CHAPTER - 6

SOME CONVERGENCE RESULTS FOR ACCRETIVE OPERATORS

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The class of locally Lipschitz multivalued $\phi$-strongly accretive operator is defined. An iterative solution for the equation $Tx = f$ is obtained for this class of mapping in real normed linear space. Picard type iteration sequence used herein is shown almost T-stable. Error estimate is also given. Next, a class of Lipschitz multivalued $\phi$-strongly quasiaccretive operator is obtained and a strong convergence theorem is proved for this class of mapping in real Banach space using Picard type iteration.

The study of accretive operators was partly motivated by its close connection to the theory of partial differential equations and partly because of its relation with nonexpansive mapping which is synergistic [50]. Recently many results on iterative construction of the solution for accretive operators were appeared in the literature of nonlinear operators. Some of them were [11], [17], [19], [21], [25], [27], [55], [57], [77], [76], [108], [109], etc. In these results either Ishikawa [40] or Mann [59] iteration method were applied for such constructions. Some of them have given error estimate also. In Chapter-3, it is said that the Picard type iteration [13] is fast as compare to Mann or Ishikawa iteration. In Chapter-4, it is already mentioned that theory of multivalued mapping has applications in nonlinear evolution equations, Game theory & Mathematical Economics etc.
In this chapter, first we introduce a class of locally Lipschitz multivalued $\phi$-strongly accretive operator and prove an strong convergence results with error estimate for this class of nonlinear operator in normed linear space using Picard type iteration sequence. Further, we prove this result almost $T$-stable, we give error estimate for this result. Finally, we furnish some results derived as consequence from our main result. Next, in this chapter, we obtain a class of Lipschitz multivalued $\phi$-strongly quasi accretive operator and prove second main result of this chapter by giving a strong convergence result for this class of operator in real Banach space having uniformly convex dual $X^*$. We use Picard type sequence of iteration for this purpose. Finally, we obtain certain consequence of this result.

Before proving main results, let us recall the following:

Let $X$ be a real normed linear space $X'$ be its dual. The normalized duality mapping $J:X \rightarrow 2^{X^*}$ is defined by

$$Jx = \{f' \in X^* : \langle x, f' \rangle = ||x||^2, ||f'|| = ||x||\}$$

where $\langle \cdot, \cdot \rangle$ denotes the normalized duality pairing between $X$ and $X^*$.

The mapping $T:X \rightarrow X$ is said to be accretive [42] mapping in Banach space $X$ in term of duality if, for each $x, y \in \text{D}(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq 0.$$  

**DEFINITION 6.1** [10]. Let $K$ be a nonempty, convex subset of $X$. $T:K \rightarrow 2^X$ is said to be multivalued accretive operator in Banach space $X$ if for all $x, y \in \text{D}(T)$, there exists $j(x-y) \in J(x-y)$ such that
\( \langle \xi' - \eta', j(x-y) \rangle \geq 0 \), where \( \xi' \in Tx, \eta' \in Ty \).

Now we define:

**DEFINITION 6.2** [37]. Let \( X \) be a real Banach space, \( K \) be a nonempty subset of \( X \), \( T:K \to 2^X \) is said to be a multivalued \( \phi \)-strongly accretive operator if for all \( x,y \in D(T) \), there exists \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \phi:[0,\infty) \to [0,\infty) \) with \( \phi(0) = 0 \) such that

\[
\langle \xi' - \eta', j(x-y) \rangle \geq \phi(\|x-y\|)\|x-y\|
\]

where \( \xi' \in Tx, \eta' \in Ty \).

We define:

**DEFINITION 6.3**. Let \( X \) be a real normed linear space, \( K \) be a nonempty subset of \( X \), \( T:K \to 2^X \) is said to be a locally Lipschitz multivalued \( \phi \)-strongly accretive operator if for all \( x,y \in D(T) \), there exists \( j(x-y) \in J(x-y) \) and a strictly increasing function \( \phi:[0,\infty) \to [0,\infty) \) with \( \phi(0) = 0 \) such that

\[
\langle \xi' - \eta', j(x-y) \rangle \geq \phi(\|x-y\|)\|x-y\|
\]

where \( \xi' \in Tx, \eta' \in Ty \) and \( \|\xi' - \eta'\| \leq L\|x-y\| \), where \( L > 0 \) be a Lipschitzian constant. Mapping \( T:K \to 2^X \) is said to be “locally” if for given \( x \in K \), there exists a neighbourhood \( B \subseteq K \) such that \( B = B_r(x^*) = \{x \in K: \|x - x^*\| \leq R \} \).

**DEFINITION 6.4**. \( T:K \to 2^X \) is said to be multivalued strongly quasipaccretive in Banach space \( X \) if for any \( x \in D(T) \), there exists \( p \in N(T) \) where \( N(T) \) (null space of \( T \)) = \( \{p \in D(T):Tp=0\} \) and \( j(x-p) \in J(x-p) \) such that
We also define:

**DEFINITION 6.5.** Let $X$ be a real Banach space, $K$ be a nonempty, convex subset of $X$. $T: K \to 2^X$ is said to be Lipschitz multivalued $\phi$-strongly quasi-accretive, bounded operator if for any $x \in \text{D}(T)$, there exists $p \in \text{N}(T)$ (null space of $T$), where $\text{N}(T) = \{ p \in \text{D}(T) : Tp = 0 \}$, $j(x-p) \in J(x-p)$ and a strictly increasing function $\phi: [0, \infty) \to [0, \infty)$ such that

$$
\langle \xi' - p, j(x-p) \rangle \geq \phi(||x-p||)||x-p||,
$$

where $\xi' \in T_x$, and

$$
||\xi' - \eta'|| \leq L ||x-y||,
$$

with $L > 0$ be Lipschitzian constant.

