CHAPTER 4

Discontinuity and common fixed points in Menger space

(40 - 57)
In this chapter, we have introduced the concept of compatible of type (A_f) (Compatible of type (A_g)) of two self mappings in non-Archimedean Menger space and prove some common fixed point theorems where self mappings are discontinuous at common fixed point. Our result improve and generalize Theorem 2 of Pant [78] and many known authors in this space.

Menger [72] introduced the notion of probabilistic metric spaces (or statistical metric spaces). The study of this space expanded rapidly with the pioneering works of Schweizer and Skalar [95], [96]. Especially the theory of probabilistic metric spaces is of fundamental importance in probabilistic functional analysis.

A vast literature exists as study of common fixed points for compatible mappings in metric space as well as in Menger space, ever since Jungck [53] introduced the notion of compatible mappings. However, the study of common fixed points of non-compatible, non-compatible of type (A_f) (non-compatible of type (A_g)), non-2-compatibility is also equally interesting. In this connection Pant ([78], [79]) initiated the study of common fixed points of non-compatible mappings satisfying contractive type conditions. On these lines, we propose in this chapter to investigate some fixed point theorems without
assuming continuity of the mappings at the common fixed point and completeness of the space.

**DEFINITION 4.1.** A distribution function is a mapping $\mathcal{F} : \mathbb{R} \to \mathbb{R}^+$ which is non-decreasing and left continuous with $\inf \mathcal{F} = 0$ and $\sup \mathcal{F} = 1$. We will denote $\mathcal{D}$ by the set of all distribution functions.

**DEFINITION 4.2.** A probabilistic metric space is an ordered pair $(X, \mathcal{F})$, where $X$ is an abstract set and $\mathcal{F}$ is a mapping of $X \times X$ into $\mathcal{D}$ i.e., $\mathcal{F}$ associates a distribution function $\mathcal{F}(p, q)$ with every pair $(p, q)$ of points in $X$. We shall denote the distribution function $\mathcal{F}(p, q)$ by $F_{p, q}$. The function $F_{p, q}$ are assumed to satisfy following conditions:

1. $F_{p, q}(x) = 1$ for all $x > 0$ if and only if $p = q$.
2. $F_{p, q}(0) = 0$.
3. $F_{p, q} = F_{q, p}$.
4. If $F_{p, q}(x) = 1$ and $F_{q, r}(y) = 1$, then $F_{p, r}(x+y) = 1$.

**DEFINITION 4.3.** A triangular norm (briefly, a t-norm) is a function $\Delta : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions.

1. $\Delta(c,d) \geq \Delta(a,b)$ for $c \geq a$, $d \geq b$.
2. $\Delta(a,b) = \Delta(b,a)$.
3. $\Delta(a,1) = a$ and $\Delta(0,0) = 0$.
4. $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$.
DEFINITION 4.4. A Menger PM-space is triplet \((X, \mathcal{F}, \Delta)\) where \((X, \mathcal{F})\) is a PM-space and \(\Delta\) satisfies
\[
P_{p,r}(x+y) = \Delta(p,q(x), q,r(y)),
\]
for all \(x, y \geq 0\) and \(p, q, r \in X\).

DEFINITION 4.5. A continuous \(\Delta\) is said to be Archimedean if \(\Delta(t,t) < t\), \(t \in (0,1)\).

DEFINITION 4.6. \((X, \mathcal{F}, \Delta)\) is called a non-Archimedean Menger PM-space (Shortley, a N.A. Menger PM-space) if \((X, \mathcal{F}, \Delta)\) is a Menger PM-space and \(\Delta\) satisfies the following conditions,
\[
P_{x,z}(\max \{t_1, t_2\}) = \Delta(p_{x,y}(t_1), p_{y,z}(t_2)),
\]
for all \(x, y, z \in X\) and \(t_1, t_2 \geq 0\).

DEFINITION 4.7. A PM-space \((X, \mathcal{F})\) is said to be of type \((c)_s\) if there exists a \(s \in \Omega\) such that
\[
s(F_{x,y}(t)) < s(F_{x,z}(t)) + s(F_{z,y}(t))
\]
for all \(x, y, z \in X\) and \(t \geq 0\), where \(\Omega = \{s : [0,1] \to (0,\infty)\}\) is continuous, strictly decreasing, \(s(1) = 0\) and \(s(0) < \infty\).

