Chapter 2
FRW Universe and Scalar Field

The Friedmann models of the universe, which are based on the Einstein’s general theory of relativity and cosmological principle, have improved our understanding about the universe. In fact, the standard cosmology is considered as a reliable model and is compatible with most of the observed features of the universe. Understanding of early universe scenario underwent remarkable progress when the Friedmann models are combined with the scalar fields. It is believed that quantum fields played important roles in the dynamical behavior of the early universe [40] - [41]. The effects of quantum field in early universe have been investigated by many authors [40] - [53]. In the present chapter we provide a brief account of the FRW universe and scalar field in curved spacetime. The basic mechanism of particle creation for a quantized scalar field in a given background is described. The chapter ends with a brief discussion of the basic properties of the coherent and squeezed state formalisms of quantum optics.

2.1 Einstein Field Equations in Cosmology

Einstein’s general theory of relativity laid the foundation of modern cosmology. It gives a set of equations, which connect geometry of the spacetime with source for it, known as the Einstein’s field equations. The applications
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of the Einstein’s field equations in cosmology are based on important assumption, called the cosmological principle. The principle states that at any given cosmic time, the universe is homogeneous and isotropic. Therefore, to formulate the Einstein field equations in the standard cosmology, we need a metric which incorporate the homogeneous and isotropic properties of the universe. The Friedmann-Robertson-Walker metric is used to describe such a universe and is given by (with $c = 1$)

$$ds^2 = -dt^2 + S^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

(2.1)

where $r, \theta$ and $\phi$ are referred to as comoving coordinates. In (2.1) $S(t)$ is called the cosmic scale factor and $k$ is the curvature parameter and can take three values $k = 0, +1$ and -1 correspondingly we get three models of the universe respectively known as flat, closed and open FRW universe. Note that in the present work small Greek indices $\mu$ and $\nu$ are running from 0,1,2, 3 and Einstein summation convention is also assumed.

The Einstein field equations in the general theory relativity can be written as [54]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

(2.2)

where $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu}$ is called the Einstein tensor and $T_{\mu\nu}$ is known as the energy momentum tensor which depends upon the distribution of matter and energy in space. In the standard cosmology, the energy momentum tensor is usually taken as a perfect fluid description of matter and is given by

$$T_{\mu\nu} = pg_{\mu\nu} + (p + \rho)u_\mu u_\nu,$$

(2.3)

where $p$ and $\rho$ are, respectively, known as pressure and energy density of the fluid and $u_\mu$ is the four velocity. The energy momentum tensor which describes the content of the universe and its pure temporal and spatial components are given by

$$T_{00} = \rho,$$

(2.4)
The energy momentum tensor satisfy the following conservation condition

$$\partial_\mu T^\mu_{\nu} = 0.$$  \hfill (2.6)

Where $\partial_\mu$ denote a covariant derivative. Thus the nontrival equations that follow from (2.1 -2.6) are

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S^2} = \frac{8\pi G}{3} \rho,$$  \hfill (2.7)

and

$$2\frac{\ddot{S}}{S} + \left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S^2} = -8\pi Gp.$$  \hfill (2.8)

Combining (2.7 and 2.8) with the Einstein equations, we can write

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}.$$  \hfill (2.9)

These set of equations are called the Einstein field equations. There have been impressive advances in developing solutions to the field equations of general relativity [2]. The first solution is obtained by Schwarzschild for a spherically symmetric mass distribution. The most important solutions of Einstein field equations are the Friedmann solutions which are used to construct different cosmological models, known as the Friedmann models of the universe.

Equations (2.7) and (2.8) appear to be independent but are related. This is because the energy density and pressure satisfy the conservation equation, thus following (2.3 -2.8) we get

$$\dot{\rho} + 3\frac{\dot{S}}{S}(\rho + p) = 0.$$  \hfill (2.10)

The Friedmann models of the universe have been discussed so far without one important aspect known as the cosmological constant. The cosmological constant $\Lambda$ was introduced to modify the Einstein field equations. The
modified field equation with cosmological constant can be written as

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \]  

(2.11)

Einstein discarded the cosmological constant but later on it has made several comebacks. Currently, the cosmological constant problem has received much attention in theory as well as observations [55]. The recent observations of Tynia supernova [56] indicate a small positive \( \Lambda \) term, however for the sake of simplicity, we consider the FRW models of the universe without the cosmological constant.

### 2.2 Quantum Field Theory in Curved Spacetime

In this section, we give a brief account of the scalar field and its quantization procedure. The physical mechanism of particle creation for the scalar field in curved spacetime is also discussed briefly. We use units \( G = c = \hbar = 1 \).

