if } i = j \\
1, & \text{if } i - j = \pm 1 \\
0, & \text{if } i - j = \pm 2 
\end{cases} \quad (4.2.5)$$

and $B_i(x_j) = 0$ for $x \geq x_{i+2}$ and $x \leq x_{i-2}$.

Similarly, it can be shown that

$$B_i'(x_j) = \begin{cases} 
0, & \text{if } i = j \\
\pm \frac{3}{h}, & \text{if } i - j = \pm 1 \\
0, & \text{if } i - j = \pm 2 
\end{cases} \quad (4.2.6)$$

and
\[
B_i''(x_j) = \begin{cases} 
-\frac{12}{h^2}, & \text{if } i = j \\
\frac{6}{h^2}, & \text{if } i - j = \pm 1 \\
0, & \text{if } i - j = \pm 2 
\end{cases}
\] (4.2.7)

Each \( B_i(x) \) is also a piecewise cubic with knots at \( \pi \), and \( B_i(x) \in X \). Let \( \Omega = \{ B_{-1}, B_0, B_1, \ldots, B_{N+1} \} \) and let \( \Phi_3(\pi) = \text{span } \Omega \). The functions \( \Omega \) are linearly independent on \( [0,1] \), thus \( \Phi_3(\pi) \) is \( (N+3) \)-dimensional. Even one can show that \( \Phi_3(\pi) \subseteq_{sp} X \). Let \( L \) be a linear operator whose domain and range is \( X \). Let \( S(x) \) be the B-spline interpolating the function \( y(x) \) at the nodal points and \( S(x) \in \Phi_3(\pi) \). Then \( S(x) \) can be written as

\[
S(x) = C_{-1}B_{-1}(x) + C_0B_0(x) + C_1B_1(x) + \ldots + C_NB_N(x) + C_{N+1}B_{N+1}(x)
\] (4.2.8)

Now, we have

\[
\varepsilon S''(x_i) + \frac{p(x_i)}{x_i} S'(x_i) + \frac{q(x_i)}{x_i^2} S(x_i) = R(x_i),
\] (4.2.9)

and

\[
S(x_0) = \gamma_0, \quad S(x_N) = \delta_0
\] (4.2.10)

On solving equation (4.2.9), we get

\[
\begin{align*}
\left( \varepsilon B_i''_{i+1}(x_i) + \frac{p(x_i)}{x_i} B_i'_{i+1}(x_i) + \frac{q(x_i)}{x_i^2} B_i_{i+1}(x_i) \right) C_{i+1} + \\
\left( \varepsilon B_i''_{i-1}(x_i) + \frac{p(x_i)}{x_i} B_i'_{i-1}(x_i) + \frac{q(x_i)}{x_i^2} B_i_{i-1}(x_i) \right) C_i + \\
\left( \varepsilon B_i''(x_i) + \frac{p(x_i)}{x_i} B_i'(x_i) + \frac{q(x_i)}{x_i^2} B_i(x_i) \right) C_{i+1} = R(x_i),
\end{align*}
\] (4.2.11)
For finding the values of $C_i, C_{i-1}, C_{i+1}$ the following equations can be used:

$$
C_{i-1} \begin{cases}
B_i''(x_j) = \frac{6}{\hbar^2}, & \text{for } i - j = \pm 1 \\
B_i'(x_j) = \pm \frac{3}{\hbar}, & \text{for } i - j = \pm 1 \\
B_i(x_j) = 1, & \text{for } i - j = \pm 1
\end{cases}
$$

$$
C_i \begin{cases}
B_i''(x_j) = -\frac{12}{\hbar^2}, & \text{for } i = j \\
B_i'(x_j) = 0, & \text{for } i = j \\
B_i(x_j) = 4, & \text{for } i = j
\end{cases}
$$

