One of the aims of this thesis is to estimate baryon asymmetry in the wake of inflationary models of reheating in the early universe. To make the thesis self contained, in this chapter, we introduce the concept of inflation, reheating and the methods we use to calculate particle production in the early universe. All these concepts will be useful in the subsequent chapters. Modern cosmology is based on the standard model of cosmology, which is well supported by experimental evidence such as CMBR (COBE, WMAP), nucleosynthesis and the Hubble expansion of the universe. It also has some serious shortcomings which need attention. The inflationary model was proposed to overcome these shortcomings. Inflation is nothing but exponential expansion, which involves the introduction of a scalar field which is displaced from its vacuum value. As this field rolls slowly into its vacuum state, determined by the inflationary potential, exponential expansion results and solves some of the problems of the
Standard model of cosmology. Single or multi scalar fields can be involved in inflation, and we shall see that they are useful in models for baryogenesis. The reheating process occurs after the termination of inflation due to the quantum fluctuations of the inflaton field. Particle production at this stage can be described by parametric resonance and squeezed states. We shall briefly describe each of these processes in this chapter.

3.1 Standard Model of Cosmology.

The standard model of cosmology (standard big bang model) is based on the cosmological principle which states that the universe is isotropic and homogeneous. Starting with Einsteins equations \[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 8\pi G T_{\mu \nu}, \] (3.1)
where \( R_{\mu \nu} \) and \( R \) are the Ricci tensor and scalar, \( T_{\mu \nu} \) is the energy momentum tensor and \( G \) is the gravitational constant and assuming a perfect fluid description of the matter

\[ T_{\mu \nu} = -pg_{\mu \nu} + (P + \rho)u_{\mu}u_{\nu}, \] (3.2)
the dynamics of the universe is given by the Friedman-Robertson-Walker (FRW) solution \[ ds^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \] (3.3)
The coordinates \( r, \theta \) and \( \phi \) are comoving spherical coordinates obtained by rescaling the radial coordinate. \( k \), is the scalar curvature of 3-space which has values \( -1, 0 \) or \( +1 \), corresponding to open, spatially flat or closed universes respectively and \( a(t) \) is the expansion factor of the universe. The physical distance is given by \( R_p = a(t) \int \frac{dr}{\sqrt{1 - kr^2}} \) and for a flat \((k = 0)\) universe \( R_p = a(t)r \). The velocity is given by \( V = HR_p \left( \frac{dr}{dt} = 0 \right) \), where \( H \) is the Hubble parameter given by \( H = \frac{\dot{a}}{a} \).
If we assume the $T_{\mu\nu} = \text{diag}[-\rho, P, P, P]$ where $\rho$ is energy density and $P$ is the pressure, then from the continuity equation we get

$$\dot{\rho} = -3H(\rho + P). \quad (3.4)$$

Combining this with Einstein’s equation (3.1) we get the Friedman equation:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3}G\rho. \quad (3.5)$$

In terms of Hubble’s constant we have:

$$\frac{k}{H^2a^2} = \frac{\rho}{3H^2/8\pi G} - 1 = \Omega - 1, \quad (3.6)$$

where $\Omega = \frac{\rho}{\rho_c}$ is the ratio between the density and the critical density required to close the universe and this critical density is given by $\rho_c = \frac{3H^2}{8\pi G}$. $\Omega > 1$, $\Omega = 1$ and $\Omega < 1$ correspond to open, flat and closed universes respectively.

Another important assumption of the standard model of cosmology is that the matter is described by a classical ideal gas, with an equation of state $P = w\rho$, where $w$ is constant of proportionality independent of time. The energy density $\rho$ goes as $\rho \propto a^{-3(1+w)}$. For the radiation dominated era $P = \frac{1}{3}\rho$, which gives $\rho \propto a^{-4}$. For the matter dominated era $P = 0$ gives $\rho \propto a^{-3}$. For the vacuum energy dominated era $P = -\rho$, which gives $\rho \propto \text{const}$.