Let us consider the action for a real, massive scalar field \( \phi \), given by [40]

\[ A = \int L \, dx^4, \]  

(2.12)

where \( L \) is the Lagrangian density of the scalar field and the field obey the following wave equation

\[(\Box - m^2)\phi = 0. \]  

(2.13)

Here \( \Box \) denotes the generally covariant d’Alembertian operator. A useful concept in this context is that of the inner product of a pair of solutions of the generally covariant Klein-Gordon equation, (2.13) and is defined as

\[ (f_1, f_2) = i \int (f_2^* \overrightarrow{\partial}_\mu f_1) \, d\Sigma^\mu, \]  

(2.14)

where \( d\Sigma^\mu = d\Sigma n^\mu \), with \( d\Sigma \) is volume element in a given spacelike hypersurface, and \( n^\mu \) is the timelike unit vector normal to this hypersurface. The
The crucial property of the inner product is that it is independent of the choice of hypersurface. That is, if $\Sigma_1$ and $\Sigma_2$ are two different non-intersecting hypersurfaces, then

$$ (f_1, f_2)_{\Sigma_1} = (f_1, f_2)_{\Sigma_2}. \quad (2.15) $$

The scalar field that we considered so far is classical in nature. The quantization of a scalar field in a curved spacetime can be carried out by canonical quantization methods. Choose a foliation of the spacetime into spacelike hypersurfaces. Let $\Sigma$ be a particular hypersurface with unit normal vector $n^\mu$ labelled by a constant value of the time coordinate $t$. The derivative of $\phi$ in the normal direction is $\dot{\phi} = n^\mu \partial_\mu \phi$, and the canonical momentum is defined as

$$ \pi = \frac{\delta L}{\delta \dot{\phi}}. \quad (2.16) $$

Next one can impose the following canonical commutation relation

$$ [\phi(x, t), \pi(x', t)] = i\delta(x, x'), \quad (2.17) $$

where $\delta(x, x')$ is a delta function in the hypersurface with the property that $\int \delta(x, x') \, d\Sigma = 1$.

Let $\{f_j\}$ be a complete set of positive normal solutions of (2.13). Then $\{f^*_j\}$ will be a complete set of negative norm solutions, and $\{f_j, f^*_j\}$ form a complete set of solutions of the wave equation in terms of which we may expand an arbitrary solution. Express the field operator $\phi$ as a sum of annihilation and creation operators

$$ \phi = \sum_j (a_j f_j + a^\dagger_j f^*_j), \quad (2.18) $$

where $[a_j, a^\dagger_{j'}] = \delta_{jj'}$. This expansion defines a vacuum state $|0\rangle$ such that $a_j |0\rangle = 0$. In flat spacetime, we take the positive norm solutions to be positive frequency solutions, $f_j \propto e^{-i \omega t}$. Regardless of the Lorentz frame in which $t$ is the time coordinate, this procedure defines the same, unique Minkowski vacuum state. In curved spacetime, the situation is quite different. There is,
in general, no unique choice of the \( \{f_j\} \), and hence no unique notion of the vacuum state. This means that we cannot identify what constitutes a state without particle content, and the notion of particle becomes ambiguous. One possible resolution of this difficulty is to choose some quantities other than particle content to label quantum states. Possible choices might include local expectation values, such as \( \langle \phi \rangle, \langle \phi^2 \rangle \), etc., in the particular case of an asymptotically flat spacetime, we might use the particle content in an asymptotic region. Even this characterization is not unique. However, this non-uniqueness is an essential feature of the theory with physical consequences, namely the phenomenon of particle creation, which we discuss in the next section.

### 2.2.1 Particle Creation in Curved Spacetime

Let us consider a spacetime which is asymptotically flat in the past and in the future, but which is non-flat in the intermediate region. Let \( \{f_j\} \) be positive frequency solutions in the past (call “in-region”), and let \( \{F_j\} \) be positive frequency solutions in the future (call “out-region”). We may choose these sets of solutions to be orthonormal, so that

\[
(f_j, f_{j'}) = (F_j, F_{j'}) = \delta_{jj'} \tag{2.19}
\]

\[
(f_j^*, f_{j'}) = (F_j^*, F_{j'}^*) = -\delta_{jj'}
\]

\[
(f_j, f_{j'}^*) = (F_j, F_{j'}^*) = 0.
\]

Though, these functions are defined by their asymptotic properties in different regions, they are solutions of the wave equation everywhere in the spacetime. We may expand the in-modes in terms of the out-modes:

