CHAPTER - 3

Common fixed point for uniformly intimate pair of mappings in semi-Hausdorff spaces and its applications

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CHAPTER - 3
COMMON FIXED POINT FOR UNIFORMLY INTIMATE PAIR OF MAPPINGS
IN SEMI-HAUSDORFF SPACES AND ITS APPLICATIONS

In this chapter, we have introduced the concept of "Uniformly intimate" pair of mappings in semi-Hausdorff d-complete topological space which is continuation of the study of common fixed point in non-metric setting. Further, some common fixed point theorems in semi-Hausdorff d-complete topological space have been proved with above condition. Also, as application, common fixed point theorems for expansion mappings have been given. Our results generalize and improve the results of Fisher [32], Kang and Rhoades [59], Popa [82], Khan et al. [66].

In 1976 Jungck [50] generalized the well known Banach fixed point theorem for a pair of commuting mappings. The concept of commuting maps was further generalized by Sessa [98] in 1982 when he introduced pair of weakly commuting maps. Commuting maps are weakly commuting but converse is not true in general. Those are further weakened by the compatibility [53] and the compatibility of type (A) [76].

We have already mentioned in chapter 2 that Hicks and Rhoades ([41], [42]), proved some fixed point theorems in a d-complete topological space (non-metric setting) where the
distance function used need not satisfy triangle inequality. Later, on this line, Cho, Sharma and Sahu [20] have established some existence results under semi-compatibility condition.

In 1965, Murdeshwar and Naimpally [75], have introduced the concept of semi-Hausdorff space deriving from topological space. Wherein they have established that the semi-Hausdorff condition is strictly stronger than the $T_1$-axioms and thus it is between the $T_1$ and $T_2$ axioms. For us, this was an interesting space structure in which the existence of fixed point could be studied. With this intention first we have introduced the concept of uniformly intimate condition and then study some existence results in semi-Hausdorff $d$-complete topological space using uniformly intimate condition.

DEFINITION 3.1 [75]. A topological space $X$ is said to be semi-Hausdorff iff every sequence in $X$ has atmost one limit.

Murdeshwar and Naimpally [75] have given the following theorems.

THEOREM A. Every Hausdorff space is semi-Hausdorff, but not conversely.

THEOREM B. Every semi-Hausdorff space is $T_1$ but not conversely.

THEOREM C. Every first countable semi-Hausdorff space is Hausdorff.

Hicks and Rhoades [42] have proved the following interesting theorem:
THEOREM D. Let \((X, t)\) be a Hausdorff d-complete topological space and \(f, h\) be \(w\)-continuous self-mappings on \(X\) satisfying 
\[ d(hx, hy) = Q(m(x, y)) \]
for all \(x, y\) in \(X\), where 
\[ m(x, y) = \max\{d(fx, fy), d(fx, hx), d(fy, hy)\} \]
and \(Q\) be a real-valued function satisfying the following conditions:

(a) \(0 < Q(y) < y\) for each \(y > 0\) and \(Q(0) = 0\),

(b) \(g(y) = \frac{y}{(y-Q(y))}\) is a non-increasing function on \((0, \infty)\),

(c) \(\int_0^{y_1} g(y)\,dy < \infty\) for each \(y_1 > 0\),

(d) \(Q(y)\) is non-decreasing.

Suppose also that

(e) \(f\) and \(h\) are commuting,

(f) \(h(X) \subseteq f(X)\).

Then \(f\) and \(h\) have a unique common fixed point in \(X\).

DEFINITION 3.2 [92]. Let \(S\) and \(T\) be self-maps of a semi-Hausdorff d-complete topological space \((X, t)\). The pair \(\{S, T\}\) is said to be \(T\)-intimate iff
\[ \alpha \, d(TSx_n, Tx_n) = \alpha \, d(SSx_n, Sx_n) \]
where \(\alpha = \lim \sup\) or \(\lim \inf\) and \(\{x_n\}\) is a sequence in \(X\) such that
\[ \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = z \]
for some \(z \in X\). 21
DEFINITION 3.3. Let $A$ and $S$ be self-maps of a semi-Hausdorff $d$-complete topological space $(X,t)$. The pair $\{A,S\}$ is said to be Uniformly $S$-intimate iff
\[ d(ASu,SSu) \leq d(Au,Su), \text{ for all } u \in X. \]