Now, equation (4.2.11) is multiplied by $\tilde{h}^2$ and simplifying it by using (4.2.5), (4.2.6) and (4.2.7), we get

$$
\left( 6\varepsilon - \frac{3p(x_i)}{x_i} \tilde{h} + \frac{q(x_i)}{x_i^2} \tilde{h}^2 \right) C_{i-1} + \left( -12\varepsilon + \frac{4q(x_i)}{x_i^2} \tilde{h}^2 \right) C_i + \left( 6\varepsilon + \frac{3p(x_i)}{x_i} \tilde{h} + \frac{q(x_i)}{x_i^2} \tilde{h}^2 \right) C_{i+1} = \tilde{h}^2 R(x_i), \quad \forall i = 1, 2, 3, \ldots, N.
$$

(4.2.12)
Now for $x = 0$, we have

\[(\varepsilon + p(x) + q(x)/2)S''(x) = R(x)\]  \hspace{1cm} \text{(4.2.13)}

On solving, we get

\[\left(6(\varepsilon + p(x_0) + q(x_0)/2) \right)C_{-1} + \left(-12(\varepsilon + p(x_0) + q(x_0)/2) \right)C_0 + \]

\[\left(6(\varepsilon + p(x_0) + q(x_0)/2) \right)C_1 = \tilde{h}^2 R(x_0)\]  \hspace{1cm} \text{(4.2.14)}

From the boundary conditions (4.2.10), we have

\[C_{-1} + 4C_0 + C_1 = \gamma_0\]  \hspace{1cm} \text{(4.2.15)}

\[C_{N-1} + 4C_N + C_{N+1} = \delta_0\]  \hspace{1cm} \text{(4.2.16)}

Now eliminating $C_{-1}$ from (4.2.14) and (4.2.15), we get,

\[-36(\varepsilon + p(x_0) + q(x_0)/2)C_0 = \tilde{h}^2 R(x_0) - \gamma_0[6(\varepsilon + p(x_0) + q(x_0)/2)]\]  \hspace{1cm} \text{(4.2.17)}

Similarly, eliminating $C_{N+1}$ from equation (4.2.12) and from equation (4.2.16), we get

\[-6 p(x_N) \tilde{h} \left\{ -6 (p(x_N) \tilde{h}) \right\} C_{N-1} + \left\{ -36 \varepsilon - 12 \frac{p(x_N) \tilde{h}}{x_N} \right\} C_N = \tilde{h}^2 R_N(x) - \]

\[\delta_0 \left( 6 \varepsilon + \frac{3 p(x_N) \tilde{h}}{x_N} + \frac{q(x_N) \tilde{h}}{x_N^2} \right)\]  \hspace{1cm} \text{(4.2.18)}

Now, we write the above system of equations in the following form:

\[\begin{bmatrix} D \end{bmatrix} x_N = T_N\] where $x_N = [C_0, C_1, \ldots, C_N]^T$ are unknown,

\[T_N = \begin{bmatrix} R_0 \tilde{h}^2 - \gamma_0 \left[ 6(\varepsilon + p(x_0)) + q(x_0)\tilde{h}^2 \right], \tilde{h}^2 R_1, \tilde{h}^2 R_2, \ldots, \tilde{h}^2 R_{N-1}, \]

\[\tilde{h}^2 R_N(x) - \delta_0 \left( 6 \varepsilon + \frac{3 p(x_N) \tilde{h}}{x_N} + \frac{q(x_N) \tilde{h}}{x_N^2} \right) \end{bmatrix}^T\]
and the coefficient matrix $D$ is given by:

