CHAPTER IV

WAVES IN PLASMA
Plasma is held in equilibrium by electric and magnetic fields. It can be supposed to be a mixture of two compressible fluids of opposite sign if treated macroscopically. Then the plasma can be treated by hydrodynamic equations of force and continuity for a charged fluid moving in electric and magnetic fields,\(^1\)\(^-\)\(^3\) while the neutral gas provides the frictional force. If plasma is considered microscopically, three types of particles, positive, negative and neutrals, moving at random and colliding with each other are to be considered. The electric field gives a small drift velocity superimposed upon the random velocity of the charges. Energy is derived from the electric field and it is possible that the three groups of particles exist in equilibrium at three different temperatures \(T_1\), \(T_e\) and \(T_n\).

A complete equilibrium is assumed in the above description which is rarely the case. There are local/superimposed departures from equilibrium conditions, for example, a momentary separation of the charges in a small volume of plasma gives rise to mutual electric field of the charges in the displaced volume producing a restoring force which will produce an oscillatory motion of the charges which may
propagates in the ionized medium as a wave. A complete
description of the various possible modes of oscillations and
waves that may propagate under a given set of conditions is
very difficult. However, a number of idealised models have
been devised and with the help of these a picture can be built
up of what can be expected when an ionized gas is subjected to
certain perturbation/excitation. A number of experimental
conditions are used in producing a plasma. This further
complicates the issue and one has to examine the situation for
specific configuration and set of conditions.

It has been emphasised that in the study of waves and
oscillations in plasma one has to begin with certain assumed
equilibrium situation. It means that the plasma under
consideration has known average of properties like charge
density, percentage ionization, temperature and magnetic field
etc. Then the behaviour of small perturbation in this
equilibrium as a function of space and time has to be studied.
This requires solutions of Maxwell's equations along with
Boltzmann's equations. In the following pages an attempt has
been made to present the propagation of waves and oscillations
in a purely physical way. Difficult mathematics has been
avoided and stress is given to physical concepts. The author
is fully aware of the inadequacy of the treatment, however,
the details may be found in excellent review articles on the
topic.\textsuperscript{4,5}

Considering infinitesimal disturbances in homogeneous
media under relatively simple condition, extensive theoretical
study of three particular types of waves in plasma has been carried out. These are 'electrostatic', 'electromagnetic' and 'hydromagnetic' waves. However, an ionized gas is capable of a wide variety of oscillatory motions which are exceedingly complex.

4.2 PLASMA OSCILLATIONS IN THE ABSENCE OF A MAGNETIC FIELD

In general, under any given set of conditions, there are four modes of infinitesimal wave propagation at each frequency, although the phase velocity of some modes may be imaginary. The absence of an external magnetic field renders the situation quite simple. Then the modes are of two types, 'electromagnetic' and 'electrostatic'.

In the electromagnetic waves, the electric vector \( \mathbf{E} \) is perpendicular to the direction of wave propagation. Corresponding to the two directions of polarization there are two electromagnetic modes. These are transverse waves. Electrons in the plasma interfere with these waves and increase their wave velocity. At a certain critical frequency called 'plasma frequency' (which increases with density), the phase velocity becomes imaginary. Then these waves cannot propagate through plasma in the absence of a magnetic field.

The other two modes are of electrostatic type. In this case the electric vector \( \mathbf{E} \) and the current density vector \( \mathbf{j} \) are parallel to the direction of wave propagation. These two modes are 'electron plasma waves' and 'ion plasma waves'. In the electron plasma waves the electrons and
positive ions generally move together. The inertia of the positive ions determines the wave velocity which is normally less than the velocity of electron plasma waves.

These longitudinal electrostatic oscillations of plasma are associated with a small momentary displacement of the charges of one sign relative to the other. Such a displacement will result into an electric field in such a direction as to oppose the displacement. These waves were first detected and observed by Tonks and Langmuir. The frequency of these oscillations is determined by the mass of the displaced charges and the 'spring constant' of the electric field produced by the displacement. Frequency of ion oscillations are much less than the electron plasma frequency and the ion plasma oscillations are not easy to identify. In recent years considerable theoretical work has been done on plasma oscillations. A brief but excellent account of the history of plasma oscillations is given by Drummond.

