CHAPTER 3

SLIDING MODE CONTROL

3.1 Introduction

Sliding mode control is a particular type of the variable structure control system (VSCS), which is characterized by a discontinuous feedback control structure that switches as the system crosses certain manifold in the state space to force the system state to reach, and subsequently to remain on a specified surface within the state space called sliding surface. The switching function (sliding variable) is a function of the states and the sliding surface represents a relationship between the state variables. The system dynamics when confined to the sliding surface is referred as an ideal sliding motion and represents the controlled system behaviour, which results in reduced order dynamics with respect to the original plant. This reduced order dynamics provides attractive advantages such as insensitivity to parameter variations and matched uncertainties and disturbances, making sliding mode control an appropriate method for robust control. Moreover, sliding mode control scheme offers a simple algorithm which can be implemented easily (Boiko et al. 2007, Fallaha et al. 2011, Gonzalez et al. 2012, Hsu et al. 2004, Husain et al. 2008, Raviraj and Sen 1997, Slotine and Li 1991, Tao et al. 2010, Utkin 1993, Utkin and Poznyak 2013, Young et al. 1999).

3.2 Theory of Sliding Mode Control

Consider the system given by Equation (3.1).

\[
\dot{X}(t) = f(X, t) + b(X, t)u(t)
\]

(3.1)
where $X(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, f(X,t) \in \mathbb{R}^{n \times n}$ and $b(X,t) \in \mathbb{R}^{n \times m}$. The component of the discontinuous feedback is given by

$$u_i(t) = \begin{cases} u_i^+(X,t), & \text{if } \sigma_i(X) > 0 \\ u_i^-(X,t), & \text{if } \sigma_i(X) < 0 \end{cases} \quad i = 1,2,\ldots,m$$ (3.2)

where $\sigma_i(X) = 0$ is the $i$-th sliding surface, and $\sigma(X) = [\sigma_1(X) \sigma_2(X) \ldots \sigma_m(X)]^T = 0$ is the sliding manifold in $\mathbb{R}^n$ determined by the intersection of ‘m’ sliding manifolds $\sigma_i(X) = 0$.

The control problem consists of developing the functions $u_i^+(X,t), u_i^-(X,t)$ and the sliding surface $\sigma(X) = 0$ so that the closed loop system exhibits a sliding mode on the sliding manifold $\sigma(X) = 0$.

The design of sliding mode control law consists of the construction of a suitable sliding surface so that the dynamics of the system confined to the sliding manifold produces a desired behaviour, and the design of a discontinuous control law which forces the system trajectory to the sliding manifold and maintains it there.

### 3.2.1 Sliding Surface

The sliding surface or switching surface $\sigma(X) = 0$ is a sliding manifold in $\mathbb{R}^n$ determined by the intersection of ‘m’ sliding surfaces $\sigma_i(X) = 0$ as shown in Figure 3.1. The switching surface is designed such that the system response confined to $\sigma(X) = 0$ has a desired behaviour. The switching surface may be either linear or non-linear, the linear ones are more prevalent in the literature.
3.2.2 Existence Condition

After switching surface design, the next important issue of sliding mode control is guaranteeing the existence of a sliding mode. Consider the system with a scalar input for which Equation (3.1) can be rewritten using Equation (3.2) as

$$\dot{X}(t) = g(X, t, u)$$ (3.3)

where

$$g(X, t, u) = \begin{cases} g^+(X, t, u^+), & \text{if } \sigma(X) > 0 \\ g^-(X, t, u^-), & \text{if } \sigma(X) < 0 \end{cases}$$ (3.4)