**REMARK 6.1 [20].** If operator $T$ is accretive thus $-T$ is dissipative.

We need following lemmas to prove our results.

**LEMMA 6.1 [83].** If $X'$ be uniformly convex dual of Banach space $X$, then there exists a continuous nondecreasing function $b: [0, \infty) \to [0, \infty)$ such that

$$
b(0) = 0, \quad b(ct) \leq cb(t) \quad \forall \quad c \geq 1
$$

and

$$
||x+y||^2 \leq ||x||^2 + 2\langle y, J(x) \rangle + \max\{||x||, ||y||\} \cdot ||y|| \cdot b(||y||) \quad \forall \quad x, y \in X.
$$

**LEMMA 6.2 [101].** Let $\{p_n\}$ be a nonnegative sequence satisfying $p_{n+1} \leq (1 - \mu_n)p_n + \sigma_n$ with $\mu_n \in [0, 1]$ and $\sigma_n = 0(\mu_n)$. Then $p_n \to 0$ as $n \to \infty$.

Now, we prove the first main result of this chapter:
THEOREM 6.1. Let $X$ be a real normed linear space with uniform convex dual $X^*$ and $K$ be a nonempty, closed, convex subset of $X$. Let $T: K \to 2^X$ be locally Lipschitz multivalued $\phi$-strongly accretive operator. For each $f \in X^*$, let $x^*$ be the unique solution of equation $Tx = f$. Then there exists a neighbourhood $B$ of $x^*$ and set $\varepsilon = 1/2(5 + 2L^2 + 6L)$ for $\varepsilon > 0$, define $T_\varepsilon : K \to 2^X$ by $\xi^\varepsilon = (1 - \varepsilon)x + \varepsilon \xi'$, where $T_\varepsilon \subseteq B_\varepsilon (x^*)$, $\xi^\varepsilon \in Tx$ and $\xi' \in Tx$ for each $x \in B \subseteq K$. Define the operator $S: K \to 2^X$ by $Sx = x - Tx + f$. For arbitrary $x_0 \in K$ and define the sequence $\{x_n\}$ in $B \subseteq K$ by $x_{n+1} = \xi^\varepsilon_n$, where $\xi^\varepsilon_n \in T_\varepsilon x_n$, $n \geq 0$ and $x_{n+1} \in B_\varepsilon (x^*)$.

Let $\{y_n\}$ be any sequence in $X$ i.e. $\{y_n\} \subseteq X$ and define $\{\lambda_n\}$ by $\lambda_n = ||y_{n+1} - (1 - \varepsilon)y_n - \varepsilon \cdot w_n||$, where $w_n \in Sy_n$.

Then

(I) sequence $\{x_n\}$ converges strongly to $x^*$ with $||x_{n+1} - x^*|| \leq \delta^{n+2} ||x_0 - x^*||$, where $\delta = (1 + \varepsilon / 2)$

(II) $\lim_{n \to \infty} y_n = x^*$ iff $\lim_{n \to \infty} \lambda_n = 0$

(III) $\sum_{n=0}^{\infty} \lambda_n < +\infty$ implies $\lim_{n \to \infty} y_n = x^*$ so that $\{x_n\}$ is almost $T$-stable.

PROOF. The existence of a solution follows from Morales [68]. Let, $S$ be Lipschitz mapping with lipschitzian constant $L_* = 1 + L$, where $L$ be lipschitzian constant of $T$. For each $f \in X^*$, since $x^*$ be the unique solution of equation $Tx = f$, then $x^*$ is also be the unique solution of the equation $Sx = f + x - Tx$. The Picard type iteration formula $x_{n+1} = \xi^\varepsilon_n$ becomes $x_{n+1} = (1 - \varepsilon)x_n + \varepsilon \cdot (f + x_n - \xi'_{n})$, where $\xi'_{n} \in Tx_n$ and $x_{n+1} \in B \subseteq K$.

Furthermore, inequality (6.1) becomes,
\[
\langle y - w, j(x - y) \rangle \leq \|x - y\|^2 - \phi(\|x - y\|) \|x - y\| \quad (6.2)
\]
where \(\gamma \in Sx\), \(w \in Sy\).

Thus Picard type iteration becomes
\[
x_{n+1} = (1 - \varepsilon)x_n + \varepsilon \gamma_n, \text{ where } \gamma_n \in Sx_n. \quad (6.3)
\]

Now, first we prove Part I.

**To prove Part I**, we have

\[
\|x_{n+1} - x^*\|^2 = \|(1 - \varepsilon)(x_n - x^*) + \varepsilon(\gamma_n - x^*)\|^2
\]
using inequality of Lemma 4.1, which gives,

\[
\|x_{n+1} - x^*\|^2 \leq (1 - \varepsilon)^2 \|x_n - x^*\|^2 + 2\varepsilon \langle \gamma_n - x^*, j(x_{n-1} - x^*) \rangle
\]
\[
\leq (1 - \varepsilon)^2 \|x_n - x^*\|^2 + 2\varepsilon \langle \gamma_n - x^*, j(x_n - x^*) \rangle
\]
\[
+ 2\varepsilon \langle \gamma_n - x^*, j(x_{n-1} - x^*) - j(x_n - x^*) \rangle
\]

using inequality (6.2), we have

\[
\|x_{n+1} - x^*\|^2 \leq (1 - \varepsilon)^2 \|x_n - x^*\|^2 + 2\varepsilon \|x_n - x^*\|^2 - 2\varepsilon \phi(\|x_n - x^*\|) \|x_n - x^*\|
\]
\[
+ 2\varepsilon \langle \gamma_n - x^*, j(x_{n-1} - x^*) - j(x_n - x^*) \rangle
\]
\[
\leq [(1 - \varepsilon)^2 + 2\varepsilon] \|x_n - x^*\|^2
\]
\[
- 2\varepsilon \phi(\|x_n - x^*\|) \|x_n - x^*\| + 2\varepsilon d_n \|x_n - x^*\|^2
\]
where
\[
d_n = \left( \frac{\gamma_n - x^*}{\|x_n - x^*\|^2} \right) \left( \frac{j(x_{n-1} - x^*) - j(x_n - x^*)}{\|x_n - x^*\|^2} \right)
\]
\[
\|x_{n+1} - x^*\|^2 \leq (1 + \varepsilon^2 + 2\varepsilon d_n) \|x_n - x^*\|^2 - 2\varepsilon \phi(\|x_n - x^*\|) \|x_n - x^*\|. \quad (6.4)
\]