The electron plasma frequency appears in the descriptions of several phenomena in a plasma. Disregarding the effect of thermal velocities, electron plasma frequency is given by:

\[ \omega_{pe} = \frac{4 \pi n_e e^2}{m} \]  \hspace{1cm} (4.1)

or

\[ f_{pe} \approx 9 \times 10^3 (n_e)^{\frac{1}{2}} \text{ cps.} \]  \hspace{1cm} (4.2)

Equation (4.1) can be derived as follows:
Let us consider a region of uniform density having \( n_e \) electrons and ions bounded between two imaginary planes perpendicular to \( x \)-axis. Let \( \xi \) be the small displacement of each electron (in the \( x \)-direction only), this displacement being zero at the two boundary planes.

The corresponding change in density \( \delta n_e \) at some point \( x \) is then given by,
\[
\delta n_e = n_e \frac{d \xi}{dx} \quad \ldots (4.3)
\]

Also from Poisson's equation for the field \( \phi \) we have,
\[
\frac{d \phi}{dx} = 4\pi n_e e \frac{d \xi}{dx} \quad \ldots (4.4)
\]

Here we have ignored the arbitrary constant, which would represent a study applied field, because \( x \) is the electric field resulting only from the displacement \( \xi \).

The force on each electron being \( -e \phi = m \ddot{\xi} \), the following equation of simple harmonic motion is obtained:
\[
m \ddot{\xi} + 4\pi n_e e^2 = 0 \quad \ldots (4.5)
\]

Eqn (4.5) provides the angular frequency of electron plasma oscillations i.e.,
\[
\omega_{pe} = \frac{4\pi n_e e^2}{m}
\]

Similarly the ion plasma frequency is given by,
\[
\omega_{pi} = \frac{4\pi n_i e^2}{M} \quad \ldots (4.6)
\]

where \( M \) is the mass of the ion.
The ion and electron oscillations are pure electrostatic effects. The field and the current fluctuations occur in the direction of propagation and not perpendicular as in the electromagnetic waves.

When the magnetic field is absent the ion plasma waves do not propagate above a cut-off frequency \( \left( \frac{m_i}{m_e} \right)^{1/2} \) times the plasma frequency, whereas the phase velocity of the electron waves becomes imaginary for frequencies less than the plasma frequency.

4.3 Plasma Oscillations in a Magnetic Field

The properties of plasma are readily altered when placed in a magnetic field. The electrons and ions cannot move freely in the direction perpendicular to the lines of force. The trajectory of each particle is a helix with its axis parallel to the magnetic field. The collisions between the particle species may cause diffusion of ions and electrons across the field lines. In each such collision the particles are displaced only through a distance of the order of Larmor radius.

A magnetic field restricts the motion of charged particles. It may be used to prevent the plasma from coming in contact with the walls of the container. In a plasma occupying a cylindrical volume inside a tube with its axis parallel to the direction of the external magnetic field (there being vacuum and magnetic lines of force in the space between the plasma surface and the walls of the container),
an internal pressure force $\mathbf{j} \times \mathbf{H} / c$ is exhibited. In the absence of the magnetic field, this pressure would immediately lead to the expansion of the cylinder of plasma, allowing the latter to come in contact with the walls. Under the action of magnetic field the plasma experiences an electrodynamic force equal to $-\nabla p$ which is equal and opposite to the forces acting on the boundary layer due to the pressure drop across it. Hence the condition for equilibrium of plasma in a magnetic field is,

$$\frac{1}{c} \mathbf{j} \times \mathbf{H} = \nabla p$$  \hspace{1cm} (4.7)

This formula is quite general, as it holds for the boundary layer as well as for any volume element. It shows that the pressure of the plasma along the lines of force is constant, because $\mathbf{j} \times \mathbf{H} / c$ is perpendicular to $\mathbf{j}$ and $\mathbf{H}$ so that $\nabla p$ is also perpendicular to $\mathbf{j}$ and $\mathbf{H}$. The constancy of pressure along the lines of force is an important consequence of this equation. However, the validity of eqn. (4.7) is restricted to equilibrium conditions only. It should be modified to eqn. (4.8) below, in order to describe non-equilibrium conditions which frequently arise, that is,

$$\rho \mathbf{a} = \frac{1}{c} \mathbf{j} \times \mathbf{H} - \nabla p$$  \hspace{1cm} (4.8)

where $\rho$ is the density (mass per unit volume) and $\mathbf{a}$ is the acceleration of plasma. All the quantities in eqn (4.8) are given per unit volume (the left hand side denotes the density multiplied by acceleration, and the right hand side contains the vector sum of forces acting on a unit volume of
plasma). Eqn. (4.8) is valid not only for plasma but also for a conducting liquid, and is the basic equation of magnetohydrodynamics.