\[
f_j = \sum_k (\alpha_{jk} F_k + \beta_{jk} F_k^*). \tag{2.20}
\]
Inserting this expansion into the orthogonality relations, (2.19), leads to the conditions
\[ \sum_k (\alpha_{jk} \alpha^*_{j'k} - \beta_{jk} \beta^*_{j'k}) = \delta_{jj'}, \quad (2.21) \]
and
\[ \sum_k (\alpha_{jk} \alpha_{j'k} - \beta_{jk} \beta_{j'k}) = 0. \quad (2.22) \]
The inverse expansion of (2.20) is
\[ F_k = \sum_j (\alpha^*_{jk} f_j - \beta_{jk} f^*_j). \quad (2.23) \]
The field operator, \( \phi \), may be expanded in terms of either the \( \{ f_j \} \) or the \( \{ F_j \} \):
\[ \phi = \sum_j (a_j f_j + a^\dagger_j f^*_j) = \sum_j (b_j F_j + b^\dagger_j F^*_j). \quad (2.24) \]
The \( a_j \) and \( a^\dagger_j \) are annihilation and creation operators, respectively, in the in-region, whereas the \( b_j \) and \( b^\dagger_j \) are the corresponding operators for the out-region. The in-vacuum state is defined by \( a_j |0\rangle_n = 0, \forall j \), and describes the situation when no particles are present initially. The out-vacuum state is defined by \( b_j |0\rangle_o = 0, \forall j \), and describes the situation when no particles are present at late times. Noting that \( a_j = (\phi, f_j) \) and \( b_j = (\phi, F_j) \), we may expand the two sets of creation and annihilation operators in terms of one another as
\[ a_j = \sum_k (\alpha^*_{jk} b_k - \beta^*_{jk} b^\dagger_k), \quad (2.25) \]
or
\[ b_k = \sum_j (\alpha_{jk} a_j + \beta_{jk} a^\dagger_j). \quad (2.26) \]
This is known as the Bogoliubov transformation, and the \( \alpha_{jk} \) and \( \beta_{jk} \) are called the Bogoliubov coefficients. Next we describe the physical phenomenon of particle creation by a time dependent gravitational field. Let us assume that no particles were present before the gravitational field is turned on. If
the Heisenberg picture is adopted to describe the quantum dynamics, then \( |0>_{in} \) is the state of the system for all time. However, the physical number operator which counts particles in the out-region is \( N_k = b_k^\dagger b_k \). Thus the mean number of particles created into mode \( k \) is

\[
\langle N_k \rangle =_{in} \langle 0|b_k^\dagger b_k|0 \rangle_{in} = \sum_j |\beta_{jk}|^2.
\]

(2.27)

If any of the \( \beta_{jk} \) coefficients are non-zero, that is if any mixing of positive and negative frequency solutions occurs, then it can be termed that particles created by the gravitational field.

### 2.3 Coherent States and Squeezed States

Coherent and squeezed states are important classes of quantum states, well known in quantum optics [57] - [61]. Coherent states are considered as most classical, that can be generated from the vacuum state \( |0> \) by the action of displacement operator. In the present study, we use single mode coherent and squeezed states only. A single mode coherent state can be defined as [61]

\[
|\alpha> = D(\alpha)|0>
\]

(2.28)

where \( D(\alpha) \) is the single mode displacement operator, given by

\[
D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a).
\]

(2.29)

Here, \( \alpha \) is a complex number and \( a, a^\dagger \) are respectively the annihilation and creation operators, satisfying \([a, a^\dagger] = 1\).

The single mode displacement operator given by (2.29) satisfy the following properties

\[
D^\dagger a D = a + \alpha
\]

\[
D^\dagger a^\dagger D = a^\dagger + \alpha^*.
\]

(2.30)
A squeezed state is generated by the action of the squeezing operator on any coherent state. Therefore, a single mode squeezed state is defined as

$$|\alpha, \xi \rangle = Z(r, \vartheta)D(\alpha)|0\rangle,$$

(2.31)

with $Z(r, \vartheta)$ the single mode squeezing operator given by

$$Z(r, \vartheta) = \exp \frac{r}{2} \left( e^{-i\vartheta}a^2 - e^{i\vartheta}a^\dagger^2 \right).$$

(2.32)

Here, $r$ is the squeezing parameter, which determines the strength of squeezing and $\vartheta$ is the squeezing angle, which determines the distribution between conjugate variable, with $0 \leq r \leq \infty$ and $-\pi \leq \vartheta \leq \pi$.

The squeezing operator satisfy the following properties

$$Z^\dagger a Z = a \cosh r - a^\dagger e^{i\vartheta} \sinh r,$$

$$Z^\dagger a^\dagger Z = a^\dagger \cosh r - a e^{-i\vartheta} \sinh r.$$  

(2.33)

By setting $\alpha = 0$ in (2.31), one obtains the squeezed vacuum state, and is defined as

$$|\xi \rangle = Z(r, \vartheta)|0\rangle.$$

(2.34)

The squeezed vacuum state is considered as many-particle state and hence the resulting field may be called classical. However, the statistical properties of these states greatly differ from the coherent states and therefore, this state is considered as highly nonclassical having no analog in classical physics. In the case of coherent states, variance of the conjugate variables are always equal to each other, while in a squeezed state one component of the noise is always squeezed with respect to the other. Therefore, in $(x, p)$ plane, the noise for the coherent state can be described by a circle and for the squeezed state, it is an ellipse.