$$
\begin{pmatrix}
-36(\varepsilon + p(x_0) + q(x_0) / 2) & 0 & 0 \\
6\varepsilon - 3 \frac{p(x_i) \tilde{h}}{x_i} + \frac{q(x_i) \tilde{h}^2}{x_i^2} & -12\varepsilon + 4 \frac{q(x_i) \tilde{h}^2}{x_i^2} & 6\varepsilon + 3 \frac{p(x_i) \tilde{h}}{x_i} + \frac{q(x_i) \tilde{h}^2}{x_i^2} \\
0 & \ddots & \ddots \\
0 & 0 & 6\varepsilon - 3 \frac{p(x_i) \tilde{h}}{x_i} + \frac{q(x_i) \tilde{h}^2}{x_i^2} \\
\vdots & \ddots & \ddots \\
0 & \ldots & 0 \\
0 & \ldots & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & \ldots & 0 \\
0 & \ldots & 0 \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
-12\varepsilon + 4 \frac{q(x_i) \tilde{h}^2}{x_i^2} & \ldots & \ddots \\
6\varepsilon - 3 \frac{p(x_{N-1}) \tilde{h}}{x_{N-1}} + \frac{q(x_{N-1}) \tilde{h}^2}{x_{N-1}^2} & -12\varepsilon + 4 \frac{q(x_{N-1}) \tilde{h}^2}{x_{N-1}^2} & 6\varepsilon + 3 \frac{p(x_{N-1}) \tilde{h}}{x_{N-1}} + \frac{q(x_{N-1}) \tilde{h}^2}{x_{N-1}^2} \\
0 & -6 \frac{p(x_N) \tilde{h}}{x_N} & -36\varepsilon - \frac{12 p(x_N) \tilde{h}}{x_N}
\end{pmatrix}
$$
\[
\begin{bmatrix}
C_0 \\
C_1 \\
\vdots \\
C_N
\end{bmatrix} = 
\begin{bmatrix}
\tilde{h}^2R_0 - \gamma_0 \left[ 6(\varepsilon + p(x_0)) + q(x_0)\tilde{h}^2 \right] \\
\tilde{h}^2R_1 \\
\vdots \\
\tilde{h}^2R_N - \delta_0 \left( 6\varepsilon + 3 \frac{p(x_N)}{x_N} + q(x_N)\frac{\tilde{h}}{x_N^2} \right)
\end{bmatrix}
\]

\Rightarrow D. \quad (4.2.19)

Now, equation (4.2.19) is solved by using LU-Decomposition method. The values of \( C_0, C_1, \ldots, C_N \) are obtained by using JAVA program.

Remark: The mesh point selection strategy is applied based on Kadalbajoo and Aggarwal (2005a).

4.3 Convergence Analysis

In this section, the convergence analysis is discussed.

Assuming

\[ h = \max[h_1, h_2]. \]

First, calculate the following relationship:
\[
\begin{align*}
    h \left[ S'(x_{i-1}) + 4S'(x_i) + S'(x_{i+1}) \right] = h \left[ C_{i-2} \left( \frac{-3}{h} \right) + C_i \left( \frac{3}{h} \right) + 4 \left( C_{i-1} \left( \frac{-3}{h} \right) + C_{i+1} \left( \frac{3}{h} \right) \right) + \right] 
\end{align*}
\]

(4.3.1)

which gives

\[
\begin{align*}
    h \left[ S'(x_{i-1}) + 4S'(x_i) + S'(x_{i+1}) \right] = 3 \left[ y(x_{i+1}) - y(x_{i-1}) \right] 
\end{align*}
\]

(4.3.2)

Similarly,

\[
\begin{align*}
    h^2 S''(x_i) = 6 \left[ S(x_{i+1}) - S(x_i) \right] - 2h \left[ 2S'(x_i) + S'(x_{i+1}) \right] 
\end{align*}
\]

(4.3.3)

and

\[
\begin{align*}
    h^3 S'''(x_i) = 12 \left[ S(x_i) - S(x_{i+1}) \right] + 6h \left[ S'(x_i) + S'(x_{i+1}) \right] 
\end{align*}
\]

(4.3.4)

where \( S'''(x_{i+1}) = S'''(x) \) in \((x_i, x_{i+1})\)

as we know that by backward difference

\[
E(y(x_i)) = y(x_{i+1}) 
\]

Equation (4.3.2) may be written as

\[
\left( E^{-1} + 4 + E \right) hS'(x_i) = 3 \left( E - E^{-1} \right) y(x_i) 
\]

and hence

\[
hS'(x_i) = \left\{ \frac{3(E - E^{-1})}{(E^{-1} + 4 + E)} \right\} y(x_i) 
\]

Put \( E = e^{hD} \) and expand in powers of \( hD \), we obtain
\[ hS'(x_i) = \left \{ \frac{3(e^{hD} - e^{-hD})}{(e^{-hD} + 4 + e^{hD})} \right \} y(x_i) \]

\[ (e^{-hD} + 4 + e^{hD}) S'(x_i) = \frac{3}{h}(e^{hD} - e^{-hD}) y(x_i) \] (4.3.5)