DEFINITION 4.8. A non-Archimedean Menger PM-space \((X, \mathcal{F}, \Delta)\) is said to be of type \((D)_s\) if there exists a \(s \in \Omega\) such that
\[
s(\Delta(t_1, t_2)) \leq s(t_1) + s(t_2)
\]
for all \(t_1, t_2 \in [0,1]\).

REMARK 4.1. (a). If N.A. PM-space \((X, \mathcal{F}, \Delta)\) is of type \((D)_s\), then \((X, \mathcal{F}, \Delta)\) is of type \((C)_s\).

(b) Throughout this chapter, let \((X, \mathcal{F}, \Delta)\) be a N.A. Menger PM-space of type \((D)_s\) with a continuous strictly increasing t-norm \(\Delta\).
DEFINITION 4.9. Let \((X,\mathcal{F},\Delta)\) be a N.A. Menger PM-space and assume that \(f, g : X \to X\) be the mappings and \(\{x_n\}\) be a sequence in \(X\) such that

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = t \text{ for some } t \text{ in } X.
\]

Then \(f\) and \(g\) are said to be

(4.1) [53] **Compatible**, if

\[
\lim_{n \to \infty} s(F_{fg x_n, g f x_n}(t)) = 0, \text{ for all } t > 0.
\]

(4.2) **Non-compatible**, if

\[
\lim_{n \to \infty} s(F_{fg x_n, g f x_n}(t)) \text{ is either nonzero or nonexistent, for all } t > 0.
\]

(4.3) [76] **Compatible of type (A)**, if

\[
\lim_{n \to \infty} s(F_{fg x_n, g g x_n}(t)) = 0, \lim_{n \to \infty} s(F_{g f x_n, f f x_n}(t)) = 0,
\]

for all \(t > 0\).

(4.4) [101] **2-compatible**, if

\[
\lim_{n \to \infty} s(F_{f f x_n, g g x_n}(t)) = 0, \text{ for all } t > 0
\]

(4.5) **Non-2-compatible**, if

\[
\lim_{n \to \infty} s(F_{f f x_n, g g x_n}(t)) \text{ is either nonzero or nonexistent, for all } t > 0.
\]
DEFINITION 4.10 [102]. Let \( f, g : X \rightarrow X \) be the mappings, \( f \) and \( g \) are said to be D-compatible if \( f z = g z \), for some \( z \in X \), implies \( fgz = gfz \).

DEFINITION 4.11. Two self mappings \( f \) and \( g \) of non-Archimedean Menger space \( X \) are said to be compatible of type \( (A_f) \) (compatible of type \( (A_g) \), if

\[
\lim_{n \to \infty} s(F f x_n, g f x_n (t)) = 0 \quad \lim_{n \to \infty} s(F g x_n, f g x_n (t)) = 0,
\]

for all \( t > 0 \), wherever \( \{x_n\} \) is a sequence in \( X \) such that

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z \text{ for some } z \in X.
\]

EXAMPLE 4.1. Let \( X [0,2] \), we define

\[
s(F x, y(t)) = |x - y| \text{ for all } t > 0 \text{ and } x, y \in X.
\]

Let \( f, g : [0,2] \rightarrow [0,2] \) such that

\[
f(x) = \begin{cases} 
  x & \text{if } x \in [0,1) \\
  2 & \text{if } x \in [1,2]
\end{cases}
\]

\[
g(x) = \begin{cases} 
  2-x & \text{if } x \in [0,1) \\
  2 & \text{if } x \in [1,2]
\end{cases}
\]