To day, magnetohydrodynamics/hydromagnetics is a rapidly advancing subject. It assumes an electrically conducting medium, which may be a liquid or an ionised gas. If the ionised gas is regarded as a continuous fluid, both can be treated on a common theory. Thus the distinction between plasma dynamics and magnetohydrodynamics is lost presently. When interest was focused on the possibility of controlled nuclear fusion, magnetic fields appeared to be the only means of controlling the highly ionised plasma. Intense interest was thus developed in magnetohydrodynamics/plasma dynamics as a laboratory subject.

The four modes described in sec. (4.2) are profoundly modified in the presence of a magnetic field. Even then the number of independent modes remains the same. In addition to electrostatic and electromagnetic waves, hydromagnetic waves of frequency much less than the ion cyclotron frequency, $\omega_c$, can propagate in an ionized gas. At a frequency less than the ion cyclotron frequency, there arises a hydromagnetic wave in the presence of magnetic field. In such waves the inertia is provided by the positive ions while the restoring force is largely magnetic. These oscillations may be regarded as resulting from the vibrations of lines of force which behave as stretched strings subject to mutual repulsion.
The lines of force are subjected to vibrations by loading and unloading of charged particles on them.

However, in general, a wave in the presence of a magnetic field involves electrostatic as well as magnetic forces. For example, a high frequency disturbance is generally a combination of a longitudinal electrostatic wave and a transverse electromagnetic wave. The new type of waves in the presence of a magnetic field are the so called magneto-hydrodynamic or hydromagnetic waves described by Alfvén.\(^3\)

The speed of propagation of transverse vibrational or torsional waves propagated along the magnetic lines of force is determined by the mass loading per unit length, that is, the ion density, and a force constant determined by the field strength. This speed is given by Alfvén speed

\[
V_A = \frac{ \sqrt{4 \pi \rho} }{ \sqrt{ \rho } } \quad \text{..(4.9)}
\]

\(( \rho = n_i m + n_e m \approx n_i m \)).

On squaring both the sides of eqn.(4.9) and dividing by the square of the velocity of light at following equation is obtained:

\[
\left( \frac{V_A}{c} \right)^2 = \frac{2(\mu^2/8\pi)}{\rho c^2} \quad \text{..(4.10)}
\]

Expression on the right hand side of eqn.(4.10) is the ratio of twice the magnetic energy density to the particle rest energy per c.c. This ratio is much less than unity for plasmas, that is, the waves are much slower than light waves.
Some of the various conditions and situations under which magnetohydrodynamic waves can propagate is given by Spitzer$^2$ and Cowling$^1$.

A fairly exhaustive study of hydromagnetic waves in ionized gases has been carried out by Benes$^{13,14}$, Piddington$^{15}$, Bazan and Fleischman$^{16}$, and Ludford$^{17}$. For simplicity non-dissipative case is considered in which the lines of force are frozen into the gas.

Mathematical treatments given by a number of authors either involve hydrodynamical approximations or the transport equations. Some other authors have on the other hand, used Boltzmann equation for the distribution function with the full force term taken as the sum of electrostatic and magnetic forces, and neglect collisions$^{12}$.