Now we have

\[
\frac{\|\gamma_n - x^*\|}{\|x_n - x^*\|} \leq L, \quad \frac{\|x_n - x^*\|}{\|x_n - x^*\|} \leq L.
\]

where \(L\) be Lipschitzian constant of \(S\) and
\[
\frac{\|x_{n+1} - x^* - x_n - x^*\|}{\|x_n - x^*\|} = \frac{\|x_{n+1} - x_n\|}{\|x_n - x^*\|}
\]
\[
= \epsilon \frac{\|y_n - x_n\|}{\|x_n - x^*\|}
\]
\[
\leq \epsilon \left( \frac{\|y_n - x^*\| + \|x_n - x^*\|}{\|x_n - x^*\|} \right)
\]
\[
\leq \epsilon (L_n + 1) \|x_n - x^*\| \|x_n - x^*\|.
\]
\[
\leq \epsilon (L_n + 1)
\]

Therefore,
\[
d_n = \left( \frac{\gamma_n - x^*}{\|x_n - x^*\|} - \frac{\beta(x_{n+1} - x^*)}{\|x_{n+1} - x^*\| - \|x_n - x^*\|} \right) \leq \epsilon \cdot L_n (L_n + 1)
\]

Substituting above in inequality (6.4), we have
\[
\|x_{n+1} - x^*\| \leq (1 + \epsilon^2 + 2\epsilon^2 L_n^2 + 2\epsilon^2 L_n) \|x_n - x^*\| \|x_n - x^*\| = 2\epsilon \cdot \phi(\|x_n - x^*\|) \|x_n - x^*\|.
\]

Suppose for \( \rho > 0 \), there exists an integer \( N > 0 \) such that
\[
\|x_n - x^*\| > \rho / 2, n \geq N.
\]

Then \( \lim \inf \phi(\|x_n - x^*\|) \geq \phi(\rho / 2) > 0 \), since \( \epsilon \) be an arbitrary and assumption \( \|x_n - x^*\| \to \rho \) implies that \( \lim \inf \phi(\|x_n - x^*\|) \to 0 \). Thus, we get the above inequality as,
\[
\|x_{n+1} - x^*\| \leq (1 + \epsilon^2 + 2\epsilon^2 L_n^2 + 2\epsilon^2 L_n) \|x_n - x^*\| \|x_n - x^*\| \leq [1 + 5\epsilon^2 + 2\epsilon^2 (L_n + 2L)] \|x_n - x^*\| \|x_n - x^*\|, \quad (\because L_n = 1 + L)
\]
\[
\|x_{n+1} - x^*\| \leq [1 + 5\epsilon^2 + 2\epsilon^2 L_n (L_n + 3)] \|x_n - x^*\| \|x_n - x^*\| \leq 8 \|x_n - x^*\| \|x_n - x^*\| \leq (1 + 4\epsilon^2 + 2\epsilon^2 L_n) \|x_n - x^*\|^2 \quad (6.5)
\]

\[
\delta = 1 + 5\epsilon^2 + 2\epsilon^2 L_n (L_n + 3), \quad \delta \geq 0
\]
\[
\|x_{n+1} - x^*\|^2 \leq \delta \cdot \|x_0 - x^*\|^2
\]
\[
\|x_{n+1} - x^*\| \leq \delta^{n/2} \|x_0 - x^*\| \to 0 \quad \text{as} \quad n \to \infty.
\]

Hence \( \lim_{n \to \infty} x_n = x^* \).
**Error Estimate**: Choose $\varepsilon = \frac{1}{2(5 + 2L^2 + 6L)}$.

Since we have, from inequality (6.5),

$$\|x_{n+1} - x^*\|^2 \leq [1 + 5\varepsilon^2 + 2\varepsilon^2 L(L + 3)]\|x_n - x^*\|^2.$$

Let $\delta = 1 + 5\varepsilon^2 + 2\varepsilon^2 L(L + 3)$,

$$\|x_{n+1} - x^*\|^2 \leq \delta\|x_n - x^*\|^2.$$

Consider $\delta = \left(1 - \frac{2}{n + 1}\right)^2$, where $n \in \mathbb{N}$,

since $\|x_{n+1} - x^*\|^2 \leq \left(1 - \frac{2}{n + 1}\right)^2 \|x_n - x^*\|^2.$ \hspace{1cm} (6.6)

If there exists a constant $m$ such that

$$\|x_{n+1} - x^*\|^2 \leq m$$ \hspace{1cm} (6.7)

Claim : $\|x_n - x^*\|^2 \leq \frac{m}{n}$ \hspace{1cm} (6.8)

for $n = 1$, inequality (6.7) holds.