Clearly \( f \) and \( g \) are not continuous at \( x = 1 \). Now consider a sequence \( \{x_n\} \subset [0,2] \) such that \( x_n \rightarrow 1 \) for all \( n \in \mathbb{N} \) and \( x_n \rightarrow 1 \). Then \( f x_n = x_n \rightarrow 1 \) from the left and \( g x_n = 2-x_n \rightarrow 1 \) from the right.
Also,

$$fg(x_n) = f(gx_n) = f(2-x_n) = 2$$

$$fg(x_n) = g(fx_n) = gx_n = 2-x_n$$

and

$$ffx_n = fx_n = x_n$$

$$ggx_n = g(2-x_n) = 2.$$  

Therefore,

$$\lim_{n \to \infty} s(Ffgx_n, gfx_n(t)) = \lim_{n \to \infty} |2-2+x_n| = 1$$

$$\lim_{n \to \infty} s(Fffx_n, ggx_n(t)) = \lim_{n \to \infty} |x_n-2| = 1$$

$$\lim_{n \to \infty} s(Fgfx_n, ffx_n(t)) = \lim_{n \to \infty} |2-x_n-x_n| = 0$$

$$\lim_{n \to \infty} s(Fgfx_n, ggx_n(t)) = \lim_{n \to \infty} |2-2| = 0$$

which show that the pair of maps $f$ and $g$ are neither compatible, nor 2-compatible. But $f$ and $g$ are compatible of type $(A_f)$, compatible of type $(A_g)$ and therefore compatible of type $(A)$. Also $f$ and $g$ are D-compatible.

REMARK 4.2. Murthy, et.al. [76] assumed both the conditions, compatible of type $(A_f)$ and compatible of type $(A_g)$ in the invention of compatible of type $(A)$, which is not necessary for the computation of fixed points.
DEFINITION 4.12. Let $R^+$ be the set of non-negative real numbers and $\Phi(\cdot)$ be the family of all mappings, $\phi:(R^+)^5 \rightarrow R^+$ satisfying the following conditions:

(i) $\phi$ is non-decreasing and upper semi-continuous in each coordinate variable.

(ii) for each $t > 0$, $\phi(t, t, t, t, t) = \psi(t)$.

where $\psi : R^+ \rightarrow R^+$ is a mapping such that $\psi(0) = 0$ and $\psi(t) < t$ for all $t > 0$. If $t \leq \psi(t)$ then $t = 0$.

LEMMA 4.1. Let $f$ and $g$ be two self mappings of non-Archimedean Menger space $X$, we have the followings:

(4.6) If $f$ and $g$ are both compatible of type $(A_f)$ and compatible of type $(A_g)$, then $f$ and $g$ are compatible of type $(A)$.

(4.7) (i) Compatibility and compatibility of type $(A)$ imply 2-compatibility,

(ii) 2-compatibility and compatibility of type $(A)$ imply compatibility.

(iii) Compatibility, 2-compatibility and compatibility of type $(A_f)$ (compatibility of type $(A_g)$) imply compatibility of type $(A_g)$ (compatibility of type $(A_f)$).

\[ \begin{align*}
\text{compatibility} & \quad \rightarrow \quad \text{compatibility of type $(A_f)$} \\
2\text{-compatibility} & \quad \rightarrow \quad \text{compatibility of type $(A_g)$} \\
\end{align*} \]

\[ \quad \rightarrow \quad \text{D-compatibility}. \]

Above lemma's can be shown by simple calculations.
LEMMA 4.2. Let $f$ and $g$ be two self mappings of non-Archimedean Menger space $X$, we have the followings:

(4.9) continuity of $f(g)$ and compatibility of type $(A_g)$ (compatibility of type $(A_f)$) of $f$ and $g$ imply continuity of $f$ and $g$.

(4.9 a) continuity of $f(g)$ and compatibility of type $(A_f)$ (compatibility of type $(A_g)$) of $f$ and $g$ imply 2-compatibility of $f$ and $g$.

(4.10) Continuity of $f(g)$ and compatibility imply compatibility of type $(A_g)$ (compatibility of type $(A_f)$) of $f$ and $g$.

(4.11) Continuity of $f(g)$ and 2-compatibility of $f$ and $g$ imply compatibility of type $(A_f)$ (compatibility of type $(A_g)$) of $f$ and $g$.

PROOF.

