CHAPTER - 3

SOME RESULTS ON ITERATIVE SOLUTION
FOR ASYMPTOTICALLY
PSEUDOCONTRACTIVE OPERATORS
(34-54)
A class of $\phi$-strongly asymptotically pseudocontractive mapping is defined. Further, a strong convergence result for this class of mapping is proved in Hilbert space using a recently developed iteration scheme [105] giving errors. Next, the class of Lipschitzian $\phi$-strongly asymptotically hemicontractive mapping is obtained. Finally, an iterative solution for this class of mapping is given with errors using said iteration scheme. Examples are given in support of newly defined mapping.

It is well known that pseudocontractive mapping has acquired an important place in the theory of nonlinear analysis because it has established firm connection between nonexpansive mapping and monotone operators [6]. In the past years, many results were established on iterative approximation of the fixed points using Mann iteration scheme [59]. Later, Ishikawa [40] introduced another iteration process for such approximation. In a paper [72], iterative approximation for strongly asymptotically pseudocontractive mapping was established for Ishikawa type iteration. Further, Liu [58] introduced Ishikawa and Mann iteration process with errors. But these processes were found to be unsatisfactory [14]. Recently Yuguang Xu [105] introduced more efficient iteration process called Ishikawa iteration sequence and Mann iteration sequence with errors.
In this chapter, we introduce a class of \( \phi \)-strongly asymptotically pseudocontractive mapping and give an example to support of our mapping structure. Then, we establish a convergence result for \( \phi \)-strongly asymptotically pseudocontractive mapping using Ishikawa iteration sequence with errors introduced by Yuguang Xu. Further, we deduce strong convergence results as corollaries of this result. In this chapter next, we obtain the class of Lipschitzian \( \phi \)-strongly asymptotically hemicontractive mapping. Secondly, we provide an iterative solution in the form of convergence theorem for this class of mapping in real p-uniformly smooth Banach space using same Ishikawa iteration sequence with errors. Finally, we obtain certain corollaries from our second main result.

The iteration sequences introduced by Yuguang Xu [105] is given below:

(a) **Ishikawa iteration sequence with errors.** Let \( X \) be a Banach space and let \( K \) be a nonempty convex subset of \( X \) and \( T: K \to K \) be a mapping, for any given \( x_1 \in K \), the sequence \( \{x_n\} \) defined iteratively by

\[
x_{n+1} = a_n x_n + b_n Ty_n + c_n u_n
\]

\[
y_n = a'_n x_n + b'_n Tx_n + c'_n v_n
\]

for all \( n \geq 1 \),

where \( \{u_n\}, \{v_n\} \) are bounded sequences in \( K \) and \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are sequences in \([0,1]\) such that \( a_n + b_n + c_n = a'_n + b'_n + c'_n = 1 \) for all \( n \geq 1 \).

(b) **Mann iteration sequence with errors.** Let \( X, K \) and \( T \) are same as in (a), for any given \( x_1 \in K \), the sequence \( \{x_n\} \) defined iteratively by

\[
x_{n+1} = a_n x_n + b_n Tx_n + c_n u_n \quad \forall \ n \geq 1
\]
where \( \{u_n\} \) is an arbitrary sequence in \( K \) and \( \{a_n\}, \{b_n\}, \{c_n\} \) are sequences in \([0,1]\).

**Remark 3.1** [13]. Chidume observed above two iterations more efficient as compared ones.

Following definitions are needed before introducing the new type of pseudocontractive mapping.

**Definition 3.1** [8]. Let \( H \) be a Hilbert space. A mapping \( T: H \to H \) is said to be pseudocontractive if

\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2
\]

for all \( x, y \in H \). 

\( T \) is said to be strongly pseudocontractive if there exists \( k \in [0,1) \) such that

\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2
\]

for all \( x, y \in H \).

**Definition 3.2** [72]. A mapping \( T: H \to H \) in Hilbert space \( H \) is said to be strongly asymptotically pseudocontractive if for \( k \in [0,1) \), there exists a sequence \( k_n \in [1, \infty) \) for all \( n \in \mathbb{N} \)

\[
\|T^nx - T^ny\|^2 \leq (2k_n - 1)\|x - y\|^2 + k\|(I - T^nx)x - (I - T^ny)y\|^2.
\]

Now we define \( \phi \)-strongly asymptotically pseudocontractive mapping in Hilbert space as below:

**Definition 3.3.** A mapping \( T: H \to H \) in Hilbert space \( H \) is said to be \( \phi \)-strongly asymptotically pseudocontractive with sequence \( k_n \in [1, \infty)^N \) for all \( n \in \mathbb{N} \), if for any \( x, y \in D(T) \) and \( k \in [0,1) \) there exists a strictly increasing function \( \phi: [0, \infty) \to [0, \infty) \) with \( \phi(0) = 0 \) such that
\[ ||T^*x - T^*y||^2 \leq (2k_n - 1)||x - y||^2 + k||(I - T^n)||x - y||. \]  

**DEFINITION 3.4** [90]. Let \( H \) be a Hilbert space, \( A \) be a nonempty subset of \( H \), \( T: A \to A \) is said to be Lipschitzian asymptotically pseudocontractive mapping, if for any \( x, y \in D(T) \), there exists a sequence \( k_n \in [1, \infty)^n \) \( \forall n \in \mathbb{N} \), with Lipschitzian mapping
\[
||T^nx - T^ny|| \leq L||x - y||,
\]
where \( L > 0 \) be Lipschitzian constant, such that
\[
||T^nx - T^ny||^2 \leq (2k_n - 1)||x - y||^2 + ||(I - T^n)x - (I - T^n)y||^2.
\]

**DEFINITION 3.5** [77]. Let \( H \) and \( A \) are as in **Definition 3.4**. \( T: A \to A \) is said to be lipschitzian \( \phi \)-strongly pseudocontractive mapping if for \( k \in [0,1] \), there exists a strictly increasing function \( \phi:[0,\infty) \to [0,\infty) \) with \( \phi(0) = 0 \) such that
\[
||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2 - \phi(||x - y||)||x - y||
\]
\( \forall \ x, y \in A \) with Lipschitzian i.e. \( ||Tx - Ty|| \leq L||x - y|| \), where \( L > 0 \) be a Lipschitzian constant.