EXISTENCE OF COMMON FIXED POINT FOR UNIFORMLY INTIMATE PAIRS OF MAPPINGS:

THEOREM 3.1. Let $A,B,S$ and $T$ be self mappings of a semi-Hausdorff $d$-complete topological space $(X,t)$, satisfying the following conditions:

(3.1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X),$

(3.2) $d(Ax,By) \leq Q(m(x,y))$ for all $x,y$ in $X$, where
\[ m(x,y) = \max \{d(Sx,Ty), d(Sx,Ax), d(Ty,By)\} \]
and $Q$ satisfy the conditions (a), (b), (c) and (d),

(3.3) The pairs $\{A,S\}$ and $\{B,T\}$ are Uniformly $S$ and $T$-intimate respectively.

(3.4) $A,B,S$ and $T$ are $w$-continuous,

Then $A,B,S$ and $T$ have unique common fixed point in $X$.

PROOF. Let $x_0$ be an arbitrary point in $X$, there exists a point $x_1$ in $X$ such that $Ax_0 = Tx_1$ as $A(X) \subseteq T(X)$. For this point $x_1$, we can choose a point $x_2$ in $X$ such $Bx_1 = Sx_2$ as $B(X) \subseteq S(X)$. Thus repeating the foregoing arguments, we define two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that
\[
\begin{align*}
Y_{2n} &= Ax_{2n} = Tx_{2n+1} \\
Y_{2n+1} &= Bx_{2n+1} = Sx_{2n+2}
\end{align*}
\]

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For simplicity, let $d_n = d(y_n, y_{n+1})$ and applying (3.2), we have

$$d_{2n+1} = d(Ax_{2n+2}, Bx_{2n+1})$$

$$\leq Q(\max \{d(Sx_{2n+4}, Tx_{2n+1}), d(Sx_{2n+4}, Ax_{2n+2}), d(Tx_{2n+1}, Bx_{2n+1})\}),$$

which implies

$$(3.6) \quad d_{2n+1} \leq Q(\max \{d_{2n}, d_{2n+1}\}).$$

Suppose that $d_{2n+1} > d_{2n}$ for some $n$. Then from (3.6), we have

$$d_{2n+1} \leq Q(d_{2n+1}) < d_{2n+1},$$

which is a contradiction. Therefore, from (3.6), we have

$$d_{2n+1} \leq Q(d_{2n}).$$

Similarly, we have

$$d_{2n+2} \leq Q(d_{2n+1}).$$

So, in general, for $d_0 > 0$, we have

$$(3.7) \quad d_n \leq Q(d_{n-1}), \quad n = 1, 2, \ldots.$$

We define a sequence $\{t_n\}$ of positive real numbers such that $t_{n+1} = Q(t_n)$ with $t_1 = d_0 > 0$. By (a), we have

$$0 < t_{n+1} < t_n < \ldots < t_1, \quad n \geq 1.$$}

Moreover, by (b) and (c), the series $\sum t_n$ converges.
We shall now show that \( d_n \leq t_{n+1} \), \( n \geq 1 \). From (3.7), for \( n = 1 \), we have \( d_1 \leq Q(d_0) = Q(t_1) = t_2 \) and the desired inequality is valid for \( n = 1 \). Assume that it is true for some \( n > 1 \). Again, from (3.7), we have

\[
d_n \leq Q(d_{n-1}) \leq Q(t_n) = t_{n+1}.
\]

Since \( \{t_n\} \) is convergent, it follows that \( \{d(y_n, y_{n+1}) \} \) is also convergent. Since \( X \) is \( d \)-complete, \( \{y_n\} \) converges to some \( z \in X \) and so the subsequences \( \{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}, \) and \( \{Tx_{2n+1}\} \) of \( \{y_n\} \) also converge to the point \( z \). Let there exist some points \( u \) and \( v \) in \( X \) such that \( x_{2n} \rightarrow u \) and \( x_{2n+1} \rightarrow v \) as \( n \rightarrow \infty \). Then \( w \)-continuity of \( A, B, S \) and \( T \) imply \( Ax_{2n} \rightarrow Au, Sx_{2n} \rightarrow Su, Bx_{2n+1} \rightarrow Bv \) and \( Tx_{2n+1} \rightarrow Tv \). Since \( X \) is semi-Hausdorff, therefore we have

\[
z = Au = Su = Bv = Tv.
\]