Since we know that

\[ e^{hD} + e^{-hD} = 2 \left [ 1 + \frac{h^2 D^2}{2} + \frac{h^4 D^4}{24} + \frac{h^6 D^6}{120} + .... \right ] \]

and

\[ e^{hD} - e^{-hD} = 2 \left [ hD + \frac{h^3 D^3}{6} + \frac{h^5 D^5}{120} + ...... \right ] \]

Putting these values in equation (4.3.5) and simplifying it, we get

\[ S'(x_i) = \left ( D - \frac{1}{180} h^4 D^5 + ....... \right ) y_i \]

Hence,

\[ S'(x_i) = y'(x_i) - \frac{1}{180} h^4 y^5(x_i) + O(h^6) \] (4.3.6)

Similarly,

\[ S''(x_i) = y''(x_i) - \frac{1}{12} h^2 y^4(x_i) + \frac{1}{360} h^4 y^6(x_i) + O(h^6) \] (4.3.7)

and

\[ S'''(x_{i+}) = y'''(x_i) + \frac{1}{2} h y^4(x_i) + \frac{1}{12} h^2 y^5(x_i) - \frac{1}{360} h^4 y^7(x_i) - \frac{1}{1440} h^5 y^8(x_i) + O(h^6) \] (4.3.8)
\[ S''(x_i) = y''(x_i) - \frac{1}{2} h y^4(x_i) + \frac{1}{12} h^2 y^5(x_i) + O(h^6) \quad (4.3.9) \]

Adding equations (4.3.8) and (4.3.9), we get

\[ S''(x_i) + S''(x_i) = 2y''(x_i) + \frac{2}{12} h^2 y^5(x_i) + O(h^3) \]

After subtracting equations (4.3.8) and (4.3.9), we get

\[ S''(x_i) - S''(x_i) = h y''(x_i) - \frac{2}{360} h^4 y^7(x_i) + O(h^6) \]

Hence, we have

\[ \frac{1}{2}(S''(x_i) + S''(x_i)) = y''(x_i) + \frac{1}{12} h^2 y^5(x_i) + O(h^3) \quad (4.3.10) \]

\[ \frac{1}{2}(S''(x_i) - S''(x_i)) = h y''(x_i) - \frac{1}{360} h^4 y^7(x_i) + O(h^5) \quad (4.3.11) \]

Now, \( e(x_i) \) is defined here in the form

\[ e(x_i) = S(x_i) - y(x_i) \]

and substituting equations (4.3.6), (4.3.7), (4.3.8) and (4.3.10) in the above expression and by using Taylor’s expansion, we get

\[ e(x_i + \varphi h) = e(x_i) + \varphi h e'(x_i) + \frac{\varphi^2 h^2}{2!} e''(x_i) + \frac{\varphi^3 h^3}{3!} e'''(x_i) + \frac{\varphi^4 h^4}{4!} e''''(x_i) + ... \]

After putting the values of \( e'(x_i) \), \( e''(x_i) \), \( e'''(x_i) \),... and simplifying it, we get

\[ e(x_i + \varphi h) = \left( S(x_i) - y(x_i) \right) - \frac{1}{180} \varphi h^5 y^5(x_i) - \frac{\varphi^2 h^4}{24} y^4(x_i) + \frac{\varphi^3 h^6}{720} y^6(x_i) + \frac{\varphi^4 h^8}{12} y^4(x_i) \]

\[ + \frac{\varphi^3 h^5}{72} y^5(x_i) - \frac{\varphi^3 h^7}{2160} y^7(x_i) - \frac{\varphi^3 h^8}{8640} y^8(x_i) + O(h^9) \quad (4.3.12) \]
As we know

\[
\max_{x_i \leq x \leq x_{i+1}} |y(x) - S(x)| \leq \frac{1}{2} M h^2
\]  