**DEFINITION 3.6** [40]. Let \( H \) and \( A \) are as in **Definition 3.4**. A mapping \( T: A \to A \) is said to be Lipschitzian pseudocontractive mapping with Lipschitzian constant \( L \) such that
\[
||Tx - Ty||^2 \leq ||x - y||^2 + ||(I - T)x - (I - T)y||^2
\]
and
\[
||Tx - Ty|| \leq L||x - y||
\]
for all \( x, y \in A \).
Following Osilike [73], we demonstrate an example in support of newly introduced mapping as below:

**EXAMPLE 3.1.** Every strongly asymptotically pseudocontractive mappings is $\phi$-strongly asymptotically pseudocontractive mapping with $\phi:[0, \infty) \to [0, \infty)$ defined by $\phi(s) = k_n s$ where $k_n \in [1, \infty)^n$ for all $n \in \mathbb{N}$.

The following example shows that the class of strongly asymptotically pseudocontractive mappings is a proper subset of the class of $\phi$-strongly asymptotically pseudocontractive mappings.

Let $H = \ell_2$ (the real with the usual norm) and $A = \{x \in H : \|x\| \leq 1\}$.

Define a mapping $T: A \to A$ by $T^nx = \frac{x}{1 + x}$.

We show that $T$ is $\phi$-strongly asymptotically pseudocontractive with $\phi:[0, \infty) \to [0, \infty)$ defined by $\phi(s) = \frac{s}{1 + s}$ and $k_n \in [1, \infty)^n$ for all $n \in \mathbb{N}$ and $k \in [0, 1)$, i.e.

$$\|T^nx - T^ny\|^2 \leq (2k_n - 1)\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2.$$

However, pick any $x \in A$ such that $0 < x < 1$ and $y = 0$, then

$$\|T^nx - T^ny\|^2 > (2k_n - 1)\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2.$$

Hence, $T$ is not strongly asymptotically pseudocontractive.

Let $X$ be a real Banach space and $p > 1$. We denote $J_p$, the generalised duality mapping from $X$ into $2^{X^*}$ given by

$$J_p(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^p \text{ and } \|f^*\| = \|x\|^{p - 1}\}.$$
where $X^*$ denotes the dual of $X$ and $\langle \cdot, \cdot \rangle$ denotes the generalised duality pairing. In particular, $J = J_x$ is called the normalised duality mapping and $J_{\rho}(x) = \|x\|^{\rho-2} J(x)$ if $x \neq 0$. $X$ is uniformly smooth if and only if $J_{\rho}$ is single valued and uniformly continuous on any bounded subset of $X$. Single valued generalised duality mapping is denoted by $j_{\rho}$.

Let us recall the following definitions in Banach space structure as below:

**DEFINITION 3.7 [6].** An operator $T$ with domain $D(T)$ and range $R(T)$ in Banach space $X$ is called strongly pseudocontractive if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ and a constant $t > 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \frac{1}{t} \|x - y\|^2$$

if $t = 1$, then $T$ is called pseudocontractive.

**DEFINITION 3.8 [73].** An operator $T: D(T) \rightarrow R(T)$ in Banach space $X$ is called $\phi$-strongly pseudocontractive if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ and a strictly increasing function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|) \|x - y\|$$

**DEFINITION 3.9 [15].** An operator $T: D(T) \rightarrow R(T)$ in Banach space $X$ is said to be hemicontractive if $F(T) = \{x \in D(T): Tx = x\} \neq \emptyset$ for all $x \in D(T)$ and $x^* \in F(T)$, there exist $j(x - x^*) \in J(x - x^*)$ such that

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2$$
DEFINITION 3.10[73]. An operator \( T: D(T) \to R(T) \) in Banach space \( X \) is called \( \phi \)-hemicontractive if \( F(T) = \{ x \in D(T) : Tx = x \} \neq \emptyset \) and for all \( x \in D(T), x^* \in F(T) \), there exist \( j(x-x^*) \in J(x-x^*) \) and a strictly increasing function \( \phi: [0, \infty) \to [0, \infty) \) with \( \phi(0) = 0 \) such that
\[
\langle Tx - x^*, j(x-x^*) \rangle \leq \|x - x^*\|^2 - \phi(\|x - x^*\|)\|x - x^*\|.
\]

DEFINITION 3.11 [92]. An operator \( T: D(T) \to R(T) \) in Banach space \( X \) is said to be asymptotically pseudocontractive with sequence \( \{k_n\} \) if \( \lim k_n = 1 \) (as \( n \to \infty \)) for all \( n \in \mathbb{N} \), and for \( x, y \in K \), there exists \( j(x-y) \in J(x-y) \) such that
\[
\langle T^n x - T^n y, j(x-y) \rangle \leq k_n \|x - y\|^2
\]
where \( J \) be normalized duality mapping.

DEFINITION 3.12 [92]. An operator \( T: D(T) \to R(T) \) in Banach space \( X \) is said to be asymptotically hemicontractive with sequence \( \{k_n\} \) if \( \lim k_n = 1 \) (as \( n \to \infty \)) for all \( n \in \mathbb{N} \) and a fixed point \( x^* \in F(T) \), for \( x \in K \), there exists \( j(x-x^*) \in J(x-x^*) \) such that
\[
\langle T^n x - x^*, j(x-x^*) \rangle \leq k_n \|x - x^*\|^2
\]
where \( J \) be normalized duality mapping.