The problems of standard model of cosmology are

- **The Horizon Problem:**
  The horizon problem arises from the fact that the transfer of information is limited by the speed of light. This results in a particle horizon which limits the separation of two regions that are in causal contact. The CMBR shows that the universe is isotropic if the universe is radiation dominated or matter dominated since the big bang. The particle horizon corresponds to approximately $R_p$ which is $2^\circ$ of the sky, so they would be no mechanism for distant objects to have same temperature through out the universe to make it homogenous and isotropic.
• The Flatness Problem:
In the equation (3.6), the observations from CMBR give a value of $\Omega \approx 1$. Therefore, since $\Omega = \frac{\rho}{\rho_c}$, the total energy density of the early universe must be approximately equal to the present critical density. The flatness problem is the question of why the initial conditions correspond to the universe being close to spatial flatness? The problem is that any small perturbation will cause a fluctuation of the energy density to grow exponentially. Thus, it requires extreme fine tuning to maintain the density of the early universe to this critical density value.

• The Entropy Problem:
The entropy problem is related to the flatness problem in the following way [51]. It is known that the entropy in a comoving volume stays constant in an adiabatic expansion. The entropy of the universe is given by $S_U = T^3H^{-3}$, where $H$ is the Hubble’s constant. Rewriting the Friedman equation in terms of entropy we get $\Omega - 1 = \frac{\kappa m^2_p}{S_U^{2/3}T^2}$. Since we have seen from the flatness problem that $\Omega \approx 1$, the value of the entropy $S_U \approx 10^{60}$, which is very very large. This reliance on fine tuned initial conditions is called flatness problem.

3.2 Inflation.
The idea of inflation [23] is proposed to overcome the problems of standard model of cosmology. Inflation is defined to be any epoch of the universe dominated by the negative pressure $\rho = -P$ vacuum density. With this condition, the Friedman equation (3.5) becomes

$$H = \frac{\dot{a}}{a} = constant,$$  \hspace{1cm} (3.7)

therefore

$$a = e^{Ht}.$$  \hspace{1cm} (3.8)
Thus inflation corresponds to an exponential expansion of the universe. With this hypothesis, the problems in the standard model of cosmology are solved. Substituting into eq (3.6), we see that with the large value of the expansion parameter, we automatically get $\Omega \approx 1$. Thus the flatness problem is solved. We have seen that the entropy problem is related to the flatness problem, thus the entropy problem is also solved. Inflation solves the Horizon problem by postulating that prior to the inflation, in a small radius of $10^{-23}$ cm corresponding to a time scale of $10^{-36}$ sec, before inflation, the entire universe is causally connected and achieves equilibrium to become isotropic. Inflation then expanded the universe rapidly, freezing in these properties, so that the universe evolved to an almost homogeneous, isotropic state. The information needed to change it from that state is not in causal contact, thus the homogeneity and isotropy is maintained. In these ways, inflation is a solution to the problems of standard model of cosmology.

In order to drive the universe into an inflationary phase, scalar field(s) known as inflaton(s) were proposed. The initial energy density of the universe is stored in the potential energy of the inflaton field. The inflaton is initially at its highest energy state. It rolls down to its minimum energy state via random fluctuations, which trigger a phase transition, and cause the inflaton releases its potential energy in the form of radiation and matter in a process known as reheating. Various inflationary scenarios have been proposed. Most of these vary with in the form of the inflaton potential, the number of inflaton fields and the order of phase transition.

In inflationary models, the inflaton field $\phi$ has a potential energy $V(\phi)$ which is flat near $\phi = 0$ and steep near $\phi = \phi_{\text{min}}$. At high cosmic temperatures $V(\phi)$ is flat due to temperature corrections. As the inflaton field rolls slowly down the potential to the state $\phi = 0$, during which the energy density is nearly constant, the scale factor grows exponentially. As the temperature drops, the effective potential $V(\phi)$ develops a small barrier separating the local minimum at $\phi = 0$ and the vacua at $\phi = \phi_{\text{min}}$. The tunneling of $\phi$ from $\phi = 0$ to $\phi = \phi_{\text{min}}$ gives rise to a phase transition. The first inflationary model proposed was based on first
order phase transition and called old inflation and was later on modified to be a second order phase transition model [52].