Although the general solutions are complicated yet they reduce to simpler forms in certain ranges of frequency, plasma density and magnetic field. One can define these ranges in terms of certain characteristic frequencies and lengths. These characteristic angular frequencies and lengths are:

- **Electron plasma frequency**, $\omega_{pe} = (\pi ne^2/m)^{1/2}$
- **Ion plasma frequency**, $\omega_{pi} = (4\pi ne^2/m)^{1/2}$
- **Electron gyro or cyclotron frequency**, $c_e \omega_e = eH/mc$
- **Ion gyro or cyclotron frequency**, $c_i \omega_{ci} = eH/mc$
Debye shielding length,

\[ \lambda_D = \left( \frac{kT}{8 \pi ne^2} \right)^{\frac{1}{2}} \approx \left( \frac{kT}{m \omega_p e} \right)^{\frac{1}{2}} \]

Electron radius of gyration,

\[ \rho_e = v_e \frac{me}{eh} \approx \left( \frac{kT_e}{m \omega e} \right)^{\frac{1}{2}} \]

Ion radius of gyration,

\[ \rho_i = v_i \frac{Mc}{eh} \approx \left( \frac{kT_i}{m \omega c_i} \right)^{\frac{1}{2}} \]

where \( \mathbf{B} \) is the magnetic field; \( v \), the particle velocity and \( c \), the velocity of light (other symbols have their usual meanings).

The above characteristic lengths have to be compared with the wavelength \( \lambda = 2\pi /k \) ( \( k \) is the magnitude of the propagation vector \( \mathbf{k} \)). In all theories it is assumed that there is no appreciable change in the applied magnetic field within these lengths and most authors assume the plasma to be a uniform neutral plasma containing ions and electrons per c.c. in a homogeneous field.

While studying the various wave modes, one must also distinguish between the longitudinal (\( \mathbf{E} \parallel \mathbf{B} \)) and transverse (\( \mathbf{E} \perp \mathbf{B} \)) waves. Electromagnetic waves propagated by electrons and magnetohydrodynamic (Alfvén) waves due to heavy ions are transverse waves. Electron-plasma oscillations (high-frequency waves) and magnetoacoustic waves \(^{12,18-21}\) (low frequency, low speed ion-sound waves) are longitudinal waves. A brief discussion of Alfvén waves and magnetoacoustic waves is given in what follows.
Let us consider a wave in which the particle velocity is parallel to the direction of propagation both being perpendicular to the magnetic field $\mathbf{B}$. Such a wave is called a magnetosonic wave which is a longitudinal hydromagnetic wave. A relation between the sound wave speed ($v_s$), Alfvén wave speed ($V_A$), and the magnetosonic wave speed ($V_m$) has been obtained as:

$$V_m^2 = \frac{V_A^2 + v_s^2}{1 + \frac{V_A^2}{c^2}} \quad \ldots (4.11)$$

Where

$$V_A = \frac{H}{\sqrt{4\pi \rho}} \quad \ldots (4.12)$$

$$v_s = \sqrt{\frac{\gamma k(T_A + 2T_e)}{M}} \quad \ldots (4.13)$$

and $c$ is the speed of light.

Usually the Alfvén wave speed is much less than the speed of light, therefore eqn. (4.11) reduces to

$$V_m^2 = V_A^2 + v_s^2 \quad \ldots (4.12)$$

In these disturbances the inertia of the positive ions is opposed by two restoring forces (i) the pressure gradient of the gas and (ii) the compressional stresses between the lines of force. If the magnetic pressure $H^2/8\pi$ is large as compared to the material pressure $p$, the speed of the magnetosonic wave is the same as Alfvén speed. If the material pressure is much greater, the compressional wave is essentially a sound wave/ion-plasma wave.
If an oblique propagation of a hydromagnetic waves with respect to the lines of force is considered, three modes of oscillation/waves are possible\(^{22,23}\). One of these three modes is an Alfvén wave with a velocity perpendicular both to the magnetic field and the propagation vector. The velocity of the wave measured normal to the wavefront is given by,

\[
V \approx \frac{\mu \cos \theta}{(4\pi \rho)^{\frac{1}{2}}}
\]

where \(\theta\) is the angle between the magnetic field and the direction of propagation.

In this type of disturbance the Poynting vector is parallel to the field direction. The energy flows along the line of force.