Now, we define Lipschitzian \( \phi \)-strongly asymptotically hemicontractive mapping in Banach space as below:

DEFINITION 3.13. An operator \( T: D(T) \to X \) in Banach space \( X \) is called Lipschitzian \( \phi \)-strongly asymptotically hemicontractive mapping with sequence \( \{k_n\} \) such that \( \lim k_n = 1, \ k_n \in [1, \infty)^n \) if for all \( x, y \in D(T) \),
There exists $j(x - x^*) \in J(x - x^*)$ and a strictly increasing function 
\[ \phi : [0, \infty) \to [0, \infty) \] 
with $\phi(0) = 0$ such that 
\[ \langle T^n x - x^*, j(x - x^*) \rangle \leq k_n \| x - x^* \|^2 - \phi(\| x - x^* \|) \| x - x^* \| \]  
for all $n \geq 1$.

And 
\[ \| T^n x - T^n y \| \leq L \| x - x^* \| \] for all $n \geq 1$, 
where $L > 0$ be a Lipschitzian constant.

Following is the example supporting our mapping structure.

**EXAMPLE 3.2.** We can show that mapping which is Lipschitzian 
$\phi$-strongly asymptotically hemicontractive is not necessarily $\phi$-strongly 
asymptotically pseudocontractive.

Let $X = \mathbb{R}$ (with the usual norm) and $K = [0, \pi]$. Define $T : K \to \mathbb{R}$ 
by $T^n x = \frac{x}{3} \cos x$, for each $x \in K$ and all $n \in \mathbb{N}$. It is clear that $F(T)$ is 
nonempty, let $x^* \in F(T)$ for each $x \in K$ and $k_n \in [1, \infty)^N$ for all $x \in \mathbb{N}$. It is 
proved that 
\[ \langle T^n x - x^*, j(x - x^*) \rangle \leq k_n \| x - x^* \|^2 - \phi(\| x - x^* \|) \| x - x^* \|. \]

So $T$ is Lipschitzian $\phi$-strongly asymptotically hemicontractive. However if 
$x = \pi$ and $y = \frac{\pi}{2}$, we get for each $x, y \in K$ and $k_n \in [1, \infty)^N$ for all $x \in \mathbb{N}$, such 
that 
\[ \langle T^n x - y, j(x - y) \rangle > k_n \| x - y \|^2 - \phi(\| x - y \|) \| x - y \|. \]
Thus $T$ is not $\phi$-strongly asymptotically pseudocontractive. Furthermore, it is 
easy to see that
\[ \|T^nx - T^ny\| \leq \left(1 + \frac{\pi}{3}\right)\|x - y\|, \quad \text{for all } x, y \in K. \]

It is clear that \( T \) is a Lipschitzian \( \phi \)-strongly asymptotically hemicontractive which is not a \( \phi \)-strongly asymptotically pseudocontractive, for a strictly increasing function \( \phi : [0, \infty) \rightarrow [0, \infty) \) defined by

\[ \phi(r) = \frac{r \cdot \cos r}{3}, \quad \forall r \in K. \]

Before proving our first main result, we shall need following lemmas:

**Lemma 3.1.** [85] Let \( H \) be a Hilbert space \( \alpha \in [0,1] \) and \( z, w \in H \). Then

\[ \|\alpha z + (1 - \alpha)w\|^2 + \alpha(1 - \alpha)\|z - w\|^2 \leq \alpha\|z\|^2 + (1 - \alpha)\|w\|^2. \]

**Lemma 3.2.** [16]. Suppose that \( \{\psi_n\}, \{\sigma_n\} \) are two sequences of nonnegative numbers and \( \{\delta_n\} \) be real sequence in \([0,1]\), such that for some real number \( N_0 \geq 1 \),

\[ \psi_{n+1} \leq (1 - \delta_n)\psi_n + \sigma_n \]

for all \( n \geq N_0 \).

If \( \sum \sigma_n < \infty \), then Sequence \( \{\psi_n\} \) converging to zero i.e. \( \lim_{n \to \infty} \psi_n = 0 \), as \( n \to \infty \).

**Lemma 3.3.** [96]. Let \( \{p_n\} \) and \( \{w_n\} \) are two sequences of nonnegative numbers such that for some real number \( N_0 \geq 1 \),

\[ p_{n+1} \leq p_n + w_n \]

for all \( n \geq N_0 \).

(i) If \( \sum w_n < \infty \), then \( p_n \) exists.
(ii) If \( \sum w_n < \infty \) and sequence \( \{p_n\} \) has a subsequence converging to zero, then \( \lim_{n \to \infty} p_n = 0. \)

**LEMMA 3.4** [102]. Let \( p > 1 \) and \( X \) be a real Banach space. Then \( X \) is \( p \)-uniformly smooth iff there exists a constant \( d_p > 0 \) such that

\[
\|x + y\|^p \leq \|x\|^p + p\langle y, j_p(x) \rangle + d_p\|y\|^p \quad \forall \ x, y \in X.
\]

Now, we state and prove first main theorem of this chapter.