Assume that $n = r$, then from (6.6), we have

$$\|x_{r+1} - x^*\|^2 \leq \left(1 - \frac{2}{r + 1}\right)^2 \|x_r - x^*\|^2 \leq \left(1 - \frac{2}{r + 1}\right)^2 \cdot \frac{m}{r} \leq m \cdot \frac{1}{(r + 1)^2}.$$

Thus (6.8) holds and by induction (6.8) holds for all positive integers, hence we get
To prove Part II, let us consider \( \lim_{n \to \infty} y_n = x^* \), then

\[
\lambda_n = ||y_{n+1} - (1 - \varepsilon)y_n - \varepsilon w_n||
\]

\[
\lambda_n \leq ||y_{n+1} - x^*|| + ||(1 - \varepsilon)(y_n - x^*) + \varepsilon(w_n - x^*)||
\]

\[
\lambda_n \leq ||y_{n+1} - x^*|| + [1 + \varepsilon^2(5 + 6L + 2L^2)]^{1/2}||y_n - x^*||
\]

since sequence \( \{y_n\} \to x^* \), i.e. \( \lim_{n \to \infty} ||y_n - x^*|| = 0 \) and \( \lim_{n \to \infty} ||y_{n+1} - x^*|| = 0 \)

as \( n \to \infty \) i.e. \( \lambda_n \to 0 \) as \( n \to \infty \).

Converse Part. Let \( \lim_{n \to \infty} \lambda_n = 0 \) as \( n \to \infty \), then we have from the following inequality

\[
||y_{n+1} - x^*|| \leq \lambda_n + [1 + \varepsilon^2(5 + 6L + 2L^2)]^{1/2}||y_n - x^*||
\]

\[
||y_{n+1} - x^*|| \leq 0 + [1 + \varepsilon^2(5 + 6L + 2L^2)]^{1/2}||y_n - x^*||
\]

\[
\leq \delta^{1/2}||y_n - x^*||, \text{ where } \delta = 1 + \varepsilon^2(5 + 6L + 2L^2).
\]

It follows that \( \lim_{n \to \infty} y_n \to x^* \) as \( n \to \infty \).

To prove Part III, let us consider \( \sum_{n=0}^{\infty} \lambda_n < \infty \) and

\[
M_i = \sup \{||w_n - Sx^*|| : n \in \mathbb{N}\}, \text{ where } w_n \in Sy_n, \text{ it is obvious that } M_i < +\infty.
\]

Then

\[
||y_{n+1} - x^*|| \leq \lambda_n + (1 - \varepsilon)||y_n - x^*|| + \varepsilon ||w_n - x^*||
\]

\[
\leq (1 - \varepsilon)||y_n - x^*|| + (\varepsilon M_i + \lambda_n).
\]

By induction \( \{||y_n - x^*||\}_{n=0}^{\infty} \) is bounded.
Let us consider \( M_0 = M + \sup\{||y_n - x^*||: n \in \mathbb{N}\} < +\infty \). Using Lemma 4.1, we have

\[
||y_{n+1} - x^*||^2 = ||\lambda_n + (1-\varepsilon)(y_n - x^*) + \varepsilon(w_n - x^*)||^2 \\
\leq ||(1-\varepsilon)(y_n - x^*) + \varepsilon(w_n - x^*)||^2 + 2\lambda_n[|| (1-\varepsilon)(y_n - x^*) + \varepsilon(w_n - x^*) || + \lambda^2_n] \\
\leq (1-\varepsilon)^2 ||y_n - x^*||^2 + 2\varepsilon( (w_n - x^*, j(y_{n+1} - x^* - \lambda_n)) + \lambda^2_n \\
+ 2\lambda_n(1-\varepsilon)(M_2 - M_1) + 2\varepsilon \lambda_n M_1 \\
\leq (1-\varepsilon)^2 ||y_n - x^*||^2 + 2\varepsilon( ||y_n - x^*||^2 \\
- \phi(||y_n - x^*||)||y_n - x^*|| \\
+ \lambda_n[\lambda_n + 2(1-\varepsilon)(M_2 - M_1) + 2\varepsilon M_1] \\
+ 2\varepsilon^2 L_\varepsilon(L_\varepsilon + 1)||y_n - x^*||^2 \\
\leq [(1 + \varepsilon^2) + 2\varepsilon^2 L_\varepsilon(L_\varepsilon + 1)||y_n - x^*||^2 \\
+ 2\lambda_n[ \lambda_n + 2(1-\varepsilon)(M_2 - M_1) + 2\varepsilon M_1 ] \\
\leq t||y_n - x^*||^2 + \sigma_n \]
\]

where \( t = 1 + \varepsilon^2 + 2\varepsilon^2 L_\varepsilon(L_\varepsilon + 1) \)

and \( \sigma_n = \lambda_n[\lambda_n + 2(1-\varepsilon)(M_2 - M_1) + 2\varepsilon M_1] \). Let \( \rho_n = ||y_n - x^*||^2 \), then inequality becomes \( \rho_{n+1} \leq t\rho_n + \sigma_n \).

Using Lemma 5.4, we get \( \rho_n \to 0 \), as \( n \to \infty \), i.e. \( \lim_{n \to \infty} y_n = x^* \).

So that \( \{x_n\} \) is almost T-stable.

This completes the proof.

Following is the direct consequence of Theorem 6.1 for dissipative operator.
THEOREM 6.2. Let $X$ be a normed linear space with uniform convex dual $X^*$ and $K$ be a nonempty, closed, convex subset of $X$. Let $T: K \rightarrow 2^X$ be locally Lipschtiz multivalued $\phi$-strongly dissipative operator. For each $f \in X^*$, let $x^*$ be the unique solution of equation $(-\lambda Tx) = f$, where $\lambda$ be a real positive constant, then there exists a neighbourhood $B$ of $x^*$ and set $\varepsilon = 1/(2(5 + 2L^2 + 6L))$ for $\varepsilon > 0$, define $T_{\varepsilon}: K \rightarrow 2^X$ by $\xi_{\varepsilon} = (1 - \varepsilon)x + \varepsilon \xi'$, where $T_{\varepsilon} \subseteq B_{R}(x^*)$, $\xi_{\varepsilon} \in T_{\varepsilon}x$ and $\xi' \in Tx$ for each $x \in B \subseteq K$. Define $S: K \rightarrow 2^X$ by $Sx = x + \lambda Tx + f$. For arbitrary $x_0 \in K$, and define the sequence $\{x_n\}$ in $B \subseteq K$ by $x_{n+1} = \xi_{\varepsilon}^\prime$, where $\xi_{\varepsilon}^\prime \in T_{\varepsilon}x_n$ and $x_{n+1} \in B_{R}(x^*)$, $n \geq 0$.