An example of a theoretical model for inflation is one with a Lagrangian density of the form [6, 33]

\[ L = g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi - V(\phi), \]  

(3.9)

where \( V(\phi) \) is the potential of the scalar field. These contributes to the energy momentum tensor by

\[ T^{\mu\nu} = \frac{\partial L}{\partial (\partial_{\mu} \phi)} \partial^{\nu} \phi - g^{\mu\nu} L. \]  

(3.10)

In the homogeneous state, the energy momentum tensor becomes a perfect fluid, which is given in eq (3.2). The value of \( \rho \) and \( P \) in terms of scalar field \( \phi \) are given by

\[ \rho = \frac{1}{2} \ddot{\phi}^2 + V(\phi), \]  

(3.11)

\[ P = \frac{1}{2} \ddot{\phi}^2 - V(\phi). \]  

(3.12)

Putting eq (3.11) and (3.12) in the Friedman equation (3.5) we get

\[ H^2 = \frac{8\pi G}{2} \left( \frac{1}{2} \ddot{\phi}^2 + V(\phi) \right). \]  

(3.13)

The equation for an expanding universe containing a homogeneous scalar field is given by

\[ \dddot{\phi} + 3H \dot{\phi} + \frac{d}{d\phi} V(\phi) = 0. \]  

(3.14)

To produce enough inflation and rapid reheating after the inflation, the potential should be flat initially and steep afterwards. The scalar field would then be expected to roll slowly on the flat potential and the \( \dddot{\phi} \) can be neglected. Then the equation of motion becomes

\[ 3H \dot{\phi} + \frac{d}{d\phi} V(\phi) = 0, \]  

(3.15)

and the Friedman equation can be approximated as

\[ H^2 = \frac{8\pi G}{2} V(\phi). \]  

(3.16)
The slow-roll [33] parameters are defined as
\[ \epsilon = -\frac{\dot{H}}{H^2} = 4\pi G \frac{\dot{\phi}^2}{H^2} = \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2, \]
and
\[ \eta(\phi) = \frac{1}{8\pi G} \frac{V''}{V}, \]
where, the first measures the slope of the potential and the second, the curvature. The necessary conditions for the slow-roll approximation to hold are
\[ \epsilon << 1; |\eta| << 1. \]
These slow-roll condition are sufficient for inflation to take place.

As stated earlier, the original inflationary model proposed by A. Guth[23] in 1981 was based on an inflationary potential that under goes first order phase transition. The early universe is supercooled and goes into a metastable state which is called the false vacuum. The equation of state in the false vacuum approaches \( p = -\rho \), and inflation occurs. After inflation, the scalar field tunnels through the false vacuum to the true vacuum. The tunneling produces a small bubble of true vacuum in the sea of false vacuum, expanding at the speed of light. The problem with the old inflationary model is that it requires a large cosmological constant. If the bubble walls expand with the speed of light, the space between the bubbles expands exponentially, and the bubbles at the present time will be much smaller than our apparent horizon. This is in contradiction with the observed isotropy of the CMBR. This model also has the problem that it will not reheat properly. When the bubbles nucleated, they do not generate any radiation. Radiation could only be generated in collisions between bubble walls.

To overcome these limitations, the new inflationary model was proposed independently by A. Linde [52], and by A. Albrecht and P. Steinhardt [53]. This model is based on a double well inflaton potential, which undergoes second order phase transition. In one version of the model the Coleman-Weinberg potential is used,
\[ V(\phi) = \frac{1}{4} (\phi^2 - \sigma^2)^2. \]
At $T < T_c$ ($T_c$ being the critical temperature), thermal fluctuation trigger the instability of field $\phi$ at $\phi(x) = 0$ and then $\phi(x)$ goes to a global minimum $\phi(x) = +\sigma$ or $\phi(x) = -\sigma$. Within the fluctuations region $\phi(x)$ is homogeneous, neglecting spatial gradient. The slow rolling conditions (3.17) and (3.18) are satisfied in this region as required for inflation.

Because of fine tuning problems, to satisfy the slow roll conditions, the inflaton must have a very small mass. Other inflationary models have been proposed, but the basic philosophy is the same. A few of these models are summarized below.

The chaotic inflationary model, proposed by A. Linde [52] in 1986, uses a potential $V(\phi) = \frac{m^2}{2} \phi^2$. This potential has a minimum at $\phi = 0$. $\phi$ oscillates at minimum if universe does not expand and the equation of motion is given by

$$\ddot{\phi} + 3H\dot{\phi} + m^2\Phi = 0,$$

(3.21)

the above equation is like a harmonic oscillator, with a friction term $3H\dot{\phi}$. Initially the scalar field $\phi$ and $H$ are too large, therefore the friction term in the equation (3.21) is large and the scalar field moves very slowly as a ball in a viscous liquid. At this stage, the energy density of scalar field, unlike the density of matter, remains almost constant, and the expansion of universe is continued with much greater speed. Due to the rapid growth of the scale factor of the universe and slow motion of the field $\phi$, we can make the following approximations $\ddot{\phi} \ll 3H\dot{\phi}$, $H^2 \gg \frac{K}{a^4}$, $\dot{\phi}^2 \ll m^2\Phi$. The equation of motion reduces to

$$3\frac{\dot{a}}{a} = -m^2\Phi.$$

(3.22)