**THEOREM 3.1.** Let \( H \) be a real Hilbert space and \( A \) be a nonempty closed, bounded, convex subset of \( H \) and let \( T : A \to A \) be completely continuous and \( \phi \)-strongly asymptotically psuedocontractive mapping with sequence \( k_n \in [1, \infty)^N \), \( d_n = (k_n - 1) \) for all \( n \in \mathbb{N} \), \( \sum_{n=1}^{\infty} (d_n^2 - 1) < \infty \), \( \{\alpha_n\}, \{\beta_n\} \in [0,1]^N \) and \( x_1 \in A \) for all \( n \in \mathbb{N} \). Define the sequence \( \{x_n\} \) by Ishikawa iteration sequence with errors \( (a) \), where \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are real sequences in \([0,1]\) satisfying the following conditions:

(i) \( a_n + b_n + c_n = a'_n + b'_n + c'_n = 1 \) for all \( n \geq 1 \)

(ii) \( \lim b_n = \lim b'_n = 0 \)

(iii) \( \sum c_n < \infty; \sum c'_n < \infty \)

(iv) \( \sum_{n=1}^{\infty} \alpha_n \beta_n = \infty; \)

(v) \( 0 \leq \alpha_n \leq \beta_n < 1 \), for all \( n \geq 1 \) where \( \alpha_n = b_n + c_n \) and \( \beta_n = b'_n + c'_n \)

Then \( \{x_n\} \) converges strongly to some fixed point of \( T \).
**PROOF.** From Schauder’s theorem we know that $T$ possesses a fixed point $x \in A$. So $F(T) \neq \emptyset$. Now we have from Ishikawa iteration sequence with errors,

$$||y_n - x||^2 = ||a'_n (x_n - x) + b'_n (T^nx_n - x) + c'_n v_n||^2$$

$$= ||(1 - \beta_n)(x_n - x) + \beta_n (T^nx_n - x) - c'_n (T^nx_n - v_n)||^2.$$ 

Using the Lemma 3.1, for some constants $M_1 \geq 0, M_2 \geq 0$, we have

$$||y_n - x||^2 \leq (1 - \beta_n)||x_n - x||^2 + \beta_n ||T^nx_n - x||^2 - \beta_n (1 - \beta_n)||x_n - T^nx_n||^2 + c'_n M_1. \quad (3.3)$$

Also,

$$||y_n - T^n y_n||^2 = ||(1 - \beta_n)(x_n - T^ny_n) + \beta_n (T^nx_n - T^ny_n) - c'_n (T^nx_n - v_n)||^2$$

$$\leq (1 - \beta_n)||x_n - T^ny_n||^2 + \beta_n ||T^nx_n - T^ny_n||^2 - \beta_n (1 - \beta_n)||x_n - T^nx_n||^2 + c'_n M_2. \quad (3.4)$$

Using inequality (3.1), we obtain,

$$||T^nx_n - x||^2 \leq d_n ||x_n - x||^2 + k||T^nx_n - x_n||^2 - \phi(||x_n - x||)||x_n - x||.$$ 

Put $y_n$ instead of $x_n$, for all $n \in \mathbb{N}$, we have from above inequality

$$||T^ny_n - x||^2 \leq d_n ||y_n - x||^2 + k||T^ny_n - y_n||^2 - \phi(||y_n - x||)||y_n - x||.$$ 

Using inequality (3.3) and (3.4),

$$||T^ny_n - x||^2 \leq (1 - \beta_n)d_n ||x_n - x||^2 + d_n \beta_n ||T^nx_n - x||^2 - d_n \beta_n (1 - \beta_n)||x_n - T^nx_n||^2$$

$$+ d_n c'_n M_1 + k(1 - \beta_n)||x_n - T^ny_n||^2 + k\beta_n ||T^nx_n - T^ny_n||^2$$

$$- k\beta_n (1 - \beta_n)||x_n - T^nx_n||^2 + kc'_n M_2 - \phi(||y_n - x||)||y_n - x||.$$ 

Consider $M_4 = d_n M_1 + kM_2 \geq 0$,

$$||T^ny_n - x||^2 \leq d_n ||x_n - x||^2 - \beta_n (1 - \beta_n)(k + d_n)||x_n - T^nx_n||^2$$

$$+ k(1 - \beta_n)||x_n - T^ny_n||^2$$

$$+ k\beta_n ||T^nx_n - T^ny_n||^2 + c'_n M_4 - \phi(||y_n - x||)||y_n - x||. \quad (3.5)$$

Therefore, for some constant $M_4 \geq 0$; by using Lemma 3.1.
\[
\|x_{n+1} - x\|^2 = \|(1 - \alpha_n)(x_n - x) + \alpha_n(T^n y_n - x) - c_n (T^n y_n - u_n)\|^2 \\
\leq (1 - \alpha_n)\|x_n - x\|^2 + \alpha_n\|T^n y_n - x\|^2 - \alpha_n (1 - \alpha_n)\|x_n - T^n y_n\|^2 + c_n M_4.
\]

Using inequality (3.5)

\[
\|x_{n+1} - x\|^2 \\
\leq (1 - \alpha_n)\|x_n - x\|^2 + d_n\|x_n - x\|^2 - \alpha_n \beta_n (1 - \beta_n)(k + d_n)\|x_n - T^n x_n\|^2 \\
+ \alpha_n \Phi (\|y_n - x\|)\|y_n - x\| - \alpha_n (1 - \alpha_n)\|x_n - T^n y_n\|^2 + c_n M_5.
\]

where \( M_5 = \max\{M_3, M_d\} \), since \( \alpha_n \leq 1 \), and \( k(1 - \beta_n) \leq (1 - \alpha_n) \), it therefore follows that

\[
\|x_{n+1} - x\|^2 \leq (1 - \alpha_n + d_n \alpha_n)\|x_n - x\|^2 - \alpha_n \beta_n (k + d_n)(1 - \beta_n)\|x_n - T^n x_n\|^2 \\
+ k \alpha_n \beta_n\|T^n x_n - T^n y_n\|^2 - \alpha_n \Phi (\|y_n - x\|)\|y_n - x\| + (\alpha_n c_n' + c_n) M_5.
\]

Suppose for \( \delta > 0 \), there exists an integer \( N > 0 \) such that

\[
\|y_n - x\| \geq \frac{\delta}{2} \quad \text{for all} \quad n \geq N.
\]

Then \( \lim \inf \phi (\|y_n - x\|) \geq \phi (\delta / 2) > 0 \).