Let $\{y_n\}$ be any sequence in $X$ i.e. $\{y_n\} \subseteq X$ and define $\{\lambda_n\}$ by

$$\lambda_n = \|y_{n+1} - (1 - \varepsilon)y_n - \varepsilon \cdot w_n\|,$$

where $w_n \in Sy_n$.

Then,

(I) sequence $\{x_n\}$ converges strongly to $x^*$ with

$$\|x_{n+1} - x^*\| \leq \delta^{n/2}\|x_n - x^*\|,$$ where $\delta = (1 + \varepsilon/2)$

(II) $\lim_{n \to \infty} y_n = x^*$ iff $\lim_{n \to \infty} \lambda_n = 0$

(III) $\sum_{n=0}^{\infty} \lambda_n < \infty$ implies $\lim_{n \to \infty} y_n = x^*$ so that $\{x_n\}$ is almost $T$-stable.

PROOF. Suppose that the existence of the solution follows from the dissipativity of $T$. Furthermore $(-\lambda T)$ is locally Lipschtiz multivalued $\phi$-strongly accretive. Thus result now follows from Theorem 6.1.

We have the following corollaries as consequence of Theorem 6.1.

COROLLARY 6.1[56]. Let $X$ and $K$ are same as in Theorem 6.1. Let $T: K \rightarrow X$ be locally Lipschtizian $\phi$-strongly accretive operator. For each
f ∈ X*, let x* be the unique solution of the equation Tx = f. Then there exists a
neighbourhood B of x*, set ε = \frac{1}{2(5 + 2L^2 + 6L)} for ε > 0, define Tε : K → X
by Tεx = (1 - ε)x + εTx for each x ∈ K and Tε ⊆ Bε(x*). Define S : K → X by
Sx = x - Tx + f. For arbitrary x₀ ∈ B ⊆ K define the sequence {xₙ} in B by
xₙ₊₁ = Tεxₙ, n ≥ 0.

Let {yₙ} be any sequence in X i.e. {yₙ} ⊆ X and defined {λₙ} by
λₙ = ||yₙ₊₁ - (1 - ε)yₙ - εSyₙ||.

Then,
(I) sequence {xₙ} converges strongly to x* with
||xₙ₊₁ - x*|| ≤ δⁿ⁺² ||x₀ - x*||, where δ = (1 + ε/2)
(II) \limₙ→∞ yₙ = x* iff \limₙ→∞ λₙ = 0
(III) \sumₙ=₀ λₙ < +∞ implies \limₙ→∞ yₙ = x* so that \{xₙ\} is almost T-stable.

PROOF. Proof follows from Theorem 6.1.

COROLLARY 6.2[17]. Let X and K are same as Theorem 6.1. Let
T : K → 2X be locally Lipschitz multivalued strongly accretive operator. For
each f ∈ X*, let x* be the unique solution of equation Tx = f. Then there exists
a neighbourhood B of x*, and set ε = \frac{4 - 3k}{2(5 + 2L^2 + 6L)}, for k > 0, ε > 0, define
Tε : K → 2X by \xiₖ = (1 - ε)x + εξ', where ξₖ ∈ Tεx and ξ' ∈ Tx for each
x ∈ B ⊆ K. Define S : K → 2X by Sx = x - Tx + f. For arbitrary x₀ ∈ B, and
define the sequence \{xₙ\} in B ⊆ K by xₙ₊₁ = ξₖxₙ, where ξₖ ∈ Tεxₙ and
Let \( \{x_n\} \) be any sequence in \( X \) i.e. \( \{y_n\} \subseteq X \) and define \( \{\lambda_n\} \) by
\[
\lambda_n = \|y_{n+1} - (1 - \varepsilon)y_n - \varepsilon \cdot w_n\|, \quad \text{where } w_n \in Sy_n.
\]
Then,
(I) sequence \( \{x_n\} \) converges strongly to \( x^* \) with
\[
\|x_{n+1} - x^*\| \leq \delta^{n/2}\|x_0 - x^*\|, \quad \text{where } \delta = \left(1 + \frac{k\varepsilon}{2}\right)
\]
(II) \( \lim_{n \to \infty} y_n = x^* \) iff \( \lim_{n \to \infty} \lambda_n = 0 \)
(III) \( \sum_{n=0}^{\infty} \lambda_n < \infty \) implies \( \lim_{n \to \infty} y_n = x^* \) so that \( \{x_n\} \) is almost \( T \)-stable.