(4.9) Suppose that $f$ and $g$ are compatible of type $(A_g)$. Let $\{x_n\}$ be a sequence in $X$ such that

$$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z \text{ for some } z \in X.$$ 

Since $f$ is continuous, $f g x_n, f f x_n \to f z$ as $n \to \infty$ and so we have

$$s(F f g x_n, g f x_n(t)) \leq s(F g f x_n, f f x_n(t)) + s(F f f x_n, f g x_n(t)) \to 0$$

for all $t > 0$, as $n \to \infty$. Therefore $f$ and $g$ are compatible.

Similarly, continuity of $g$ and compatibility of type $(A_f)$ imply compatibility of $f$ and $g$. 

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(4.9a) Suppose that \( f \) and \( g \) are compatible of type \((A_f)\) and \( f \) is continuous. Then,
\[
s(F_{ffx_n,ggx_n}(t)) + s(F_{fgx_n,ffx_n}(t)) + s(F_{fgx_n,ggx_n}(t)) > 0
\]
for all \( t > 0 \), as \( n \to \infty \). Therefore \( f \) and \( g \) are 2-compatible.

(4.10) Suppose \( f \) is continuous and \( f \) and \( g \) are compatible, then
\[
s(F_{fgx_n,ffx_n}(t)) \leq s(F_{fgx_n,fgx_n}(t)) + s(F_{fgx_n,ffx_n}(t)) \to 0
\]
for all \( t > 0 \), as \( n \to \infty \).
Therefore \( f \) and \( g \) are compatible of type \((A_g)\).

(4.11) Suppose that \( f \) and \( g \) are 2-compatible and \( f \) is continuous, then
\[
s(F_{fgx_n,ggx_n}(t)) \leq s(F_{fgx_n,fgx_n}(t)) + s(F_{fgx_n,ggx_n}(t)) \to 0
\]
for all \( t > 0 \), as \( n \to \infty \).
Therefore \( f \) and \( g \) are compatible of type \((A_f)\).

**Lemma 4.3.** Let \( f \) and \( g \) be two self-mappings of non-Archimedean Menger space \( X \) such that

(4.12) \( f(X) \subset g(X) \)

(4.13) \( s(F_{fx, fy}(t)) \leq \phi(s(F_{gx, gy}(t)), s(F_{fx, qx}(t)),
\quad s(F_{fy, gy}(t)), s(F_{fx, gy}(t)),
\quad s(F_{fy, gx}(t))) \),

for all \( t > 0 \) and \( x, y \in X \), where \( \phi \in \Phi(5) \).
If \( f \) and \( g \) are D-compatible and there is a sequence \( \{x_n\} \) in \( X \)
such that
\[ \lim_{n \to \infty} x_n = \lim_{n \to \infty} g x_n = z \]
for some \( z \in X \). Then \( z \) is unique common fixed point of \( f \) and \( g \).

**PROOF.** Since \( z \in f(X) \) and \( f(X) \subseteq g(X) \), then there exists \( u \in X \) such that \( gu = z \). Using (4.13), we get

\[ s(F_{fu}, fx_n(t)) \leq \phi(s(F_{gu}, gx_n(t)), s(F_{fu}, gu(t)), s(F_{fx_n}, gx_n(t)), s(F_{fu}, gx_n(t)), s(F_{fx_n}, gu(t))). \]

Letting \( n \to \infty \), we have

\[ s(F_{fu}, z(t)) \leq \phi(0, 0, s(F_{fu}, z(t)), 0, s(F_{fu}, z(t)), 0). \]

Which is a contradiction. Thus \( s(F_{fu}, z(t)) = 0 \). Therefore, \( fu = z \). Hence \( u \) is a coincidence point of \( f \) and \( g \). That is \( fu = gu = z \). By D-compatibility of \( f \) and \( g \), we have

\[ fz = fg u = gf u = gz. \]

If \( fz \neq z \), using (4.13), we get

\[ s(F_{fz}, fu(t)) \leq \phi(s(F_{gz}, gu(t)), s(F_{fz}, gz(t)), s(F_{fu}, gu(t)), s(F_{fz}, gu(t)), s(F_{fu}, gz(t))). \]

which implies

\[ s(F_{fz}, z(t)) \leq \phi(s(F_{fz}, z(t)), 0, 0, s(F_{fz}, z(t)), s(F_{fz}, z(t))). \]