The above inequality becomes

\[
\|x_{n+1} - x\|^2 \leq (1 - \alpha_n + d_n \alpha_n)\|x_n - x\|^2 - \alpha_n \beta_n (1 - \beta_n)\|T^n x_n\|^2 + k \alpha_n \beta_n\|T^n x_n - T^n y_n\|^2 - \alpha_n \Phi (\|y_n - x\|)\|y_n - x\| + (\alpha_n c_n' + c_n) M_5
\]

(3.6)

since \( T \) is completely continuous, \( \{\|x_n - T^n x_n\|\} \) is a bounded sequence.

Let \( \lim \inf_{n \to \infty} \|x_n - T^n x_n\| = \rho > 0 \).
Claim $\rho = 0$, consider that the claim is false i.e. $\rho > 0$. Then there exists an integer $N_1 > 0$ such that

$$||x_n - T^nx_n|| \leq \frac{\rho}{2} \quad \forall \ n \geq N_1.$$ 

From Ishikawa iteration sequence with errors, we have,

$$y_n = a'_nx_n + b'_nT^nx_n + c'_nv_n = (1 - \beta_n)x_n + (\beta_n - c'_n)T^nx_n + c'_nv_n$$

$$(x_n - y_n) = \beta_n(x_n - T^nx_n) + c'_n(T^nx_n - v_n)$$

$$||x_n - y_n|| \leq \beta ||x_n - T^nx_n|| + c'_n||T^nx_n - v_n|| \leq \text{diam}(A). (\beta_n + c'_n) \to 0$$

as $n \to \infty$.

The continuity of $T$ implies $||T^nx_n - T^ny_n|| \to 0$ as $n \to \infty$.

Thus, there exists an integer $N_2 > 0$ such that $||T^nx_n - T^ny_n|| \leq \frac{\rho}{4} \quad \forall \ n \geq N_2$. By condition (ii) and (iii), there exists an integer $N_3$. Let $N = \max\{N_1, N_2, N_3\}$. Then inequality (3.6) yields,

$$||x_{n+1} - x||^2 \leq (1 - \alpha_n + d_n\alpha_n)||x_n - x||^2 - \alpha_n\beta_n(1 - \beta_n)(k + d_n)\frac{\rho^2}{4}$$

$$+ k\alpha_n\beta_n\frac{\rho^2}{36} + (\alpha_n c'_n + c_n)M_5.$$ 

$$||x_{n+1} - x||^2 \leq (1 - \alpha_n + d_n\alpha_n)||x_n - x||^2$$

$$- \alpha_n\beta_n\frac{\rho^2}{4} [(1 - \beta_n)(k + d_n) - \frac{k}{9}] + (\alpha_n c'_n + c_n)M_5.$$ 

Let,

$$t_n = (1 - \alpha_n + d_n\alpha_n), \lambda = \frac{\rho^2}{4} [(1 - \beta_n)(k + d_n) - \frac{k}{9}].$$

Thus,

$$\alpha_n\beta_n\lambda \leq t_n||x_n - x||^2 - ||x_{n+1} - x||^2 + M_5(\alpha_n c'_n + c_n).$$

By summing
\[
\lambda \sum_{j=1}^{n} \alpha_j \beta_j \leq t_n \|x_n - x\|^2 - \|x_{n+1} - x\|^2 + M \sum_{j=1}^{n} (\alpha_j + c'_j + c_n).
\]

It implies that \( \sum_{j=N}^{n} \alpha_j \beta_j < \infty \), which contradicts the hypothesis (iv). Thus \( \rho = 0 \)
i.e.
\[
\liminf_{n \to \infty} \|x_n - T^nx_n\| = 0.
\]

Then from inequality (3.6)
\[
\|x_{n+1} - x\|^2 \leq (1 - \alpha_n + d_n \alpha_n)\|x_n - x\|^2 + k \alpha_n \beta_n \|T^nx_n - T^ny_n\|^2
+ (\alpha_n c'_n + c_n)M,
\]

\[
\|x_{n-1} - x\|^2 \leq (1 - \delta_n)\|x_n - x\|^2 + \alpha_n,
\]

where \( \alpha_n = k \alpha_n \beta_n \|T^nx_n - T^ny_n\|^2 + (\alpha_n c'_n + c_n)M \) and \( \delta_n = \alpha_n (1 - d_n) \).
Thus the above inequality reduces to Lemma 3.2 and we have,
\[
\psi_{n+1} \leq (1 - \delta_n)\psi_n + \alpha_n,
\]

Then \( \lim \{\psi_n\} \) converges to zero i.e. \( \lim \psi_n = 0 \) as \( n \to \infty \), where
\[
\psi_n = \|x_n - x\|^2.
\]
This implies that \( \|x_n - x\| = 0 \).
This completes the proof.

Following corollaries are direct consequence of Theorem 3.1.
COROLLARY 3.1[90]. Let \( H \) and \( A \) are as in Theorem 3.1. Let \( T : A \to A \)
be completely continuous and asymptotically pseudocontractive mapping
with sequence \( \{\alpha_n\}, \{\beta_n\} \in [0, 1]^N \) \( k_n \in [1, \infty)^N \), \( d_n = (2k_n - 1) \) for all \( n \in N \),
\[
\sum_{n=1}^{\infty} (d_n^2 - 1) < \infty, \text{ and } x_n \in A \text{ for all } n \in N.
\]
Define the sequence \( \{x_n\} \)
by Ishikawa iteration sequence with errors (a), where \( \{a_n\}, \{b_n\}, \{c_n\}, \)
\{a_n\}, \{b_n\}, \{c_n\} are sequences in \([0,1]\) satisfying the conditions as in Theorem 3.1. Then \(\{x_n\}\) converges strongly to some fixed point of \(T\).