For the sliding mode to exist, the phase trajectories of the two substructures corresponding to the two different values of the vector function $g$ given by Equation (3.4) must be directed towards the sliding surface $\sigma(X, t) = 0$. While approaching the sliding surface from the points which satisfy $\sigma(X) > 0$, the vector $g^+$ must be directed towards the sliding surface and for the points which satisfy $\sigma(X) < 0$, the vector $g^-$ must be
directed towards the sliding surface. The normal vectors \((g_N^+, g_N^-)\) of the function \(g\) are orthogonal to the sliding surface. Therefore,

\[
\begin{align*}
\lim_{\sigma \to 0^+} g_N^+ &< 0 \implies \lim_{\sigma \to 0^+} \nabla \sigma g^+ < 0 \\
\lim_{\sigma \to 0^-} g_N^- &> 0 \implies \lim_{\sigma \to 0^-} \nabla \sigma g^- > 0
\end{align*}
\]  
(3.5)

Since

\[
\frac{d\sigma}{dt} = \sum_{i=1}^n \frac{\partial \sigma}{\partial X_i} \frac{dX_i}{dt} = \nabla \sigma g
\]  
(3.6)

The existence condition of the sliding mode becomes

\[
\begin{align*}
\lim_{\sigma \to 0^+} \frac{d\sigma}{dt} &< 0 \implies \lim_{\sigma \to 0} \sigma \frac{d\sigma}{dt} < 0 \\
\lim_{\sigma \to 0^-} \frac{d\sigma}{dt} &> 0
\end{align*}
\]  
(3.7)

If the inequality given in Equation (3.7) is satisfied in the entire state space, this condition is sufficient for the system to reach the sliding surface.

### 3.2.3 Reaching Condition

The reaching condition is the condition for the system to reach the sliding surface within finite time. Consider the system given by Equation (3.1) for which the scalar discontinuous input \(u(t)\) is given by Equation (3.8).

\[
u(t) = \begin{cases} 
    u^+(X, t) & \text{for } \sigma(X) > 0 \\
    u^-(X, t) & \text{for } \sigma(X) < 0
\end{cases}
\Rightarrow
u(t) = -Lsign(\sigma(X))
\]  
(3.8)

where \(L\) is a positive real number and \(sign()\) is the signum function.

Let \([X^+]\) and \([X^-]\) be the steady-state representative point corresponding to the inputs \(u^+(X, t)\) and \(u^-(X, t)\). Then, a sufficient
condition for the system to reach the sliding surface is given by Equation (3.9).

\[
\begin{align*}
[X^+] \in \sigma(X) &< 0 \\
[X^-] \in \sigma(X) &> 0
\end{align*}
\] (3.9)

Equation (3.9) implies that if the steady-state point for one sub-structure belongs to the region of the state space reserved to the other sub-structure, then sooner or later the system representative point will hit the sliding surface as shown in Figure 3.2 (Utkin 1993).

![Figure 3.2 Attractiveness of the sliding manifold](image)

3.2.4 Reaching Time

For a domain to be that of a sliding mode, it is sufficient that in some n-dimensional domain $\Omega$, there exists a function $V(X,t,\sigma)$ differentiable with respect to $X,t,\sigma$, satisfying the following conditions (Utkin 1977, Utkin 1993).
1. \( V(X, t, \sigma) > 0 \), with \( \sigma \neq 0 \), and arbitrary \( X, t \) and \( V(X, t, 0) = 0 \), ie., \( V(X, t, \sigma) \) is positive definite with respect to \( \sigma \) and on \( \|\sigma\| = \rho \), for all \( X \in \Omega \) and any \( t \)

\[
\inf_{\|\sigma\|=\rho} V(X, t, \sigma) = h_\rho, \quad h_\rho > 0
\]

\[
\sup_{\|\sigma\|=\rho} V(X, t, \sigma) = H_\rho, \quad H_\rho > 0
\]

(3.10)\( (3.11) \)

where \( h_\rho \) and \( H_\rho \) depend upon \( \rho \).

2. \( \dot{V}(X, t, \sigma) \) has a negative supremum for all \( X \in \Omega \) except for \( X \) on the switching surface.