Since the pair \( \{A, S\} \) is Uniformly \( S \)-intimate, therefore

\[
d(ASu, SSu) \leq d(Au, Su).
\]

which implies that \( Az = Sz \). Now using (3.2), we have

\[
d(Az, z) = d(Az, Bv)
\]

\[
\leq Q(\max \{d(Sz, Tv), d(Sz, Az), d(Tv, Bv)\})
\]

\[
\leq Q(d(Az, z)) < d(Az, z)
\]

which is a contradiction, so \( Az = z \).

Similarly when the pair \( \{B, T\} \) is Uniformly \( T \)-intimate, we obtain \( z = Bz = Tz \). Therefore \( z \) is a common fixed point of \( A, B, S \) and \( T \). Uniqueness easily follows from (3.2).
THEOREM 3.2. Let \( A, R, S \) and \( T \) be the mappings from a semi-Hausdorff d-complete topological space \((X, \tau)\) into itself satisfying the conditions (3.1), (3.3), (3.4) and the following:

\[(3.8) \quad d(Ax, By) \leq k(m(x, y)) \quad \text{for all } x, y \in X,\]

where \( k : [0, \infty) \to (0, \infty) \) is a function such that \( k(0) = 0 \), \( k(t) < t \) for all \( t \in (0, \infty) \) and \( k \) is non-decreasing with \( \sum_{n=1}^{\infty} k^n(t) < \infty \) for all \( t \in (0, \infty) \) (\( k \) is not assumed to be continuous).

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

PROOF. For an arbitrary point \( x_0 \) in \( X \), by (3.1), we can choose a sequence \( \{y_n\} \) defined by (3.5) and from (3.5) and (3.8), we obtain

\[d_{2n+1} \leq k(d_{2n}) \quad \text{and} \quad d_{2n+2} \leq k(d_{2n+1})\]

for all \( n = 0, 1, 2, \ldots \).

By induction, for \( d_0 > 0 \),

\[d_n \leq kd_{n-1} \leq \cdots \leq k^n(d_0)\]

for all \( n = 1, 2, \ldots \).

Now \( \sum_{n=1}^{\infty} d(y_n, y_{n+1}) < \infty \) implies that \( \{y_n\} \) is a Cauchy sequence in \( X \), which means that it converges to some point \( z \) in \( X \), since \( X \) is semi-Hausdorff d-complete. As in the proof of Theorem 3.1, it can be shown that \( z \) is a unique common fixed point of \( A, B, S \) and \( T \) in \( X \). This completes the proof.
One can also have Theorem 3.1 as a special case of Theorem 3.2, where \( k \) forces the w-continuity condition for all \( A, B, S \) and \( T \).

In the following theorems, all of \( A, B, S \) and \( T \) need not be w-continuous.

Let \( \mathcal{F} \) be a family of mappings \( f \) from \([0,\infty)^3 \) into \([0,\infty) \) such that each \( f \) is upper semi-continuous, non-decreasing in each coordinate variable and for any \( t > 0, f(t,t,t) = \gamma(t) < t \) and \( \sum_{n=1}^{\infty} \gamma^n(t) < \infty \), where \( \gamma \) be a mapping from \([0,\infty) \) into itself.

**Theorem 3.3.** Let \( A, B, S \) and \( T \) be mappings from a semi-Hausdorff \( d \)-complete topological space \((X, \tau)\) into itself satisfying the conditions \((3.1), (3.3)\) and the followings:

\[(3.9)\] there exists an \( f \in \mathcal{F} \) such that
\[d(Ax, By) \leq f(d(Sx, Ty), d(Sx, Ax), d(Ty, By))\]
for all \( x, y \in X \),

\[(3.10)\] one of \( S \) and \( T \) is w-continuous.