**PROOF.** From the Schauders fixed point theorem, \(F(T) \neq \emptyset\). Let \(x \in F(T)\) be a fixed point of \(T\). Using inequality (3.6) of Theorem 3.1 we have,

\[
\|T^n x_n - T^n y_n\| \leq \|x_n - y_n\| \leq \|\beta_n (x_n - T^n x_n)\| + c_n M_n,
\]

\[
\|T^n x_n - T^n y_n\|^2 \leq \beta_n^2 \|x_n - T^n x_n\|^2 + c_n M_n,
\]

for some constants \(M_n \geq 0\) and \(M_n \geq 0\).

Therefore inequality (3.6) becomes, for some constant \(M_n \geq 0\),

\[
\|x_{n+1} - x\|^2 \leq (1 - \alpha_n + d_n \alpha_n) \|x_n - x\|^2 - \alpha_n \beta_n (1 - \beta_n) (1 + d_n) \|x_n - T^n x_n\|^2 + k \alpha_n \beta_n \|x_n - T^n x_n\|^2 + c_n M_n,
\]

\[
\|x_{n+1} - x\|^2 \leq (1 - \alpha_n + d_n \alpha_n) \|x_n - x\|^2 - \alpha_n \beta_n (1 - \beta_n) (1 + d_n) \|x_n - T^n x_n\|^2 + c_n M_n.
\] (1)

Remaining part of proof follows from proof of Theorem 3.1. Thus we obtain \(x_n \to x\) as \(n \to \infty\).

**COROLLARY 3.2[9].** Let \(H\) and \(A\) are same as in Theorem 3.1. Let \(T: A \to A\) be \(\phi\)-strongly pseudocontractive mapping and \(k \in [0,1]\). Define the sequence \(\{x_n\}\) by Ishikawa iteration sequence with errors (a), where \(\{a_n\}\), \(\{b_n\}\), \(\{c_n\}\), \(\{a'_n\}\), \(\{b'_n\}\), \(\{c'_n\}\) are sequences in \([0,1]\) satisfying the conditions as in Theorem 3.1. Then \(\{x_n\}\) converges strongly to some fixed point of \(T\).
PROOF. Proof of the Corollary 3.2 will be same as Corollary 3.1, let,

\[ ||Tx_n - Ty_n|| \leq ||x_n - y_n||. \]

Rest of the proof follows from Corollary 3.1.

COROLLARY 3.3[14]. Let \( H \) and \( A \) are same as in Theorem 3.1 and let a mapping \( T : A \rightarrow A \) be strongly pseudocontractive. Define the sequence \( \{x_n\} \) by Ishikawa iteration sequence with errors \( (a) \), where \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are sequences are in \([0,1]\) satisfying the conditions as in Theorem 3.1. Then \( \{x_n\} \) converges strongly to some fixed point of \( T \).

PROOF. If we substitute \( \lim k_n = 1 \) for all \( n \in \mathbb{N} \), in the proof of Corollary 3.1, we get from inequality \((1)\), for \( d_n = 1, \forall \ n \in \mathbb{N} \),

\[ ||x_{n+1} - x||^2 \leq ||x_n - x||^2 - \alpha_n \beta_n [2(1 - \beta_n) - k\beta_n] ||x_n - Tx_n||^2 + c_n'M_n. \]

i.e. \( ||x_{n+1} - x||^2 \leq ||x_n - x||^2 + w_n \)

where \( w_n = c'_n'M_n - \alpha_n \beta_n [2(1 - \beta_n) - k\beta_n] ||x_n - Tx_n||^2. \)

If we consider

\[ ||x_{n+1} - x||^2 = \rho_{n+1}. \]

and

\[ ||x_n - x||^2 = \rho_n. \]

Then using Lemma 3.3, above inequality reduces to,

\[ \rho_{n+1} \leq \rho_n + w_n. \]

If \( \sum w_n < \infty \), then \( \lim \rho_n \) exists and converges to zero i.e. \( \lim \rho_n = 0. \) It implies that

\[ ||x_n - x|| = 0. \] as \( n \rightarrow \infty. \)

Now, we prove second main result of this chapter, for Lipschitzian \( \phi \)-strongly asymptotically hemicontractive mapping in \( p \)-uniformly smooth Banach space as below:
THEOREM 3.2. Let $p > 1$ and $X$ be a real $p$-uniformly smooth Banach space. Let $K$ be nonempty, closed, convex subset of $X$ and let $T : K \to X$ be completely continuous, Lipschitzian $\phi$-strongly asymptotically hemicontractive operator with $k_n \in [1, \infty)^N$ and fixed point $x^* \in F(T)$. Let \( \{\alpha_n\}, \{\beta_n\} \in [0,1]^N \) and $x_i \in A$ for all $n \in \mathbb{N}$, define the sequence \( \{x_n\}_{n=1}^\infty \) by

$$x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n, \quad y_n = a'_n x_n + b'_n T^n x_n + c'_n v_n \quad (3.7)$$

\( \forall \ n \geq 1 \), where \( \{u_n\}, \{v_n\} \) are bounded sequences in $K$, and \( \{a_n\}, \{b_n, c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are the real sequences in $[0,1]$ satisfying the following conditions.