Consider the function given by Equation (3.12).

\[
V(X, t, \sigma) = \frac{1}{2} \sigma^2(X)
\]

(3.12)

which is positive definite. Taking the derivative of \( V(X, t, \sigma) \) with respect to time,

\[
\dot{V}(X, t, \sigma) = \sigma \frac{\partial \sigma}{\partial t}
\]

(3.13)

Using Equation (3.7) in Equation (3.13) gives

\[
\dot{V}(X, t, \sigma) = \sigma \frac{\partial \sigma}{\partial t} < 0
\]

(3.14)

The condition given by Equation (3.14) is called reachability condition and ensures that the sliding manifold is reached asymptotically (Utkin 1977). This condition is often replaced by \( \eta \)-reachability condition (Utkin 1977, Young et al. 1999) as

\[
\dot{V}(X, t, \sigma) = \sigma \frac{\partial \sigma}{\partial t} \leq -\eta |\sigma| < 0
\]

(3.15)

By solving Equation (3.15),

\[
|\sigma[X(t)]| - |\sigma[X(0)]| \leq -\eta t
\]

(3.16)
Equation (3.16) shows that the time needed to reach the sliding surface, starting from the initial condition \(\sigma[X(0)]\) is bounded by

\[
t_r = \frac{|\sigma[X(0)]|}{\eta}
\]

Equation (3.17) shows that the condition given by Equation (3.15) ensures the finite time convergence to \(\sigma(X) = 0\).

### 3.3 System Description in Sliding Mode

Consider the system given by Equation (3.18).

\[
\dot{X}(t) = f(X, t) + b(X, t)u(t)
\]

The scalar control input \(u(t)\) is discontinuous on the sliding surface \(\sigma(X, t) = 0\), while \(f(X, t)\) and \(b(X, t)\) are continuous function vectors. Under sliding mode control, the system trajectories stay on the sliding surface.

\[
\sigma(X, t) = 0 \implies \dot{\sigma}(X, t) = 0
\]

where

\[
\dot{\sigma}(X, t) = \frac{d\sigma}{dt} = \sum_{i=1}^{n} \frac{\partial\sigma}{\partial X_i} \frac{dx_i(t)}{dt} = \nabla\sigma \dot{X}(t) = G\dot{X}(t)
\]

where \(G\) is a \(1 \times n\) matrix, the elements of which are the derivatives of the sliding surface with respect to the state variables (gradient vector). Substituting Equation (3.18) and Equation (3.19) in Equation (3.20) gives Equation (3.21).

\[
G\dot{X}(t) = Gf(X, t) + Gb(X, t)u_{eq}(t) = 0
\]
where \( u_{eq}(t) \) represents an equivalent continuous control input which maintains the system evolution on the sliding surface. Substituting \( u_{eq}(t) \) from Equation (3.21) for \( u(t) \) in Equation (3.18) gives Equation (3.22).

\[
\dot{X}(t) = \left[ I - b(X, t)(Gb(X, t))^{-1}G \right] f(X, t)
\]

(3.22)

The system motion under sliding mode control is represented by Equation (3.22). As the system motion is constrained to be on the sliding surface under sliding regime, the matrix \([I - b(Gb)^{-1}G]\) is less than the full rank. As a consequence, the equivalent system described by Equation (3.22) is of the order \( n-1 \). This reduced order dynamics provides attractive advantages such as insensitivity to parameter variations, matched uncertainties and disturbances (Gonzalez et al. 2012, Itkis 1976, Komurcugil 2012, Utkin 1993).

### 3.4 Sliding Mode Control Algorithm

As discussed in previous sections, the sliding mode control design consists of two steps, the construction of the desired sliding surface and the sliding mode enforcement capable of driving and confining the system motion on the sliding surface. Hence, the control input consists of two components, a discontinuous component \( u_n(t) \) to drive the system states on to the sliding surface and a continuous component \( u_{eq}(t) \) which ensures the motion of the system on the sliding surface whenever it is on the surface to force the error variables to the origin.

\[
u(t) = u_n(t) + u_{eq}(t)
\]

(3.23)
The discontinuous component $u_d(t)$ is of the form given by Equation (3.8) and the continuous one $u_{eq}(t)$ is of the form which satisfies Equation (3.21).