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** For an arbitrary point \( x_0 \in X \), we can choose a sequence \( \{y_n\} \) defined by \((3.5)\). From \((3.5)\) and \((3.9)\), we have
\[d_{2n+1} = d(Ax_{2n+2}, Bx_{2n+1}) \leq f(d(Sx_{2n+2}, Tx_{2n+1}), d(Sx_{2n+2}, Ax_{2n+2}), d(Tx_{2n+1}, Bx_{2n+1})), \]

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which implies

\[(3.11) \quad d_{2n+1} \leq f(d_{2n}, d_{2n+1}, d_{2n}).\]

Suppose \(d_{2n+1} > d_{2n}\) for some \(n\). Then from \((3.11)\), we have

\[0 < d_{2n+1} \leq f(d_{2n+1}, d_{2n+1}, d_{2n+1}) \]

\[< \gamma(d_{2n+1})\]

which is a contradiction. Hence from \((3.11)\), we have

\[d_{2n+1} \leq \gamma(d_{2n})\]

for all \(n = 0, 1, 2, \ldots\).

Similarly, we have

\[d_{2n+2} \leq \gamma(d_{2n+1})\]

for all \(n = 0, 1, 2, \ldots\).

So, in general, for \(d_0 > 0\),

\[d_n < \gamma(d_{n-1}) \leq \cdots \leq \gamma^n(d_0)\]

for all \(n = 0, 1, 2, \ldots\). Since \(\sum_{n=1}^{\infty} \gamma^n(t)\) is convergent for each \(t > 0\), it follows then \(\sum_{n=1}^{\infty} d(y_n, y_{n+1})\) is convergent. Since \(X\) is semi-Hausdorff \(d\)-complete, \(\{y_n\}\) converges to some point \(z\) in \(X\) and hence the subsequences \(\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\}\) and \(\{Tx_{2n+1}\}\) of \(\{y_n\}\) also converge to the point \(z\).
Now, suppose that $T$ is $w$-continuous. Since the pair \( \{B, T\} \) is uniformly $T$-intimate and the subsequences \( \{Bx_{2n+1}\} \) and \( \{Tx_{2n+1}\} \) of \( \{y_n\} \) also converge to the point $z$. Therefore, we have
\[
d(BTx_{2n+1}, TTx_{2n+1}) \leq d(Bx_{2n+1}, Tx_{2n+1}).
\]
Letting $n \to \infty$, we obtain
\[
BTx_{2n+1}, TTx_{2n+1} \to Tz.
\]
Putting $x = x_{2n}$ and $y = Tx_{2n+1}$ in (3.9), we have
\[
(3.12) \quad d(Ax_{2n}, BTx_{2n+1}) \leq f(d(Sx_{2n}, TTx_{2n+1}), d(Sx_{2n}, Ax_{2n}),
\]
\[d(TTx_{2n+1}, BTx_{2n+1})).
\]
Letting $n \to \infty$, we have
\[
d(z, Tz) \leq f(d(z, Tz), 0, 0)
\]
\[\leq \gamma(d(z, Tz))
\]
\[< d(z, Tz),
\]
which is a contradiction and so $Tz = z$.

Again, replacing $x$ by $x_{2n}$ and $y$ by $z$ in (3.9), respectively, we have
\[
(3.13) \quad d(Ax_{2n}, Bz) \leq f(d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz)).
\]
As $n \to \infty$, we have
\[
d(z, Bz) \leq f(0, 0, d(z, Bz))
\]
\[\leq \gamma(d(z, Bz))
\]
\[< d(z, Bz),
\]
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which means that \( Bz = z \). Since \( B(X) \subseteq S(X) \), there exists a point \( u \) in \( X \) such that

\[ Bz = Su = z. \]

By (3.9), we have

\[
d(Au, z) = d(Au, Bz) \\
\leq f(d(Su, Tz), d(Su, Au), d(Tz, Bz)) \\
= f(0, d(z, Au), 0) \\
< d(z, Au),
\]

which is a contradiction and so \( Au = z \). Since the pair \( \{A, S\} \) is Uniformly \( S \)-intimate and \( Au = Su = z \), then we have

\[
d(ASu, SSu) \leq d(Au, Su)
\]

which implies that \( ASu = SSu \) and hence \( Az = Sz \).