**PROOF.** Proof follows from Theorem 6.1.

**COROLLARY 6.3** [74]. Let \( X \) and \( K \) are same as in Theorem 6.1. Let 
\( T : K \to 2^X \) Lipschitz multivalued \( \phi \)-strongly accretive operator. For each \( f \in X^* \), let \( x^* \) be the unique solution of equation \( Tx = f \). Set 
\[
\varepsilon = \frac{1}{2(5 + 2L^2 + 6L)}, \quad \text{for } \varepsilon > 0, \text{ define } T_\varepsilon : K \to 2^X \text{ by } \tilde{\xi}_n^\varepsilon = (1 - \varepsilon)\xi_n + \varepsilon \tilde{\xi}'_n,
\]
where \( \tilde{\xi}_n^\varepsilon \in T_\varepsilon x_n \) and \( \tilde{\xi}'_n \in Tx \) for each \( x \in K \). Define 
\( S : K \to 2^X \) by \( S(x) = x - Tx + f \). For arbitrary \( x_0 \in K \), and define the sequence 
\( \{x_n\} \) in \( K \) by \( x_{n+1} = \tilde{\xi}_n^\varepsilon \), where \( \tilde{\xi}_n^\varepsilon \in T_\varepsilon x_n, \ n \geq 0 \). Let \( \{y_n\} \) be any sequence in \( X \) i.e. \( \{y_n\} \subseteq X \) and define \( \{\lambda_n\} \) by
\[
\lambda_n = \|y_{n+1} - (1 - \varepsilon)y_n - \varepsilon \cdot w_n\|, \quad \text{where } w_n \in Sy_n.
\]
Then,
(I) sequence \( \{x_n\} \) converges strongly to \( x^* \) with
\[ \|x_{n+1} - x^*\| \leq \delta^{n/2} \|x_0 - x^*\|, \text{ where } \delta = (1 + \varepsilon/2) \]

(II) \( \lim_{n \to \infty} y_n = x^* \iff \lim_{n \to \infty} \lambda_n = 0 \)

(III) \( \sum_{n=0}^{\infty} \lambda_n < +\infty \) implies \( \lim_{n \to \infty} y_n = x^* \) so that \( \{x_n\} \) is almost T-stable.

**PROOF.** Proof follows from Theorem 6.1.

**COROLLARY 6.4[56].** Let \( X \) and \( K \) are same as Theorem 6.2. Let \( T : K \to X \) be locally Lipschitian \( \phi \)-strongly dissipative operator. For any given \( f \in X^* \), let \( x^* \) be the unique solution of equation \((\lambda T x) = f\). Then, there exists a neighbourhood \( B \) of \( x^* \) and set \( \varepsilon = 1/2(5 + 2L^2 + 6L) \) for \( \varepsilon > 0 \), define \( T_\varepsilon : K \to X \) by \( T_\varepsilon x = (1 - \varepsilon)x + \lambda T x \) for \( x \in K \) and \( T_\varepsilon \subseteq B_r(x^*) \). Define \( S : K \to X \) by \( Sx = x + \lambda T x + f \). For arbitrary \( x_0 \in K \), and define the sequence \( x_n \) in \( B \) by \( x_{n+1} = T_\varepsilon x_n \), \( n \geq 0 \). Let \( \{y_n\} \) be any sequence in \( X \) i.e. \( \{y_n\} \subseteq X \) and define \( \{\lambda_n\} \) by

\[ \lambda_n = \|y_{n+1} - (1 - \varepsilon)y_n - \varepsilon S y_n\|. \]

Then,

(I) sequence \( \{x_n\} \) converges strongly to \( x^* \) with

\[ \|x_{n+1} - x^*\| \leq \delta^{n/2} \|x_0 - x^*\|, \text{ where } \delta = (1 + \varepsilon/2) \]

(II) \( \lim_{n \to \infty} y_n = x^* \iff \lim_{n \to \infty} \lambda_n = 0 \)

(III) \( \sum_{n=0}^{\infty} \lambda_n < +\infty \) implies \( \lim_{n \to \infty} y_n = x^* \) so that \( \{x_n\} \) is almost T-stable.

**PROOF.** Proof follows from Theorem 6.2.

Now, we prove second main result of Chapter-6.
**THEOREM 6.3.** Let $X$ be a real Banach space with uniformly convex dual $X'$. Let $K$ be a closed, convex subset of $X$ and $T:K \to 2^X$ be Lipschitz multivalued $\phi$-strongly quasi-accretive and bounded operator. Let $p$ be the unique solution of the equation $Tx = 0$ with $p \in N(T)$. Define $T_c = K \to 2^X$ by $\xi'' = x - \varepsilon \cdot \xi'$ for each $x \in K$, where $\xi'' \in T_c x$ and $\xi' \in T x$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}_{n=0}^\infty$ in $K$ by $x_{n+1} = \xi''_n$, $n \geq 0$ where $\xi''_n \in T_c x_n$.

Then $\{x_n\}_{n=0}^\infty$ converges strongly to the unique solution of the equation $Tx = 0$ with

$$\|x_{n+1} - p\| \leq \delta \|x_n - p\|,$$

where $\delta = \left(1 - \frac{1}{2} \varepsilon\right)$, $\varepsilon \in (0,1]$ and $\varepsilon > 0$ is defined by $\varepsilon = \frac{1 + 2b(\varepsilon) \cdot M_2}{2[1 + b(\varepsilon) \cdot M_2]}$.

where $b$ be nondecreasing function, $M_2 = \frac{L}{\max\{LM_1,1\}}$, $M_1 = \sup\{x_n - p : p \in N(T)\}$ and $L$ be Lipschitzian constant.

**PROOF.** Existence of a fixed point follows from Deimling [22]. Define $S:(I-T)$ by $Sx = (I-T)x$ where $I$ be the identity operator, i.e.

$$\langle \eta - p, j(x - p) \rangle \leq \|x - p\|^2 - \phi(\|x - p\|)(\|x - p\|)$$

(6.9)

where $p$ be the solution of the equation $Tx = 0$ and $\eta \in Sx$, $p \in N(T)$.