### 3.5 Sliding Mode Control of Uncertain Systems

The model identification of practical systems introduces parameter errors and hence the models often contain uncertain parameters which often lie within certain bounds.

Consider the following state dynamics with uncertainties in parameters.

$$\dot{X}(t) = [f(X, t) + \Delta f(X, t, \gamma)] + [b(X, t) + \Delta b(X, t, \gamma)]u(t)$$

(3.24)

where $\gamma$ is a time-varying vector function of uncertain parameters, the values of which lies within some closed and bounded set, $\Delta f$ and $\Delta b$ are the plant uncertainties which lie in the image of $b(t, X)$ for all values of $X$ and $t$. This requirement is called matching condition.

Assuming that the matching conditions are satisfied, the total plant uncertainty can be lumped into a single vector $\epsilon(t, X(t), \gamma(t), u(t))$ and the uncertain system can be represented by the following dynamics.

$$\dot{X}(t) = f(X, t) + b(X, t)u(t) + b(X, t) \epsilon(t, X(t), \gamma(t), u(t))$$

(3.25)

with initial condition $X(t_0) = X_0$. Let $X(t) : [t_0, \infty) \rightarrow \mathbb{R}^n$ be a solution of Equation (3.25). Then $X(t)$ is uniformly bounded if for each $X_0$, there is a positive finite constant, $q(X_0)$ such that $\|X(t)\|_2 < q(X_0)$ for all $t \in [t_0, \infty]$ where $\|\cdot\|_2$ is the Euclidean vector norm. Also, the solution $X(t)$ is
uniformly ultimately bounded within some closed bounded set \( S \subset \mathbb{R}^n \) if for each \( X_0 \), there is a non-negative constant \( T(X_0) < \infty \) such that \( X(t) \in S \) for all \( t > t_0 + T(X_0, S) \).

The problem is to find \( u(t) \) such that for any initial condition \( X_0 \) and for all uncertainties \( \gamma(t) \), a uniformly bounded solution \( X(t): [t_0, \infty] \rightarrow \mathbb{R}^n \) exists. Gutman and Palmor (1982) proposed a solution given by Equation (3.26).

\[
u(t) = -\frac{b^T(x,t)\nabla_x V(x,t)}{\|b^T(x,t)\nabla_x V(x,t)\|_2} \rho(X, t) \text{ if } b^T(X, t)\nabla_x V(X, t) \neq 0 \quad (3.26)
\]

If \( b^T(X, t)\nabla_x V(X, t) \neq 0 \), then

\[
U/U \in \mathbb{R}^m \text{ and } |U| \leq \rho(X, t)
\]

where \( \rho(X, t) \) is a scalar function satisfying \( \rho(X, t) \geq \|e(t, X(t), \gamma(t), u(t))\|_2 \).

If the input is scalar, Equation (3.26) reduces to the form

\[
u(t) = -\text{sign}\left(b^T(X, t)\nabla_x V(X, t)\right) \quad (3.28)
\]

where \( b^T(X, t)\nabla_x V(X, t) \) can be seen as switching surface.