(4.3.13)

where \( h = x_{i+1} - x_i \), \( i = 0, 1, 2, \ldots, n \) and \( M = \max \{|y''(x)|\} \)

and also we know that

\[
y''(x_i) = S''(x_i) + \frac{1}{12} h^2 y'''(x_i) + O(h^4)
\]  

(4.3.14)

Putting the values of equations (4.3.11)-(4.3.12) in equation (4.3.10), and simplifying it, we get,

\[
e(x_i + \varphi h) = \frac{-\varphi^2 (\varphi - 1)^2 h^4 y^4(x_i)}{24} + \frac{\varphi (\varphi^2 - 1)(3\varphi^2 - 2)h^5 y^5(x_i)}{360} + \\
\frac{\varphi^4 h^4 y^4(x_i)}{24} + \frac{\varphi^5 h^5 y^5(x_i)}{120} + O(h^6) \ldots
\]  

(4.3.15)

where \( 0 \leq \varphi \leq 1 \), which shows that our method has error of order \( h^4 \).

### 4.4 Numerical Illustrations

In this section, two examples are given to demonstrate the accuracy and efficiency of the method in solving the considered problems. These examples have been chosen because the exact solutions are available for comparison.

**Example 4.1:** Consider the following singular singularly perturbed boundary value problem:

\[
\epsilon y''(x) + \frac{1}{x} y'(x) + \frac{1}{x^2} y(x) = f(x),
\]

having boundary conditions

\[
y(0) = 0, \quad y(1) = 0,
\]
where

\[ f(x) = 6\varepsilon x - 12\varepsilon x^2 + 4x - 5x^2. \]

The exact solution is given by

\[ y(x) = x^3 - x^4. \]

The error analysis for the given problem 4.1 is mentioned into the Table 4.1 for different values of \( \varepsilon \) and \( N \).

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<th>( \varepsilon = 2^{-K} )</th>
<th>16</th>
<th>32</th>
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<td>5.71795117E-21</td>
<td>1.13498391E-21</td>
</tr>
</tbody>
</table>
Table 4.1: Maximum Absolute Error of Example 4.1 (continued)

<table>
<thead>
<tr>
<th>$\varepsilon = 2^{-K}$</th>
<th>$N$</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>256</td>
<td>512</td>
<td>1024</td>
</tr>
<tr>
<td>K=4</td>
<td>1.67891855E-8</td>
<td>1.68687240E-9</td>
<td>9.06057728E-11</td>
</tr>
<tr>
<td>K=6</td>
<td>1.13611524E-8</td>
<td>1.37012435E-9</td>
<td>1.65152483E-10</td>
</tr>
<tr>
<td>K=8</td>
<td>1.34344258E-11</td>
<td>2.3650327E-12</td>
<td>5.36132149E-11</td>
</tr>
<tr>
<td>K=10</td>
<td>2.22719632E-13</td>
<td>3.95331351E-14</td>
<td>6.76042967E-15</td>
</tr>
<tr>
<td>K=12</td>
<td>3.53194732E-15</td>
<td>6.28189296E-16</td>
<td>1.07641713E-16</td>
</tr>
<tr>
<td>K=14</td>
<td>5.53908573E-17</td>
<td>9.85672967E-18</td>
<td>1.68982691E-18</td>
</tr>
<tr>
<td>K=16</td>
<td>8.66326883E-19</td>
<td>1.54181109E-19</td>
<td>2.64359774E-20</td>
</tr>
<tr>
<td>K=18</td>
<td>1.35421551E-20</td>
<td>2.41018741E-21</td>
<td>4.13265097E-22</td>
</tr>
<tr>
<td>K=20</td>
<td>2.11774872E-22</td>
<td>3.76912786E-23</td>
<td>6.46282227E-24</td>
</tr>
</tbody>
</table>
Example 4.2: Consider the following singular singularly perturbed boundary value problem:

\[ \varepsilon y''(x) + \frac{1}{x} y'(x) + \frac{1}{x^2} y(x) = f(x) \]

having boundary conditions

\[ y(0) = 0, \quad y(1) = 0, \]

where

\[ f(x) = \sin \pi x \left( 3 + \varepsilon \left( 2 - x^2 \pi^2 \right) \right) + x \pi \cos \pi x (1 + 4 \varepsilon). \]