By using (3.9), we have

\[
d(Az, z) = d(Az, Bz) \\
\leq f(d(Sz, Tz), d(Sz, Az), d(Tz, Bz)) \\
= f(d(z, Az), 0, 0) \\
\leq \gamma d(Az, z) \\
< d(Az, z),
\]

which is a contradiction, so \( Az = z \). Therefore,

\[ Az = Bz = Sz = Tz = z, \]

that is, \( z \) is a common fixed point of \( A, B, S \) and \( T \). The
uniqueness of the common fixed point \( z \) follows easily from (3.9). Similarly, we can prove the theorem when \( S \) is \( w \)-continuous. This completes the proof.

**COROLLARY 3.1.** Let \( A, B, S \) and \( T \) be mappings from a semi-Hausdorff \( d \)-complete topological space \( (X, t) \) into itself satisfying the conditions (3.1), (3.3), (3.10) and following:

(3.14) there exists an \( \alpha \in (0,1) \) such that

\[
d(Ax, By) \leq \alpha m(x, y)
\]

for all \( x, y \) in \( X \).

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**PROOF.** If we define a mapping \( f : [0, \infty)^3 \to [0, \infty) \) by

\[
f(t_1, t_2, t_3) = \alpha \max \{t_1, t_2, t_3\},
\]

then \( f \in \mathcal{F} \) and so, by Theorem 3.3, this corollary follows.

Let now \( \Phi \) denote the family of all functions, \( \phi : [0, \infty) \to [0, \infty) \) which is non-decreasing, upper semi-continuous from the right with \( \phi(0) = 0 \),

\[
\phi(t) < t \text{ and } \sum_{n=1}^{\infty} \phi^n(t) < \infty \text{ for each } t > 0.
\]

**COROLLARY 3.2.** Let \( A, B, S \) and \( T \) be mappings from a semi-Hausdorff \( d \)-complete topological space \( (X, t) \) into itself satisfying the conditions (3.1), (3.3), (3.10) and following:

(3.15) \( d(Ax, By) \leq \phi(m(x, y)) \)

for all \( x, y \) in \( X \), where \( \phi \in \Phi \).

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).
PROOF. If we define a mappings \( f : [0, \omega)^3 \rightarrow [0, \omega) \) by

\[
f(t_1, t_2, t_3) = \phi \left( \max \{t_1, t_2, t_3\} \right),
\]

then \( f \in \mathcal{Y} \) and so, by Theorem 3.3, this corollary follows.

REMARK 3.1. If we replace the condition (3.3) by \( ASx = SAx \) and \( BTx = TBx \) for all \( x \in X \) in Corollary 3.1, then we get the original theorem of this type, proved by Fisher [32] for a complete metric space, in this new setting.

REMARK 3.2. All corollaries and Theorem 2 of [32] can be formulated with respect to the stronger condition (3.9) and the uniformly intimate concept in this new setting.

REMARK 3.3. It can be verified by the examples in [32] to see that the condition (3.14), the hypothesis of uniformly intimate pair and the condition that one of mappings \( S \) and \( T \) is \( w \)-continuous can not be omitted from Theorem 3.3.

REMARK 3.4. If we let \( m(x, y) = d(Sx, Ty) \) in Corollary 3.2, and \( S \) and \( T \) are surjective, then we obtain a result under the uniformly intimate concept in this new setting. The original theorem of this type was proved by Kang and Rhoades [59] under the compatibility concept in a complete metric space.
APPLICATIONS

Fixed points for expansion mappings:

Let \((X, t)\) be a semi Hausdorff topological space, \(N\) the set of all positive integers and \(\psi\) the family of all functions \(\psi: [0,\infty)^3 \to [0,\infty)\) satisfying the following properties:

\((\psi-1)\) \(\psi\) is continuous in \(([0, \infty))^3\)

\((\psi-2)\) \(\psi(1,1,1) = h > 1\) where \(h \in [0, \infty)\)

\((\psi-3)\) let \(\alpha, \beta \in [0, \infty)\) be such that

\((\psi-4)\) \(\alpha = \psi(\beta, \beta, \alpha) = h \cdot \beta\)

\((\psi-5)\) \(\alpha = \psi(\beta, \alpha, \beta) = h \cdot \beta\)

\((\psi-6)\) \(\psi(\alpha, 0, 0) > \alpha\) for all \(\alpha \neq 0\).