### 3.6 Design Procedure of Sliding Mode Control for Uncertain Systems

Consider an uncertain system given by Equation (3.25). The sliding mode controller for the system is

\[
u(t) = u_{eq}(t) + u_n(t) \quad (3.29)
\]
where $u_{eq}(t)$ is the equivalent control, assuming that all the uncertainties are zero, i.e., $\epsilon(t, X(t), \gamma(t), u(t)) = 0$ and $u_n(t)$ is designed by considering non-zero uncertainties. Considering the switching surface $\sigma(X, t) = 0$ gives

$$u_{eq} = -\left[\frac{\partial \sigma}{\partial x} b\right]^{-1} \left[\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x} f\right]$$  \hspace{1cm} (3.30)

$u_n(t)$ accounting for the uncertainties is given by Equation (3.26). Substituting Equations (3.26) and (3.30) in Equation (3.29) gives

$$u = -\left[\frac{\partial \sigma}{\partial x} b\right]^{-1} \left[\frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x} f\right] - \frac{b^T(x, t)v_x v(x, t)}{\|b^T(x, t)v_x v(x, t)\|^2} \rho(X, t)$$  \hspace{1cm} (3.31)

Assume that $\|\epsilon(t, X(t), \gamma(t), u(t))\|_2 \leq \rho(X, t)$ where $\rho(X, t)$ is a non-negative scalar function. Consider a scalar function given by Equation (3.32).

$$\bar{\rho}(X, t) = \alpha + \rho(X, t), \alpha > 0.$$  \hspace{1cm} (3.32)

Consider the Lyapunov function

$$V(X, t) = \frac{1}{2} \sigma^T(X, t) \sigma(X, t)$$  \hspace{1cm} (3.33)

In order to assure the existence of a sliding mode and the attractiveness to the surface, the reaching condition $\frac{dV}{dt}(X, t) = \dot{V} = \sigma^T \dot{\sigma} < 0$ must be satisfied whenever $\sigma(X, t) \neq 0$. We have

$$\dot{\sigma}(X, t) = \frac{\partial \sigma}{\partial t} + \frac{\partial \sigma}{\partial x} \dot{X}(t)$$  \hspace{1cm} (3.34)

The gradient of the Lyapunov function is

$$\nabla_X V(X, t) = \left[\frac{\partial \sigma(X, t)}{\partial x}\right]^T \sigma(X, t)$$  \hspace{1cm} (3.35)

If $\sigma(X, t) = 0$, then $u(X, t) = u_{eq}(X, t)$. Taking the derivative of Lyapunov function of Equation (3.33) and using Equation (3.25),
\[ \dot{V} = \sigma^T \frac{\partial \sigma}{\partial t} + \sigma^T \frac{\partial \sigma}{\partial x} (f + bu + b \epsilon) \]  

(3.36)

where \(X\) and \(t\) arguments have been omitted for the simplicity of the notation. Using Equation (3.31) in Equation (3.36) gives

\[ \dot{V} = - \left\| b^T \left( \frac{\partial \sigma}{\partial x} \right)^T \sigma \right\|_2 \rho + \sigma^T \frac{\partial \sigma}{\partial x} (b \epsilon) \leq -\alpha \left\| b^T \left( \frac{\partial \sigma}{\partial x} \right)^T \sigma \right\|_2 \]  

(3.37)

Equation (3.37) implies that \(\dot{V}\) is negative-definite, which establishes the attractiveness to the switching surface.

### 3.7 Sliding Mode Control of Second Order Systems

In this section, the design procedure of sliding mode control scheme for single-input second order systems is presented.

Consider the single-input second order system

\[
\begin{cases}
    \dot{x}_1(t) = x_2(t) \\
    \dot{x}_2(t) = f(X, t) + b(X, t)u(t)
\end{cases}
\]  

(3.38)

where \(X = [x_1 \ x_2]^T\) is the state vector, \(f(X, t)\) and \(b(X, t)\) are the functions representing the dynamics of the system, \(u(t)\) is the scalar control input.