The exact solution is given by

\[ y(x) = x^2 \sin \pi x. \]

The error analysis for the given problem 4.2 is mentioned into the Table 4.2 for different values of \( \varepsilon \) and \( N \).
Table 4.2: Maximum Absolute Error of Example 4.2

<table>
<thead>
<tr>
<th>$\varepsilon = 2^{-K}$</th>
<th>16</th>
<th>32</th>
<th>64</th>
<th>128</th>
</tr>
</thead>
<tbody>
<tr>
<td>K=4</td>
<td>1.18027513E-3</td>
<td>1.29591270E-4</td>
<td>1.39598566E-5</td>
<td>1.47265982E-6</td>
</tr>
<tr>
<td>K=6</td>
<td>2.61463678E-6</td>
<td>1.12095036E-8</td>
<td>3.54638125E-7</td>
<td>9.42000021E-7</td>
</tr>
<tr>
<td>K=8</td>
<td>2.66036677E-8</td>
<td>5.45257670E-9</td>
<td>1.16474811E-9</td>
<td>2.28708315E-10</td>
</tr>
<tr>
<td>K=10</td>
<td>5.53382337E-10</td>
<td>8.81824036E-11</td>
<td>1.89953530E-11</td>
<td>3.76029864E-12</td>
</tr>
<tr>
<td>K=12</td>
<td>3.23975127E-11</td>
<td>1.38992402E-12</td>
<td>3.00000412E-13</td>
<td>5.95084026E-14</td>
</tr>
<tr>
<td>K=14</td>
<td>5.72529941E-12</td>
<td>2.17742411E-14</td>
<td>4.70002574E-15</td>
<td>9.32775494E-16</td>
</tr>
<tr>
<td>K=16</td>
<td>1.34935726E-12</td>
<td>3.42480147E-16</td>
<td>7.3490729E-17</td>
<td>1.45869524E-17</td>
</tr>
<tr>
<td>K=18</td>
<td>3.33274252E-13</td>
<td>5.84299838E-18</td>
<td>1.14871080E-18</td>
<td>2.28011364E-19</td>
</tr>
<tr>
<td>K=20</td>
<td>8.31011042E-14</td>
<td>2.15750519E-19</td>
<td>1.79634763E-20</td>
<td>3.56565746E-21</td>
</tr>
</tbody>
</table>
Table 4.2: Maximum Absolute Error of Example 4.2 (continued)

<table>
<thead>
<tr>
<th>( \varepsilon = 2^{-K} )</th>
<th>256</th>
<th>512</th>
<th>1024</th>
</tr>
</thead>
<tbody>
<tr>
<td>K=4</td>
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<td>1.56658002E-8</td>
<td>1.68804961E-9</td>
</tr>
<tr>
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<td>1.38024049E-8</td>
<td>1.666068857E-9</td>
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<tr>
<td>K=8</td>
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<td>5.50548937E-10</td>
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<tr>
<td>K=10</td>
<td>6.99733614E-13</td>
<td>1.24200923E-13</td>
<td>2.12388879E-14</td>
</tr>
<tr>
<td>K=12</td>
<td>1.10960960E-14</td>
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<td>3.38167905E-16</td>
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<tr>
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<td>4.84374480E-19</td>
<td>8.30510954E-20</td>
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<tr>
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<td>7.57182800E-21</td>
<td>1.29831068E-21</td>
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<td>6.65310420E-22</td>
<td>1.18410647E-22</td>
<td>2.03035553E-23</td>
</tr>
</tbody>
</table>

4.5 Conclusion

In this chapter, a B-spline method for solving singular singularly perturbed boundary value problems is described. In the tables of given examples 4.1 and 4.2, maximum absolute error can be seen. As the value of \( \varepsilon \) decreases to study the behavior of the solution at the boundary layer, a large number of mesh points in that region is required. Therefore the uniform mesh is not a good technique for such problems because a number of mesh points in the boundary layer region should be much higher than that from the outer region. It can be observed that the exact and approximate values are better agreement with each other. The errors of the approximate solutions are monotonically decreasing with the increasing of nodal points.