Which is a contradiction. Thus \( fz = z \). Hence \( z \) is a common fixed point of \( f \) and \( g \). Unicity of \( z \) easily follows from (4.13).
EXISTENCE OF COMMON FIXED POINT FOR NON-COMMUTING MAPPINGS:

**THEOREM 4.1.** Let $f$ and $g$ be non-compatible self-mappings of non-Archimedean Menger space $X$ satisfying the conditions $(4.12)$, $(4.13)$ and the following:

$$(4.14) \quad f \text{ and } g \text{ are compatible of type } (A_g).$$

Then $f$ and $g$ have a unique common fixed point and the fixed point is a point of discontinuity.

**PROOF.** Non-compatibility of $f$ and $g$ implies that there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z, \text{ for some } z \in X.$$ 

But $\lim_{n \to \infty} s(F_{fgx_n, gfx_n}(z))$ is either nonzero or non-existent for all $t > 0$. Since $f$ and $g$ are compatible of type $(A_g)$, which implies by $(4.8)$, $f$ and $g$ are D-compatible. Therefore by Lemma 4.3, $z$ is unique common fixed point of $f$ and $g$.

Now, suppose $f$ is continuous and since $f$, $g$ are compatible of type $(A_g)$, then by $(4.9)$ $f$ and $g$ are compatible which is a contradiction of our assumption that $f$ and $g$ are non-compatible. Hence $f$ is discontinuous at the point $z$.

Next, suppose that $g$ is continuous. Then for the sequence $\{x_n\}$, we get $\lim_{n \to \infty} gfx_n = gz = z$ and $\lim_{n \to \infty} ggx_n = gz = z$. In view of these limits, we have from $(4.13)$.
Letting $n \to \infty$, we have

\[ s(F_{fz}, fg_n(t)) \leq \phi(s(F_{gz}, gg_n(t)), s(F_{fz}, gz(t)), s(F_{fg_n}, zg(t))), \]

\[ s(F_{fg_n}, gz(t)) \leq \psi(s(F_{fg_n}, zg(t))), \]

\[ < s(F_{fg_n}, z(t)), \]

which yields a contradiction unless $\lim_{n \to \infty} fg_n = fz = gz = z$.

But $\lim_{n \to \infty} fg_n = gz$ and $\lim_{n \to \infty} gfx_n = gz$, contradicts the fact that $\lim_{n \to \infty} s(F_{fg_n}, gfx_n(t))$ is either nonzero or non-existent for all $t > 0$. Thus, both $f$ and $g$ are discontinuous at their common fixed point.

**COROLLARY 4.1.** Let $f$ and $g$ be non-compatible self mappings satisfying conditions (4.12) and (4.14) and the following:

\[ s(F_{fz}, fy(t)) \leq \max \{s(F_{gx}, gy(t)), \]

\[ 2^{-1}[s(F_{fx}, gx(t)) + s(F_{fy}, gy(t))], \]

\[ 2^{-1}[s(F_{fx}, gy(t)) + s(F_{fy}, gx(t))]. \]

Then $f$ and $g$ have a unique common fixed point and the fixed point is a point of discontinuity.
PROOF. If we define \( \phi(R^+)^5 \rightarrow R^+ \) such that
\[
\phi(t_1, t_2, t_3, t_4, t_5) = \max \{ t_1, \frac{t_2 + t_3}{2}, \frac{t_4 + t_5}{2} \}
\]
where \( t_1, t_2, t_3, t_4, t_5 \in R^+ \).
Thus, the proof now follows on similar lines as in Theorem 4.1.

REMARK 4.3. Corollary 4.1 is an improvement of the Theorem 2 of Pant [76].

THEOREM 4.2. Let \( f \) and \( g \) be non 2-compatible self-mappings of non-Archimedean Menger space \( X \) satisfying the conditions (4.12), (4.13) and following:
(4.14a) \( f \) and \( g \) are compatible of type \( (A_f) \).
Then \( f \) and \( g \) have a unique common fixed point and the fixed point is a point of discontinuity.