Now from Picard type iteration formula,

$$x_{n+1} = \xi''_n,$$

where $\xi''_n \in T_c x_n$

$$= x_n - \varepsilon \cdot \xi'$$

$$= x_n - \varepsilon (x_n - \eta_n),$$

where $\eta_n \in Sx_n$

$$x_{n+1} = (1 - \varepsilon)x_n + \varepsilon \cdot \eta_n.$$  

(6.10)
Using Lemma 6.1 and inequality (6.10), we have
\[ \|x_{n+1} - p\|^2 = \|(1-\varepsilon)x_n + \varepsilon \cdot \eta_n - p\|^2 \]
\[ = \|(1-\varepsilon)(x_n - p) + \varepsilon (\eta_n - p)\|^2 \]
\[ \leq (1-\varepsilon)^2 \|x_n - p\|^2 + 2\varepsilon (1-\varepsilon) (\eta_n - p, j(x_n - p)) \]
\[ + \max \{ (1-\varepsilon)\|x_n - p\|, 1 \} \cdot \varepsilon \|\eta_n - p\| \cdot b(\varepsilon). \]

Using inequality (6.9), above becomes
\[ \|x_{n+1} - p\|^2 \leq (1-\varepsilon)^2 \|x_n - p\|^2 + 2\varepsilon (1-\varepsilon) \|\eta_n - p\|^2 \cdot \max \{ (1-\varepsilon)\|x_n - p\|, 1 \} \cdot \varepsilon \|\eta_n - p\| \cdot b(\varepsilon). \]

for any \( r > 0 \), there exists an integer \( N > 0 \) such that \( \|x_n - p\| \geq r/2 \). Then
\[ \liminf \Phi(||x_n - p||) \geq \Phi(r/2) > 0. \]
Since \( \varepsilon \) be an arbitrary and assumption
\[ \|x_n - p\| \to r \) implies \( \liminf \Phi(||x_n - p||) \to 0 \) our inequality becomes
\[ \|x_{n+1} - p\|^2 \leq (1-\varepsilon)^2 \|x_n - p\|^2 + (\max \{ (1-\varepsilon)\|x_n - p\|, 1 \} \cdot \varepsilon \|\eta_n - p\| \cdot \max \{ (1-\varepsilon)\|x_n - p\|, 1 \} \cdot b(\varepsilon). \]

(6.11)

Since
\[ \|\eta_n - p\| \leq 1 \|x_n - p\|, \]
\[ \|\eta_n - p\| \leq L M_1. \]
For \( M_1 \geq 0, M_2 \geq 0 \), let \( M_1 = \sup \{ x_n - p : p \in N(T) \} \) and \( M_2 = L M_1 \cdot \max \{ L M_1, 1 \} \). The inequality (6.11) becomes
\[ \|x_{n+1} - p\|^2 \leq (1-\varepsilon)^2 \|x_n - p\|^2 + \max \{ (1-\varepsilon)\|x_n - p\|, 1 \} \cdot \varepsilon \cdot M_2 \cdot b(\varepsilon). \]
(6.12)

Now we shall consider the two cases.

Case I. For \( x \in K \), there exists an integer \( N \) such that
\[ \max \{ (1-\varepsilon)\|x_n - p\|, 1 \} = 1. \]
(6.13)
It follows that \( (1-\varepsilon) \|x_n - p\| \leq 1 \), by using (6.10)
\[ \|x_{n+1} - p\| \leq (1-\varepsilon)\|x_n - p\| + \varepsilon\|\eta_n - p\| \]
\[ \leq 1 + M_1\varepsilon \cdot L \]
\[ \leq M_3 \]

for some constant \( M_3 \geq 0 \). Now, we have to show that \( \|x_n - p\| \leq M_3 \), for all \( n > N \). By induction, consider \( \|x_{n-1} - p\| \leq M_3 \). Using (6.10), we have

\[ \|x_n - p\| \leq (1-\varepsilon)\|x_{n-1} - p\| + \varepsilon\|\eta_{n-1} - p\| \]
\[ \leq (1-\varepsilon)M_3 + \varepsilon M_3 \]
\[ \leq M_3. \]

Thus \( \|x_n - p\| \leq M_3 \) for all \( n \geq N \). Hence \( \{x_n\} \) is bounded. Using (6.13), inequality (6.12) we have

\[ \|x_{n+1} - p\|^2 \leq (1-\varepsilon^2)\|x_n - p\|^2 + 1 \cdot \varepsilon M_3 \cdot b(\varepsilon). \]

Let \( \rho_n = \|x_n - p\|^2 \), \( \sigma_n = \varepsilon \cdot M_3 \cdot b(\varepsilon) \), \( \mu_n = \varepsilon^2 \), then above inequality will be

\[ \rho_{n+1} \leq (1-\mu_n)\rho_n + \sigma_n. \]

Hence by Lemma 6.2, \( \lim_{n \to \infty} \rho_n = \infty \) i.e. \( \lim_{n \to \infty} x_n = p \).

Case II. For all \( n \geq 0 \) such that

\[ \max \{(1-\varepsilon)\|x_n - p\|, 1\} = (1-\varepsilon)\|x_n - p\| \]

we have to show that case II is false. For this, we consider that case II is not false, then we have

\[ 1 \leq (1-\varepsilon)\|x_n - p\| \leq \|x_n - p\|, \quad \forall \, n \geq 0. \]

From inequality (6.12),

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\[ \|x_{n+1} - p\|^2 \leq (1 - \varepsilon^2)\|x_n - p\|^2 + (1 - \varepsilon)\varepsilon \cdot M \cdot b(\varepsilon) \]
\[ \|x_{n+1} - p\|^2 \leq (1 - \varepsilon^2)\|x_n - p\|^2 + (1 - \varepsilon)\varepsilon \cdot M \cdot b(\varepsilon)\|x_n - p\|^2 \]
\[ \leq [(1 - \varepsilon^2) + (1 - \varepsilon)\varepsilon \cdot M \cdot b(\varepsilon)]\|x_n - p\|^2 \]