**PROPOSITION 3.1.** Let \(A, B, S\) and \(T\) be mappings from semi-Hausdorff topological space \((X, t)\) into itself such that the pairs \(\{A, S\}\) and \(\{B, T\}\) are uniformly \(S\) and \(T\)-intimate respectively. Suppose that for all \(x, y\) in \(X\) and \(\psi \in \Psi\) such that:

\((3.16)\) \(d(Sx, Ty) \leq \psi(d(Ax, By), d(Ax, Sx), d(By, Ty))\).

If there exist \(u, v\) and \(z\) in \(X\) such that

\(Au = Su = Bv = Tv = z\), then \(Az = Bz = Sz = Tz = z\).

**PROOF.** Since the pair \(\{A, S\}\) is uniformly \(S\)-intimate and \(Au = Su = z\), then,

\(d(ASu, SSu) \leq d(Au, Su)\),

which implies that \(Az = Sz\). From \((3.16)\), we have

\(d(Sz, z) = d(Sz, Tv) \leq \psi(d(Az, Bv), d(Az, Sz), d(Bv, Tv))\)

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\[\psi(d(Sz, z), 0, 0) > d(Sz, z) \quad \text{by property (}\psi \text{-6)},\]

which is a contradiction, so that Sz = z. By symmetry we have Bz = T z = z.

**Theorem 3.4.** Let A, B, S and T be mappings from a semi-Hausdorff topological space \((X, t)\) into itself satisfying the conditions \((3.1), (3.3), (3.16)\) and the following:

\[(3.17) \quad S(X) \text{ is } d\text{-complete.} \]

Then A, B, S and T have a unique common fixed point in X.

**Proof.** For an arbitrary point \(x_0\) in X, we can choose a sequence \(\{y_n\}\) defined by \((3.5)\). Now from \((3.5)\) and \((3.16)\), we have

\[d_{2n} = d(Sx_{2n+2}, Tx_{2n+1}) = \psi(d(Ax_{2n+2}, Bx_{2n+1}), d(Ax_{2n+2}, Sx_{2n+2}), d(Bx_{2n+1}, Tx_{2n+1}))\]

\[= \psi(d_{2n+1}, d_{2n+1}, d_{2n})\]

\[\geq h \cdot d_{2n+1} \quad \text{by property (}\psi \text{-4).} \]

This implies

\[d_{2n+1} \leq 1/h \cdot d_{2n}.\]

Similarly, we can get

\[d_{2n+2} \leq 1/h d_{2n+1}.\]

So, in general, we have for \(d_0 > 0\)

\[d_n \leq 1/h \cdot d_{n-1} \leq \ldots \leq 1/h^n \cdot d_0.\]
for all $n \in N$. Since $h > 1$, this implies that $\lim_{n \to \infty} d_n = 0$.

It follows that $\sum_{n=1}^{\infty} d(y_n, y_{n+1})$ is convergent.

Since in addition, $S(X)$ is $d$-complete, $\{y_n\}$ converges to some $z$ in $S(X)$ and hence the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to $z$.

Let $Su = z$ for some $u$ in $X$.

Putting $x = u$ and $y = x_{2n+1}$ in (3.16), we obtain

$$d(Su, Tx_{2n+1}) \geq \psi(d(Au, Bx_{2n+1}), d(Au, Su), d(Bx_{2n+1}, Tx_{2n+1})).$$

Letting $n \to \infty$, we have

$$0 \geq \psi(d(Au, z), d(Au, z), 0) \geq h \cdot d(Au, z) \quad \text{by property (}\psi-4),$$

which implies $Au = z$.

Since $z = Au \in A(X) \subseteq T(X)$, there exists a point $v$ in $X$ such that $Au = Tv$. Again, replacing $x$ by $u$ and $y = v$ in (3.16), we obtain

$$0 = d(Su, Tv) \geq \psi(d(Au, Bv), d(Au, Su), d(Bv, Tv))$$

$$= \psi(d(z, Bv), 0, d(Bv, z))$$

$$\geq h \cdot d(Bv, z), \quad \text{by property (}\psi-5),$$

which implies $Bv = z$.

Therefore, we have $Au = Su = Bv = Tv = z$ and hence by Proposition 3.1, it follows that $z$ is a common fixed point of $A$, $B$, $S$ and $T$. 