PROOF. Non-2-compatibility of \( f \) and \( g \) implies that there exists a sequence \( \{x_n\} \) in \( X \) such that
\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z, \text{ for some } z \text{ in } X.
\]
But \( \lim_{n \to \infty} s(F_{ffx_n}^n, gx_n(t)) \) is either nonzero or non-existent.
Since \( f \) and \( g \) are compatible of type \( (A_f) \). Thus (4.8) implies that \( f \) and \( g \) are D-compatible. Therefore by Lemma 4.3, \( z \) is unique common fixed point of \( f \) and \( g \).

Now, suppose \( f \) is continuous and since \( f \) and \( g \) are compatible of type \( (A_f) \), then by (4.9a) \( f \) and \( g \) are 2-compatible which is a contradiction of non-2-compatibility of \( f \) and \( g \).
Hence $f$ is discontinuous at the fixed point.

Next, suppose that $g$ is continuous then for the sequence $\{x_n\}$, we get

$$\lim_{n \to \infty} gfx_n = \lim_{n \to \infty} ggx_n = gz = z.$$ In view of these limits, we have from (4.13)

$$s(F, f) = s(F, f) = s(F, f) = s(F, f) = s(F, f).$$

Letting $n \to \infty$, we have

$$s(F, f) = s(F, f) = s(F, f) = s(F, f) = s(F, f),$$

which is a contradiction unless $\lim_{n \to \infty} ffx_n = fz = z$. But $\lim_{n \to \infty} ffx_n = fz = z$ and $\lim_{n \to \infty} ggx_n = gz = z$, contradicts the fact that $\lim_{n \to \infty} s(F, f) = gz = z$, for all $t > 0$, is either nonzero or non-existent. Thus, both $f$ and $g$ are discontinuous at the common fixed point.

**Theorem 4.3.** Let $f$ and $g$ be, non-compatible of type $(A_g)$, self-mappings of non-Archimedean Menger space $X$ satisfying the condition (4.12) and followings:

(4.15) $f$ and $g$ are compatible,
\{4.16\} \quad s(F_{fx, fy}(t)) = \phi_1(s(F_{gx, gy}(t)), s(F_{fx, gy}(t));
\quad s(F_{fy, gx}(t)))

for all \( t > 0 \) and \( x, y \in X \). Where \( \phi_1 \in \Phi(3) \).

Then \( f \) and \( g \) have a unique common fixed point and the fixed point is a point of discontinuity.

PROOF. Non-compatibility of type \((A_g)\) of \( f \) and \( g \) implies that there exists a sequence \( \{x_n\} \) in \( X \) such that

\[(4.17) \quad \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = z, \text{ for some } z \in X,\]

but \( \lim_{n \to \infty} s(F_{gx_n, fx_n}(t)) \) is either nonzero or non-existent.

Since \( f \) and \( g \) are compatible, then by \((4.8)\), \( f \) and \( g \) are D-compatible. If we define

\( \phi(t_1, t_2, t_3, t_4, t_5) = \phi_1(t_1, t_4, t_5), \) for all \( t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}^+ \),

then it follows from Lemma 4.3 that \( z \) is unique common fixed point of \( f \) and \( g \).

We now claim that \( f \) and \( g \) are discontinuous at the common fixed point \( z \). If possible, suppose \( f \) is continuous and since \( f \) and \( g \) are compatible. Then by \((4.10)\), \( f \) and \( g \) are compatible of type \((A_g)\), which contradicts our assumption that \( f \) and \( g \) are non-compatible of type \((A_g)\). Hence \( f \) is discontinuous at the point \( z \).