Since \( p \) is continuous and \( \varepsilon > 0 \), therefore \( b(\varepsilon) \to 0 \). Hence, there exists an integer \( N_0 \), such that for all \( n > N_0 \), let \( b(\varepsilon) \cdot M \leq a \), where \( a \geq 0 \) be some constant. Then we have,

\[ \|x_{n+1} - p\|^2 \leq (1 - \varepsilon^2)\|x_n - p\|^2 + (1 - \varepsilon)\varepsilon \cdot M \cdot b(\varepsilon)\|x_n - p\|^2 \]
\[ \leq [1 - \varepsilon^2 + \varepsilon(1 + a)]\|x_n - p\|^2 \]
\[ \leq \delta\|x_n - p\|^2 \text{, where } \delta = 1 - \varepsilon^2(1 + a) + \varepsilon a \]
\[ \leq \delta^n\|x_0 - p\|^2 \to 0 \text{ as } n \to \infty. \]

Hence \( x_n \to p \) as \( n \to \infty \), which contradicts the assumption \( \|x_n - p\| \geq 1 \).

This completes the proof.

Following are the direct consequences of Theorem 6.3:

**Theorem 6.4.** Let \( X \) and \( K \) are same as in **Theorem 6.3**. Let \( T:K \to 2^X \) be Lipschitz multivalued \( \phi \)-strongly quasi-dissipative operator. Let \( p \) be the unique solution of the equation \( Tx = 0 \) with \( p \in N(T) \). Define \( T_\varepsilon:K \to 2^X \) by \( \xi_{\varepsilon} = x + \varepsilon \cdot \xi \) for each \( x \in K \), where \( \xi_{\varepsilon} \in T_\varepsilon x \in \xi \in Tx \). For arbitrary \( x_0 \in K \), define the sequence \( \{x_n\} \) in \( K \) by \( x_{n+1} = T_\varepsilon x_n \), \( n \geq 0 \) where \( \xi_{\varepsilon} \in T_\varepsilon x_n \).

Then \( \{x_n\} \) converges strongly to the unique solution of the equations. \( -Tx = 0 \) with \( \|x_{n+1} - p\| \leq \delta^n\|x_n - p\| \) where \( \delta = \left(1 - \frac{1}{2}\varepsilon\right)\) and \( \varepsilon > 0 \) is defined by

\[ \varepsilon = \frac{1 + 2b(\varepsilon) \cdot M_2}{2[1 + b(\varepsilon) \cdot M_2]} \]
where \( b \) be the nondecreasing function, \( M = LM_1 \cdot \max \{ LM_1, 1 \} \), \( M_1 = \sup \{ x_n - p : p \in N(T) \} \) and \( L > 0 \) be Lipschitzian constant.

**PROOF.** Since \( T \) be Lipschitzian multivalued \( \phi \)-strongly quasidissipative. Furthermore if \( T \) be quasi-accretive then \((-T)\) be quasidissipative. Then proof follows from Theorem 6.3.

**THEOREM 6.5.** Let \( X \) and \( K \) are same as Theorem 6.3 and \( T: K \rightarrow X \) be lipschitz \( \phi \)-strongly quasi-accretive, bounded operator. Let \( p \) be the unique solution of the equation \( Tx = 0 \) with \( p \in N(T) \). Define \( T_x = K \rightarrow X \) by \( T_{x} x = x - \varepsilon . T x \). For arbitrary \( x_0 \in K \), define the sequence \( \{ x_n \}^\infty_{n=0} \) in \( K \) by \( x_{n+1} = T_{\varepsilon} x_n, n \geq 0 \).

Then \( \{ x_n \}^\infty_{n=0} \) converges strongly to the unique solution of the equation \( Tx = 0 \) with

\[
||x_{n+1} - p|| \leq \delta^{n+1} ||x_0 - p||,
\]

where \( \delta = \left( 1 - \frac{1}{2} \varepsilon \right), \delta \in [0, 1] \) & \( \varepsilon > 0 \) is defined by \( \varepsilon = \frac{1 + 2b(\varepsilon) \cdot M_2}{2[1 + b(\varepsilon) \cdot M_2]} \), where \( b \) be nondecreasing function,

\[
M_2 = LM_1 \cdot \max \{ LM_1, 1 \}, M_1 = \sup \{ x_n - p : p \in N(T) \}
\]

and \( L > 0 \) be Lipschitzian constant.

**PROOF.** Proof follows from Theorem 6.3.

**THEOREM 6.6.** Let \( X \) and \( K \) are same as Theorem 6.3 and \( T: K \rightarrow X \) be lipschitzian \( \phi \)-strongly quasi-dissipative, bounded operator. Let \( p \) be the unique solution of the equation \( -Tx = 0 \) with \( p \in N(T) \). Define \( T_{\varepsilon} = K \rightarrow X \) by
$T_x x = x + \varepsilon \cdot T x$ for each $x \in K$. For arbitrary $x_0 \in K$, define the sequence 
\[ \{x_n\}_{n=0}^{\infty} \] in $K$ by $x_{n+1} = T_n x_n$, $n \geq 0$.

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique solution of the equation $-T x = 0$ with
\[ ||x_{n+1} - p|| \leq \delta^{2n+1} ||x_0 - p||, \]
where $\delta = \left(1 - \frac{1}{2} \varepsilon^2\right)$, $\delta \in [0,1]$ & $\varepsilon > 0$ is defined by 
\[ \varepsilon = \frac{1 + 2b(\varepsilon) \cdot M_1}{2[1 + b(\varepsilon) \cdot M_2]}, \]
where $b$ be nondecreasing function,
\[ M_2 = LM_1 \cdot \max \{LM_1, 1\}, \ M_1 = \sup \{x_s - p : p \in N(T)\} \]
and $L > 0$ be Lipschitzian constant.

**PROOF.** Proof follows from Theorem 6.4.