The other two modes involve compression of the gas. Considering \(V_A \ll c\) we have,

\[
1 = \frac{V^2_A + V^2_g}{2V^2_A V^2_g} \sec^2 \theta \times \left[ 1 + \left( 1 - \frac{4V^2_A V^2_g \cos^2 \theta}{(V^2_A + V^2_g)^2} \right)^{\frac{1}{2}} \right] \quad \text{...(4.14)}
\]

when \(\theta = 90^\circ\), considering the minus sign in eqn.(4.14), we get,

\[
V^2 = V^2_A + V^2_g
\]

which is the same as eqn.(4.12)( here \(V\) denotes \(V_m\)).

When \(\theta = 0\), this mode becomes either a sound wave \((V = V_g)\), or an Alfvén wave \((V = V_A)\), whichever has the greater velocity.

If the plus sign is considered one gets the other mode. This gives a velocity which vanishes as \(\cos \theta\) as with
the Alfvén wave. This is usually called a modified Alfvén wave.

The variation of $V$ with $\theta$ for two different values of $V_s/V_A$ are depicted in Figures (4.1(a) and (4.1(b)).

From these figures it is clear that the mode with the highest velocity (fast wave) is a modified Alfvén wave if $V_s < V_A$. Similarly for the wave of the lowest velocity (slow wave), when $V_s > V_A$. The modified Alfvén wave cannot be distinguished from pure Alfvén wave.

4.4 Plasma-Wave Experiments

Experimental researches on plasma waves has proceeded along many different paths. The investigations on self excited oscillations (or resonances) and the use of externally excited waves for propagation studies, are the most prominent among these researches. In the older experiments the first method was frequently employed. The principal disadvantage of such experiments is that the frequency cannot be varied over a wide range. Accurate volume attenuation measurements are generally not possible with this method due to the presence of boundary effects.

The use of externally excited waves is advantageous from the standpoint of basic propagation studies. For the determination of phase velocity and attenuation as a function of frequency, the waves should have plane wave-fronts and should be unaffected by the walls. The knowledge of phase velocity and attenuation in such experiments enables one to understand fully the more complex phenomena such as propagation in wave guides and
resonances in enclosures. However, experiments under the above mentioned perfect conditions require very special techniques, particularly for the excitation of the waves. Until 1960 almost no such experiments were reported. Recent experiments thereafter, have been used with great success for measurements of almost all kinds of plasma waves. A brief account of some experiments on plasma waves is given below.

**Electrostatic-wave experiments:** Self-sustained ion-plasma oscillations were first observed by Tonks and Langmuir. Later experiments of Linder and Hernquist and by Hernquist with electron beam plasmas confirmed the expected dependence of the frequency of these oscillations on the ion density and ion mass. An extensive experimental study of the dependence of self-excited oscillations on electron temperature, ion mass, neutral gas pressure, and system dimensions has been published more recently by Alexeff and Wieldigh.

The first dispersion measurements of externally excited ion waves in weakly ionized plasma were reported by Maita and Sato and by Little. Landau damping and collisional damping were investigated respectively by Wong et al and Sessler. In their investigations, the experimental conditions were such that the boundary effects could be minimized. Ion-plasma waves exhibit other properties also, such as interference and wall effects, which were studied by Jones and Alexeff, Crawford and Kuhlman, Little and Jones and others. They have also investigated the interaction of
ion-plasma waves with bounded plasmas. The boundary conditions become important if the wavelength of the ion plasma waves were comparable to the tube diameter. Little and Jones obtained a non-zero cut-off frequency for the lowest mode. They explained this by a pressure node at the wall. At higher modes, however, they observed another mode with lower phase velocity which might be interpreted at the lowest (dispersionless) mode. Hence they found it necessary to assume a pressure node at the walls. This was in agreement with Crawford and Kuhlcr's explanation of their experiments.

Standing electron-acoustic waves near plasma-frequency range were observed a long time ago by Penning and Tonks, but the experiments on electron-acoustic waves were by recently reported by Hallberg and Wharton, by Van den Bergh and by Derfler and Simonen. Other experiments involve either standing waves or resonances of plasma cylinders.

Electromagnetic-wave experiments: Electromagnetic waves may be launched into a plasma, propagated through it and received on the opposite side of the sending antenna under certain plasma conditions. An electromagnetic wave can propagate through an isotropic plasma slab many wavelengths thick and having uniform electron density between sharp vacuum boundaries without magnetic field, provided $\omega/\omega_p > 1$. Launching is easily possible by means of electromagnetic horns which give them directivity. The plasma interface causes an impedance discontinuity, which may lead to both reflection and
transmission. A finite electron collision rate or Landau damping within the plasma slab would mask the propagation, thus a part of the energy of the wave is reflected internally from the backside interface. The power transmitted is then found to be a function not merely of the absorption coefficient but of the plasma reflection and transmission also.