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Let us suppose that there exists a second common fixed point \( w \) of \( A, B, S \) and \( T \). Then from relation (3.16), we have

\[
d(z, w) = d(Sz, Tw)
\]

\[\geq \psi(d(Az, Bw), d(Az, Sz), d(Bw, Tw))\]

\[= \psi(d(z, w), 0, 0)\]

\[> d(z, w) \quad \text{by property } (\psi-6),\]

which arises a contradiction.

Hence \( w \) cannot exits and \( z \) is a unique common fixed point of \( A, B, S \) and \( T \). This completes the proof.

**REMARK 3.5**. Theorem 3.4 improves and generalizes Theorem 1 of Popa [82] and Theorem 3 of Khan, Khan and Sessa [66], in semi-Hausdorff \( d \)-complete topological space under uniformly intimate condition.

**COROLLARY 3.3.** Let \( A, B, S \) and \( T \) be mappings from semi-Hausdorff topological space \((X, t)\) into itself such that the pairs \( \{A, S\} \) and \( \{B, T\} \) are uniformly \( S \) and \( T \)-intimate respectively and satisfying the conditions (3.1), (3.17) and following:

(3.19) there exists \( a, b, c \in [0, \infty) \) with \( a > 1, b < 1, c < 1 \) and \( a + b + c > 1 \) such that:

\[
d^r(Sx, Ty) \geq a \cdot d^r(Ax, By) + b \cdot d^r(Ax, Sx) + c \cdot d^r(By, Ty)
\]

for all \( x, y \) in \( X \), where \( r > 0 \).

Then \( A, B, S \) and \( T \) have unique common fixed point in \( X \).
PROOF. Let us further define the $\psi: (0, \infty)^3 \rightarrow [0, \infty)$ as follows:

$$\psi(t_1, t_2, t_3) = (a \cdot t_1^r + b \cdot t_2^r + c \cdot t_3^r)^{1/r}$$

Then, $\psi \in \Psi$, and thus, by Theorem 3.4, this corollary follows.

REMARK 3.6. It should be noted that Corollary 3.3 improves and generalizes Theorem 1 of Popa [83], in non-metric setting.

If we put $b = c = 0$ in Corollary 1, we obtain the following.

COROLLARY 3.4. Let $A, B, S$ and $T$ be mappings from semi-Hausdorff $d$-complete topological $(X,t)$ into itself such that the pairs $\{A, S\}$ and $\{B, T\}$ are uniformly $S$ and $T$-intimate respectively, satisfying the condition (3.1), (3.17) and the following:

(3.20) there exits a constant $\lambda \in [0, \infty)$ with $\lambda > 1$ such that

$$d(Sx, Ty) = \lambda \cdot d(Ax, By)$$

for all $x, y$ in $X$.

Then $A, B, S$ and $T$ a unique common fixed point in $X$.

REMARK 3.7. If we define $\psi$ as in proof of corollary 3.3, then the result obtained in this new setting, which improves and generalizes Theorem 2.4 of Pathak, Kang and Ryu [81]. The original theorem of this type was proved by these authors in a complete metric space.
THEOREM 3.5. Let \( A, B, S \) and \( T \) be mappings from semi-Hausdorff space \((X, t)\) into itself satisfying the conditions (3.1), (3.3), (3.17) and following:

\[ (3.21) \quad \phi(d(Sx, Ty)) \leq \max \{d(Ax, By), d(Ax, Sx), d(By, Ty)\} \]

for all \( x, y \) in \( X \), where \( \phi \in \Phi \).