The objective of the control input \(u(t)\) is to force the states of the system to a desired trajectory \(X_d(t)\). Let the error between the actual trajectory and the desired trajectory be \(e(t)\).

\[ e(t) = x_1(t) - x_{1d}(t) \]  

(3.39)

Taking the time derivative of \(e(t)\),

\[ \dot{e}(t) = x_2(t) - x_{2d}(t) \]  

(3.40)
where $x_{1d}(t)$ and $x_{2d}(t)$ are the desired trajectories for the states $x_1(t)$ and $x_2(t)$ respectively. The sliding function can be defined as the linear combination of error variables given by equation (3.41).

$$\sigma(X, t) = \dot{e}(t) + \lambda e(t), \lambda > 0$$  \hspace{1cm} (3.41)

where $\lambda$ represents the slope of the sliding surface. The sliding mode is represented by the Equation (3.42).

$$\dot{e}(t) = -\lambda e(t)$$  \hspace{1cm} (3.42)

The solution of the Equation (3.42) is given by Equation (3.43).

$$e(t) = e(0)\exp(-\lambda t)$$  \hspace{1cm} (3.43)

From Equation (3.43), it is clear that the dynamics of the system depends on $\lambda$. A low value of $\lambda$ leads to slower error convergence and longer tracking time, whereas a high value of $\lambda$ leads to faster convergence and lesser tracking time. But if the value of $\lambda$ is too high, it can cause an overshoot in the system states and degrade the tracking accuracy. Taking the time derivative of sliding function gives Equation (3.44).

$$\dot{\sigma}(t) = \lambda \dot{e}(t) + \ddot{e}(t)$$  \hspace{1cm} (3.44)

ie.,

$$\dot{\sigma}(t) = \lambda \dot{e}(t) + \dot{x}_2(t) - \dot{x}_{2d}(t)$$  
$$= \lambda \dot{e}(t) + f(X, t) + b(X, t)u(t) - \ddot{x}_{1d}(t)$$  \hspace{1cm} (3.45)

Rearranging Equation (3.45),

$$\dot{\sigma}(t) = \varphi(X, \dot{X}, t) + b(X, t)u(t)$$  \hspace{1cm} (3.46)

where

$$\varphi(X, \dot{X}, t) = \lambda \dot{e}(t) + f(X, t) - \ddot{x}_{1d}(t)$$  \hspace{1cm} (3.47)
When the system is in sliding mode $\dot{\sigma}(t) = \sigma(t) = 0$. Equating (3.45) to zero and solving for $u(t)$ gives the equivalent control expression given by Equation (3.48).

$$u_{eq}(t) = \frac{1}{b(x,t)} [\ddot{x}_{1d}(t) - f(X,t) - \lambda \dot{e}(t)], \quad b(X,t) \neq 0 \quad (3.48)$$

The equivalent control forces the error variables to move towards zero immediately. As a result, the desired behaviour can be achieved after the sliding mode starts. The additional control action needed to move the error variables to sliding mode is given by (3.8). Therefore, the total control signal is

$$u(t) = \frac{1}{b(x,t)} [\ddot{x}_{1d}(t) - f(X,t) - \lambda \dot{e}(t)] - L \text{sign}(\sigma(X)) \quad (3.49)$$

3.8 Chattering

Sliding mode controller with the discontinuous control of the form given in Equation (3.8) suffers from high frequency switching near the sliding surface as shown in Figure 3.3. This high frequency component of the control propagates through the system, therefore exciting the unmodeled fast dynamics and results in undesired oscillations which affect the system. This effect is called chattering (Boiko et al. 2007, Fridman 2001, Gonzalez et al. 2012, Levant 2007, Liang et al. 2012, Perruquetti and Barbot 2002, Zong et al. 2010). The chattering effect can degrade the system performance or may even lead to instability and may even damage the plant (Levant 2007, Slotine and Li 1991, Utkin et al. 2009, Young et al. 1999).
The chattering can be reduced by the continuous approximation of the discontinuous control of sliding mode control (Husain et al. 2008, Levant 2007, Sira-Ramirez 1992). The widely used approach for this is by introducing a boundary layer of width $\emptyset$ and replacing signum function of (3.8) by a saturation function given in (3.50).