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$

(ii) $\lim b_n = \lim b'_n = 0$

(iii) $\sum c_n < \infty, \sum c'_n < \infty$,

(iv) $0 \leq \alpha_n \leq \beta_n < 1$

(v) $0 \leq \beta_n \leq \alpha_n^{p-1}, \sum \alpha_n (1 - \alpha_n)^{p-1} = \infty, \sum \alpha_n^p < \infty$

\( \forall \ n \geq 1 \), where $\alpha_n = b_n + c_n$ and $\beta_n = b'_n + c'_n$.

Then \( \{x_n\} \) converges strongly to a fixed point of $T$.

PROOF. Let $F(T) \neq \emptyset$ i.e. $T$ possesses a fixed point $x^* \in K$ and $L > 0$ be Lipschitzian constant of $T$. Using Iteration scheme defined in the hypothesis for some constant $M_i > 0$, we have

$$\|y_n - x^*\|^p = \|a'_n (x_n - x^*) + b'_n (T^n x_n - x^*) + c'_n v_n\|^p$$

$$= \|(1 - \beta_n) (x_n - x^*) + \beta_n (T^n x_n - x^*) - c'_n (T^n x_n - v_n)\|^p.$$

Applying Lemma 3.4 in the above inequality, we have
\[ \|y_n - x^*\|^p \leq (1 - \beta_n)^p \|x_n - x^*\|^p + p\beta_n (1 - \beta_n)^{p-1} \langle T^n y_n - x^*, j_p(x_n - x^*) \rangle \\
+ \beta_n d_p \|T^n x_n - x^*\|^p + c_n M_1, \quad \text{using inequality (3.2)} \]
\[ \|y_n - x^*\|^p \leq (1 - \beta_n)^p \|(x_n - x^*)\|_p^p + p\beta_n (1 - \beta_n)^{p-1} \|k_n\|_p \|x_n - x^*\|^p \\
- \phi(\|x_n - x^*\|) + \beta_n d_p \|x_n - x^*\|^{p-1} + c_n M_1 \]
\[ \|y_n - x^*\|^p \leq (1 - \beta_n)^p + \beta_n (1 - \beta_n)^{p-1} k_n + \beta_n d_p L_p \|x_n - x^*\|^p \\
- p\beta_n (1 - \beta_n)^{p-1} \phi(\|x_n - x^*\|) \|x_n - x^*\|^{p-1} + c_n M_1. \quad (3.8) \]

Observe that,
\[ \langle T^n y_n - x^* , j_p(x - x^*) \rangle = \langle T^n y_n - T^n x_n , j_p(x_n - x^*) \rangle \\
+ \langle T^n x_n - x^* , j_p(x_n - x^*) \rangle \leq \beta_n L (L + L) \|x_n - x^*\|^p + k_n \|x_n - x^*\|^p \\
- \phi(\|x_n - x^*\|) \|x_n - x^*\|^{p-1} \]
\[ \langle T^n y_n - x^* , j_p(x - x^*) \rangle \leq [\beta_n L (L + L) + k_n] \|x_n - x^*\|^p \\
- \phi(\|x_n - x^*\|) \|x_n - x^*\|^{p-1}. \quad (3.9) \]

Also for some constant \( M_2 \geq 0 \), from the definition of Ishikawa iteration sequence, we have
\[ \|x_{n+1} - x^*\|^p = \|a_n (x_n - x^*) + b_n (T^n y_n - x^*) + c_n u_n\|^p \\
= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n (T^n y_n - x^*) - c_n (T^n y_n - u_n)\|^p \]

Applying Lemma 3.4, in above equation we have,
\[ \|x_{n+1} - x^*\|^p \leq (1 - \alpha_n)^p \|(x_n - x^*)\|^p + p\alpha_n (1 - \alpha_n)^{p-1} \langle T^n y_n - x^* , j_p(x_n - x^*) \rangle \\
+ \alpha_n^2 d_p \|T^n y_n - x^*\|^p + c_n M_2 \]
\[ \leq (1 - \alpha_n)^p \|x_n - x^*\|^p + p\alpha_n (1 - \alpha_n)^{p-1} \langle T^n y_n - x^* , j_p(x_n - x^*) \rangle \\
+ \alpha_n^2 d_p L_p \|y_n - x^*\|^p + c_n M_2 \quad (3.10) \]

Now, using inequalities (3.8) and (3.9) in (3.10), we obtain
\[ \|x_{n+1} - x^*\| \leq (1 - \alpha_n)^p \|x_n - x^*\|^p + p\alpha_n (1 - \alpha_n)^{p-1} \{[(\beta_n L(1 + L) + k_n)] \|x_n - x^*\|^p \\
- \phi(\|x_n - x^*\|) \|x_n - x^*\|^{p-1} \} + \alpha_n^p d_p L^p \{(1 - \beta_n)^p \\
+ p\beta_n (1 - \beta_n)^{p-1} k_n + \beta_n^p d_p L^p \} \|x_n - x^*\|^p \\
- p\beta_n (1 - \beta_n)^{p-1} \phi(\|x_n - x^*\|) \|x_n - x^*\|^{p-1} \} + \alpha_n^p d_p L^p \|x_n - x^*\|^{p-1} \} + \alpha_n^p d_p L^p c_n M_1 + c_n M_2. \] 

(3.11)

Since \( p - 1 > 0 \), condition (v) implies \( \alpha_n/\beta_n \leq \alpha_n/\beta_n, n \geq 0, \)

\[(1 - \alpha_n)^p + pk_n \alpha_n (1 - \alpha_n)^{p-1} \leq 1, \]

and,

\[(1 - \beta_n)^p + pk_n \beta_n (1 - \beta_n)^{p-1} \leq 1. \]

Using above inequalities in (3.11), we have

\[ \|x_{n+1} - x^*\| \leq (1 + p\alpha_n L(1 + L) + \alpha_n^p d_p L^p (1 + d_p L^p)) \|x_n - x^*\|^p \\
- [p\alpha_n (1 - \alpha_n)^p + \alpha_n^p d_p L^p p\beta_n (1 - \beta_n)^{p-1} \phi(\|x_n - x^*\|) \|x_n - x^*\|^{p-1} \} + (\alpha_n^p d_p L^p c_n + c_n) M_3, \]

where \( M_3 = \max(M_1, M_2) \).