Similar reasoning is applicable to the phase shift of a transmitted signal. The wave impedance of the plasma interface, in general, being complex, a phase shift will occur due to the penetration of the interface and another would occur due to the changing refractive index within the plasma. The total phase shift would thus be the sum of all the phase shifts along the path.

In many fields of plasma research, quantitative spectroscopic techniques are employed. These techniques require the knowledge of atomic spectroscopy and quantum theory of perturbations.

Phase change and attenuation resulting from the insertion of plasma into an electromagnetic wave beam make it possible to determine the plasma density and electron-neutral collision frequency (Belfour et al). The highest microwave frequencies which are commonly available with sufficient power enable a convenient measurement of the upper electron density \( \sim 10^{14} \) per c.c. Details of microwave technique are nicely described by Heald and Wharton. Today the subject of interaction of electromagnetic waves with ionized gases is undergoing an explosive growth.
hydromagnetic-wave experiments: Most of the experimental work in this field has been carried out on Alfvén waves. The only transverse mode propagating along a magnetic field at frequencies less than the ion cyclotron frequency, \( \omega_{ci} \), are called 'slow Alfvén waves'. When the frequencies are comparable to or smaller than \( \omega_{ci} \), the waves are known as 'fast Alfvén waves' (their high frequency continuation are called 'Helicon waves'). The mode of propagation at right angles to the direction of magnetic field is what is called 'magnetosonic waves'.

Slow Alfvén waves were first investigated in liquid mercury and sodium by Landquist\(^{10}\) and by Lehnert\(^{11}\) who excited these waves mechanically. The first measurements of standing slow Alfvén waves, in gaseous conductors, were made by Bostick and Levine\(^{12}\). Later measurements were made by Allen et al\(^{13}\) and Jephcott\(^{14}\) in 1959. They have found an expected proportionality between the phase velocity and magnetic field; which was substantiated by the further experiments of Wilcox et al\(^{15}\). These investigations also involve the first attenuation of slow Alfvén waves in gas plasmas and their discussion in terms of plasma conductivity and ion-neutral collision. Nagao and Sato\(^{16}\) observed the relation between phase velocity and total particle density. Experiments have been carried out employing pulse discharges of Argon and Neon and of Hydrogen by Jephcott and Stocker\(^{17}\) and by Brown\(^{18}\) respectively.

Fast Alfvén waves were first observed by Kovan et al\(^{19}\) and by Hook et al\(^{20}\). More recent experiments with these waves at
low frequencies comparable to the ion-cyclotron frequency were reported by Swanson et al.\textsuperscript{51} and by Jephcott and Malein\textsuperscript{52}. The former workers produced pulsed waves in a cylindrical plasma and obtained the phase shift and attenuation constants as a function of frequency by Fourier analysis of the pulsed signal. The plasma conditions, due to the use of pulsed discharges were not highly reproducible and hence Fourier analysis became necessary. The most extensive measurements of Fast Alfvén waves were reported by Scott and Malein\textsuperscript{52}

Magnetosonic waves have only recently been investigated by Hostettler and Schneider\textsuperscript{53}. The principal difficulty associated with the experimental study of magnetosonic waves is the inability of their propagation in the axial direction in discharge tubes or wave guides, if one produces a magnetic field by a coil surrounding the tube, as a general practice.

Employing a radial magnetic field in a cylindrical pulsed discharge in Argon, Hostettler and Schneider\textsuperscript{53} measured the phase velocity and attenuation length of magnetosonic waves. In their experiment, a wire loop movable in radial direction was used to excite magnetosonic waves and the latter were detected by a pick up coil placed opposite to the transmitting probe. The experiment has successfully demonstrated the propagation of magnetosonic waves across a magnetic field.
An experimental study of magnetosonic waves and the determination of their speed, by the new technique is reported in the next chapter.
REFERENCES


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