$$sat \left( \frac{\sigma(X)}{\emptyset} \right) = \begin{cases} sgn \left( \frac{\sigma(X)}{\emptyset} \right) & \text{if } \left| \frac{\sigma(X)}{\emptyset} \right| \geq 1 \\ \frac{\sigma(X)}{\emptyset} & \text{if } \left| \frac{\sigma(X)}{\emptyset} \right| < 1 \end{cases} \quad (3.50)$$

The consequence of this approach is that the system possesses robustness that is a function of the boundary layer width. If the thickness of the boundary layer increases, the time required to enter the sliding mode increases, thereby decreasing the robustness. This method is highly sensitive to the unmodeled fast dynamics and may lead to unacceptable performance (Husain et al. 2008, Slotine and Li 1991). Moreover, there will be steady-state error that is proportional to the boundary layer thickness (Slotine and Li 1991). The most recent and efficient approach for chattering elimination

### 3.9 Simulation

The performance of the sliding mode controller is studied for the testbed system shown in Figure 3.4 (Gonzalez et al. 2012).

![Mass-Spring-Damper System](image)

**Figure 3.4 Mass-Spring-Damper System**

The system consists of two masses, three springs and one damper with uncertainties in physical parameters, i.e., the physical parameters masses, spring constants and the damping coefficient are not known exactly, but the values are within certain known intervals. The dynamics of the system is
The nominal values of the system parameters are \( M_1 = 1.28 \text{ kg}, \ M_2 = 1.05 \text{ kg}, \ K_1 = 190 \text{ N/m}, \ K_2 = 780 \text{ N/m}, \ K_3 = 450 \text{ N/m}, B = 15 \text{ Ns/m}. \) The range of values of the parameters are \( M_1: 1.024 \text{ kg} - 1.536 \text{ kg}, \ M_2: 0.84 \text{ kg} - 1.26 \text{ kg}, K_1: 152 \text{ N/m} - 228 \text{ N/m}, K_2: 624 \text{ N/m} - 936 \text{ N/m}, K_3: 360 \text{ N/m} - 540 \text{ N/m}, B: 12 \text{ Ns/m} - 18 \text{ Ns/m}. \)

The control objective is to maintain the position of mass \( M_1 \) constant at \( x_{1d} \), irrespective of the behaviour of mass \( M_2 \), which can be considered as a perturbation as shown in Figure 3.5. The perturbation effect of mass \( M_2 \) on mass \( M_1 \) is

\[
W(t) = K_2 [x_3(t) - x_1(t)]
\]  

(3.52)
3.9.1 Controller Design

The dynamics of mass $M_1$ system under control is

$$
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{K_1}{M_1} x_1(t) + \frac{1}{M_1} W(t) + \frac{1}{M_1} F(t)
\end{align*}
$$
(3.53)

Comparing Equation (3.53) with the standard form of the second order uncertain system model given by Equation (3.25),

$$
f(X, t) = -\frac{K_1}{M_1} x_1(t), \quad b(X, t) = \frac{1}{M_1}, \quad d(t) = \frac{1}{M_1} W(t)
$$
(3.54)

where $d(t)$ is the disturbance.

The error function is defined as

$$
e(t) = x_1(t) - x_{1d}
$$
(3.55)

The sliding surface is selected as $\sigma = \dot{e}(t) + \lambda e(t), \quad \lambda = 4$. The sliding mode controller given in Equation (3.8) with the controller gain $L = 10$ is selected. The equivalent controller designed using Equation (3.54) in Equation (3.48) is

$$
u_{eq}(t) = M_1 \left[ \ddot{x}_{1d}(t) + \frac{K_1}{M_1} x_1(t) - \lambda \dot{e}(t) \right]
$$
(3.56)

Since $x_{1d}$ is constant,

$$
\begin{align*}
\ddot{x}_{1d}(t) &= 0 \\
\dot{e}(t) &= \dot{x}_1(t) = x_2(t)
\end{align*}
$$
(3.57)

Using Equation (3.57), $\lambda = 4$ and nominal values of the system parameters in Equation (3.56), the equivalent control is

$$
u_{eq}(t) = 190x_1(t) - 5.12x_2(t)
$$
(3.58)
The control force is limited between -10 N and 10 N.