Next, suppose that \( g \) is continuous, then for the sequence \( \{x_n\} \) of \((4.17)\), we get

\[ \lim_{n \to \infty} g x_n = \lim_{n \to \infty} g g x_n = gz = z. \]
In view of these limits, using (4.16), we get

\[ s(F_{z, ffx_{n}}(t)) = s(F_{z, gfx_{n}}(t)) \]

\[ \leq \phi_{1}(s(F_{z, gfx_{n}}(t)), s(F_{z, gfx_{n}}(t))) \]

\[ \leq \psi(s(F_{z, gfx_{n}}(t))) \]

\[ < s(F_{z, gfx_{n}}(t)) \]

letting \( n \to \infty \), we have

\[ s(F_{z, ffx_{n}}(t)) < \psi(s(F_{z, gfx_{n}}(t))) \]

which yields a contradiction unless \( \lim_{n \to \infty} F_{z, ffx_{n}}(t) = 1 \) for all \( t > 0 \), that is \( \lim_{n \to \infty} ffx_{n} = z \). Hence

\[ \lim_{n \to \infty} gfx_{n} = \lim_{n \to \infty} ffx_{n} = z. \]

Which is a contradiction of non-compatibility of type \((A_{g})\).

Thus, both \( f \) and \( g \) are discontinuous at the point \( z \).

**Theorem 4.4.** Let \( f \) and \( g \) be non-compatible of type \((A_{f})\), self-mappings of non-Archimedean Menger space \( X \) satisfying the conditions (4.12), (4.13) and the following:

\[(4.18) \quad f \text{ and } g \text{ are 2-compatible.}\]

Then \( f \) and \( g \) have a unique common fixed point and the fixed point is a point of discontinuity.
PROOF. Non-compatibility of type \((A_f)\) of \(f\) and \(g\) implies that there exists a sequence \(\{x_n\}\) in \(X\) such that
\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = z, \text{ for some } z \in X.
\]
But \(\lim_{n \to \infty} s(F_{f(x_n),g(x_n)}(z))\) is either nonzero or non-existent for all \(t > 0\). Since \(f\) and \(g\) are 2-compatible which implies by (4.8), \(f\) and \(g\) are D-compatible. Therefore by Lemma 4.3, \(z\) is unique common fixed point of \(f\) and \(g\).

Now, suppose \(f\) is continuous and since \(f, g\) are 2-compatible, then by (4.11) \(f\) and \(g\) are compatible of type \((A_f)\) which is a contradiction of our assumption that \(f\) and \(g\) are non-compatible of type \((A_f)\). Hence \(f\) is discontinuous at the point \(z\).

Rest of the proof follows from Theorem 4.1, which gives \(\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n)\). Thus we obtain a contradiction of the fact that \(\lim_{n \to \infty} s(F_{f(x_n),g(x_n)}(t))\) is either nonzero or non-existent for all \(t > 0\). Hence, both \(f\) and \(g\) are discontinuous at their common fixed point.

REMARK 4.3. All the above results are also true in metric space. These can be prove in similar fashion.

EXAMPLE 4.2. Let \(X = [2,20]\) and we define
\[s(F_{X,Y}(t)) = |x-y| \text{ for all } t > 0 \text{ and } x,y \in X.\]
Let \(f,g : X \to X\) such that
\[
f(x) = \begin{cases} 
2 & \text{if } x = 2 \text{ or } x > 5 \\
6 & \text{if } 2 < x \leq 5 
\end{cases}
\]
Then \( f \) and \( g \) satisfy all the conditions of Theorem 4.1 and have a unique common fixed point at \( x = 2 \).

Since \( f(X) = \{2\} \cup \{6\} \) and \( g(X) = [2,7] \cup \{12\} \).

Therefore \( fX \subset gX \).

Let us consider the sequence \( \{x_n\} \) in \( X \) such that

\[
x_n = 5 + \frac{1}{n}
\]

for all \( n \geq 1 \).

Then obviously

\[
\lim_{n \to \infty} fx_n = 2 = \lim_{n \to \infty} gx_n
\]

\[
\lim_{n \to \infty} fgx_n = 6, \quad \lim_{n \to \infty} gfX_n = 2
\]

\[
\lim_{n \to \infty} ffx_n = 2
\]

which imply

\[
\lim_{n \to \infty} s(Ffgx_n,gfx_n (t)) \neq 0 \text{ for all } t > 0.
\]

and

\[
\lim_{n \to \infty} s(Fgfx_n,ffx_n (t)) = 0 \text{ for all } t > 0.
\]

Hence, \( f \) and \( g \) are non-compatible but compatible of type \( (Ag) \).

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