Then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

PROOF. For an arbitrary point \( x_0 \in X \), we can choose a sequence \( \{y_n\} \) as defined by (3.5). Then, by (3.5) and (3.21), we have

\[ (3.22) \quad \phi(d_{2n}) = \phi(d(Sx_{2n+2}, Tx_{2n+1})) \]

\[ \leq \max \{d(Ax_{2n+2}, Bx_{2n+1}), d(Ax_{2n+2}, Sx_{2n+2}), \]

\[ d(Bx_{2n+1}, Tx_{2n+1})\} \]

\[ = \max \{d_{2n+1}, d_{2n}\}. \]

Now, suppose \( \max \{d_{2n+1}, d_{2n}\} = d_{2n} \), then from (3.22), we have:

\[ \phi(d_{2n}) \leq d_{2n} \]

which is a contradiction. Hence, we have:

\[ \max \{d_{2n+1}, d_{2n}\} = d_{2n+1}. \]

Therefore form (3.22), we have:

\[ d_{2n+1} = \phi(d_{2n}) \]

and similarly

\[ d_{2n+2} = \phi(d_{2n+1}) \]

So, in general for \( d_0 > 0 \), and \( n \in \mathbb{N} \),
\[
d_n \leq \phi(d_{n-1}) \leq \ldots \leq \phi^i(d_{n-i}) \leq \ldots \leq \phi^n(d_0).
\]

Since \( \sum_{n=1}^\infty \phi^n(t) \) is convergent for each \( t > 0 \), it follows that \( \sum_{n=1}^\infty d(y_n, y_{n+1}) \) is convergent. In addition since \( S(X) \) is \( d \)-complete, sequence \( \{y_n\} \) converges to some \( z \) in \( S(X) \) and hence the subsequences \( \{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n}\} \) and \( \{Tx_{2n+1}\} \) of \( \{y_n\} \) also converge to \( z \).

Let \( Su = z \) for some \( u \) in \( X \). Putting \( x = u \) and \( y = x_{2n+1} \) in inequality (3.21) and then, letting limits as \( n \to \infty \), we obtain:

\[
\phi(0) = 0 \geq d(Au, z)
\]

which implies \( Au = z \). Since in addition \( A(X) \subseteq T(X) \), there is a point \( v \) in \( X \) such that \( Au = Tv = z \). Again, replacing \( x \) by \( u \) and \( y \) by \( v \) in inequality (3.21), we obtain:

\[
\phi(0) = 0 = d(Su, Tv) \geq d(Bv, z)
\]

which means that \( Bv = z \).

Therefore, \( Au = Su = Bv = Tv = z \). Since, the pair \( \{A, S\} \) is uniformly \( S \)-intimate. Then

\[
d(ASu, SSu) \leq d(Au, Su)
\]

which implies that \( ASu = SSu \), that is \( Az = Sz \).

By property (3.21), we have:

\[
\phi(d(Sz, z)) = \phi(d(Sz, Tv))
\]

\[
\geq \max \{d(Az, Bv), d(Az, Sz), d(Bv, Tv)\}
\]

\[
= d(Sz, z)
\]

which is a contradiction, since for each \( t > 0, \phi(t) < t \).
Therefore, S\(z = z\), and by symmetry, B\(z = Tz = z\), which shows the existence of \(z\) as a common fixed point of A, B, S and T. Uniqueness of common fixed point is obvious.

**COROLLARY 3.5.** Let A, B, S and T be mappings from semi-Hausdorff \(d\)-complete topological \((X, t)\) into itself such that the pairs \(\{A, S\}\) and \(\{B, T\}\) are uniformly S and T-intimate respectively, satisfying (3.1), (3.17) and at least one of the following conditions:

for all \(x, y \in X\),

\[(3.23) \quad \phi(d(Sx, Ty)) \leq d(Ax, By)\]

\[(3.24) \quad \phi(d(Sx, Ty)) \leq 1/2 \left[d(Ax, By) + d(Ax, Sx)\right]\]

\[(3.25) \quad \phi(d(Sx, Ty)) \leq 1/2 \left[d(Ax, By) + d(By, Ty)\right]\]

\[(3.26) \quad \phi(d(Sx, Ty)) \leq 1/3 \left[d(Ax, By) + d(Ax, Sx) + d(By, Ty)\right].\]

Then A, B, S and T have a unique common fixed point in \(X\).

**EXAMPLE 3.1.** Let \(X = [0,1]\) with symmetric function \(d\) on \(X\) defined by \(d(x, y) = |x - y|\).

Let A, B, S, T : \(X \rightarrow X\) such that

\[Ax = Bx = \frac{x^2}{2} \quad \text{and} \quad Sx = Tx = x^2.\]

It is easy to verify that all the conditions of **Theorem 3.1** are satisfied and 0 is unique common fixed point of A, B, S and T.