Now,

\[ \|x_{n+1} - x^*\| \leq (1 + M_3\alpha_n^p) \|x_n - x^*\|^p - Q\phi(\|x_n - x^*\|) \|x_n - x^*\|^{p-1} \} + (\alpha_n^p d_p L^p c_n + c_n) M_3 \]

where \( M = 1 + pL(1 + L) + d_p L^p (1 + d_p L^p) \) and

\[ Q = p\alpha_n (1 - \alpha_n)^p + \alpha_n^p d_p L^p p\beta_n (1 - \beta_n)^{p-1}. \]

Since \( T \) be completely continuous, it follows that \( \{\|x_n - x^*\|\}_{n=0}^\infty \) is bounded.

Suppose, \( \|x_n - x^*\|^p \leq S \), for all \( n \geq 0 \). Then, we have
\[ ||x_{n+1} - x^*||^p \leq ||x_n - x^*||^p + M \cdot \alpha_n S - Q \Phi(||x_n - x^*||)||x_n - x^*||^p \]
\[ + (\alpha_n a_p L' c_n + c_n)M_j. \]  

Therefore, \[ ||x_{n+1} - x^*||^p \leq ||x_n - x^*||^p + M \cdot \alpha_n S. \] Using condition (iii), (iv) and (v), from Lemma 3.3, we have \[ ||x_n - x^*|| \] exists. Consider
\[ \lim_{n \to \infty} ||x_n - x^*|| = a, \text{ for } a \geq 0. \]

Suppose that the claim is false i.e. \( a > 0 \). Then there exists an integer \( N > 0 \) such that \[ ||x_n - x^*|| \geq \frac{a}{2}, \] for all \( n \geq N \). Then,
\[ \liminf_{n \to \infty} \phi(||x_n - x^*||) \geq \phi\left(\frac{a}{2}\right) > 0 \]  

From inequality (3.12) we obtain
\[ ||x_{n+1} - x^*||^p \leq ||x_n - x^*||^p - \left( p \sum_{n=0}^{N} \alpha_n (1 - \alpha_n) \right) \]
\[ + d_p L' p \sum_{n=0}^{N} \alpha_n \beta_n (1 - \beta_n)^{p-1} |\phi(||x_n - x^*||)||x_n - x^*||^{p-1} \]
\[ + (d_p L' p \sum_{n=0}^{N} \alpha_n \sum_{n=0}^{N} \beta_n + \sum_{n=0}^{N} c_n)M_j + M \sum_{n=0}^{N} \alpha_n \]

Now using condition (v) in above inequality for \( n \to \infty \), we obtain
\[ \left[ p \sum_{n=0}^{N} \alpha_n (1 - \alpha_n) \right] + d_p L' p \sum_{n=0}^{N} \alpha_n \beta_n (1 - \beta_n)^{p-1} |\phi(||x_n - x^*||)||x_n - x^*||^{p-1} < \infty \]

Thus from our assumption \[ ||x_n - x^*|| = a \geq 0 \], it therefore, follows that
\[ \liminf_{n \to \infty} \phi(||x_n - x^*||) = 0, \]
which contradicts (3.13). Thus \( a = 0 \).

This completes the proof.

\[ ||y_n - x^*||^p \]

Following corollaries are direct consequence of Theorem 3.2.

**COROLLARY 3.4[72].** Let \( p > 1 \) and \( X \) be a \( p \)-uniformly smooth Banach space. Let \( K \) be a nonempty, closed, convex subset of \( X \) and let \( T: K \to X \) be
a Lipschitzian strongly asymptotically pseudocontractive mapping. Let \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are sequences in \([0,1]\) and \( \{u_n\}, \{v_n\} \) are bounded sequences in \( K \), satisfying conditions (i)-(v) of Theorem 3.2. Then the sequence \( \{x_n\} \) generated from any \( x_0 \in K \) by (3.7), converges strongly to the fixed point of \( T \).

**PROOF.** \( T \) possesses a fixed point from Deimling [17]. Since \( F(T) \) is nonempty and \( T \) is \( \phi \)-strongly asymptotically hemicontractive. Hence the result follows from Theorem 3.2.

**COROLLARY 3.5[15].** Let \( X \) and \( K \) are as in Theorem 3.2. Let \( T: K \to X \) be Lipschitzian \( \phi \)-strongly hemicontractive mapping, with fixed point \( x^* \in F(T) \). Let \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are sequences in \([0,1]\) and \( \{u_n\}, \{v_n\} \) are bounded sequences in \( K \), satisfying conditions (i)-(v) of Theorem 3.2, then \( \{x_n\} \) converges strongly to \( x^* \).

**PROOF.** Since mapping \( T \) is Lipschitzian \( \phi \)-strongly hemicontractive, hence proof follows from Theorem 3.2.

**COROLLARY 3.6[36].** Let \( X \) and \( K \) are as in Theorem 3.2. Let \( T: K \to X \) be Lipschitzian strongly hemicontractive mapping with fixed point \( x^* \in F(T) \). Let \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\} \) are sequences in \([0,1]\) and \( \{u_n\}, \{v_n\} \) are bounded sequences in \( K \), satisfying conditions (i)-(v) of Theorem 3.2, then \( \{x_n\} \) converges strongly to \( x^* \).

**PROOF.** Since \( T \) be the Lipschitzian strongly hemicontractive mapping, hence, proof follows from Theorem 3.2.