3.9.2. Simulation Results

The Simulink model of sliding mode control based system is given in Figure 3.6.

![Simulink model of sliding mode control](image)

Figure 3.6 Simulink model of sliding mode control

The plots of the response of the system are given in Figure 3.7 to Figure 3.11.
Figure 3.7 Position of mass $M_1$ using the conventional SMC

Figure 3.8 Velocity of mass $M_1$ using the conventional SMC
Figure 3.9 Control force of the system using the conventional SMC

Figure 3.10 Zoom in part of the position response of the system using the conventional SMC
Figure 3.11 Zoom in part of the velocity response of the system using the conventional SMC

The position of mass $M_1$ is shown in Figure 3.7. The controller is able to exhibit transient response with rise time of 0.947 s and settling time of 1.62 s. The overshoot is zero. The velocity of mass $M_1$ is shown in Figure 3.8. The plot of the control effort given in Figure 3.9 shows a high frequency switching of control signal. This results in high frequency oscillations called chattering. The effect of chattering is evident in the zoom in parts of the position response and velocity response shown in Figure 3.10 and Figure 3.11 respectively.

The performance of sliding mode controllers with boundary layers of different thickness are simulated and compared with the performance of the conventional sliding mode controller. The comparison of the responses is given in Figure 3.12 to Figure 3.14.
Figure 3.12 Comparison of the responses of the position of mass $M_1$ for the conventional SMC and SMC with different boundary layers.

Figure 3.13 Zoom in part of the velocity responses of the system using sliding mode controllers with different boundary layers.
Figure 3.14 Control efforts of sliding mode controllers with different boundary layers

The rise time and settling time are respectively 0.947 s and 1.62 s for the system with the conventional sliding mode controller. The rise time and settling time for the system using sliding mode controller with a boundary layer of thickness 0.002 are 1 s and 2.5 s respectively. The rise time for the system using sliding mode controller with a boundary layer of thickness 0.01 is 1.517 s and the position response fails to reach 98% of the desired value, which is not desirable. It is found that IAE indices of the position are respectively 0.00418, 0.00574 and 0.0117 for the conventional sliding mode control, sliding mode control with a boundary layer of thickness 0.002 and sliding mode control with a boundary layer of thickness 0.01, whereas ITAE indices are respectively 0.00204, 0.0104 and 0.0426. The analysis of IAE and ITAE indices confirms that the increase in boundary layer thickness degrades the tracking accuracy, the transient response slows
down and the steady-state error increases. Sliding mode control with boundary layer reduces chattering which is evident from the comparison of zoom in parts of velocity responses of the conventional sliding mode controller and sliding mode controller with boundary layer shown in Figure 3.11 and Figure 3.13 respectively. Figure 3.14 shows that the control efforts of sliding mode controllers with boundary layers are smooth compared to that of the conventional sliding mode control shown in Figure 3.9, which confirms the chattering reduction property of sliding mode controller with boundary layer. It is clear that as the boundary layer thickness increases, the chattering effect decreases, but at the cost of degrading the tracking performance.

3.10 Summary

In this chapter, the basic principles and the design procedure of the conventional sliding mode control are discussed. The main disadvantage of the conventional sliding mode control is the dangerous chattering phenomenon. The chattering effect can be reduced by sliding mode control with a boundary layer in which the discontinuous sign function of the sliding mode control is replaced by a saturation function. However, sliding mode controller with a boundary layer degrades the tracking performance. The tracking time and tracking error increases with an increase in the boundary layer thickness. If the boundary layer thickness is not properly chosen, the controller may lead to unacceptable performance.