The Friedman equation is given by

$$(\frac{\dot{a}}{a})^2 = 2\pi G(\dot{\phi}^2),$$

(3.23)

leading to,

$$\frac{\dot{a}}{a} = 3m\Phi G(\sqrt{\frac{\pi}{3}}),$$

(3.24)

and gives

$$a = e^{Ht},$$

(3.25)
where \( H = 3m\Phi G(\sqrt{\frac{3}{\pi}}) \). Thus it gives rise to inflation.

The disadvantage of the above inflationary models is that they require tiny parameters in order to reproduce the present observations. To overcome this A. Linde proposed hybrid inflation with two scalar fields \( \phi \) and \( \chi \). The field \( \phi \) provides the vacuum energy density which drives inflation and field \( \chi \) is the slowly varying field during inflation. The effective potential is given by

\[
V(\phi, \chi) = k^2(M^2 - \frac{\phi^2}{4}) + \frac{\lambda^2\phi^2\chi^2}{4} + \frac{m^2\chi^2}{2},
\]

(3.26)

where \( k, \lambda > 0 \) are dimensionless constants and \( M, m > 0 \) are mass parameters. The vacua lie at \( \phi = \pm 2M, \phi = 0 \). For \( m = 0 \), \( V \) has a flat direction at \( \phi = 0 \), where \( V = k^2M^4 \) and the mass of \( \phi \) is \( m^2 = -k^2M^4 + \frac{\lambda^2\chi^2}{2} \). So, for \( \phi = 0 \) and \( |\chi| > \chi_c \equiv \frac{2kM}{\lambda} \), we obtain a flat valley of minima. For \( m \neq 0 \), the valley acquires a slope and the system can inflate as the field \( \chi \) slowly rolls down this valley. The slowroll conditions are satisfied and inflation sets in and continues till \( \chi \) reaches \( \chi_c \), then the inflation terminates. It is followed by a waterfall i.e. a sudden entrance into an oscillatory phase about a global minimum at which point reheating starts.

3.3 Reheating.

During inflation, the universe expanded exponentially and the temperature of the universe dropped very low, leaving it devoid of particles. For matter and radiation to form, the universe must have gone through a period of reheating after the inflation. Successful inflation must transfer energy in the inflaton field to radiation and reheat the universe to at least 1 MeV for Nucleosynthesis to occur. Quasiperiodic evolution of the inflaton field leads to the creation of particles. The process by which the large potential energy density present during inflation gets converted into radiation and matter at the end of inflation is known as reheating. This is an important concept for the ideas presented in this thesis, since matter
and antimatter must be generated in the reheating epoch. The first theory of particle production during reheating [18] was proposed using first order Born approximation for an oscillating inflaton field. This approach had some shortcomings and did not give sufficient particle production. In order to rectify this situation, parametric resonance effects were included. Parametric resonance effects during reheating can enhance the rate of particle production considerably.

Particle creation or annihilation generally occurs in a time-dependent background, the number of created particles grows exponentially, either when the background is periodic in time, or, when the coupling constant changes suddenly. This results in parametric amplification of the long wavelength modes, which in turn give rise to far from equilibrium particle production.

The first reheating model in which particle production is obtained through parametric resonance phenomenon, was proposed by Y. Shtanov, J. Traschen and R. H. Brandenberger [18]. They considered particle production using the interaction Lagrangian

$$L_{\text{int}} = -f \phi \bar{\psi} \psi - (\sigma \phi + h \phi^2) \chi^2.$$  \hspace{1cm} (3.27)

Here \( \bar{\psi} \) and \( \psi \) describe spinor particles and \( \chi \) describes scalar particles with masses given by \( m_{\psi} \) and \( m_{\chi} \), respectively. The coupling constants \( f \) and \( g \) are dimensionless, while \( \sigma \) has the dimensions of mass. The inflaton field \( \phi \) is treated as a classical external field. The scalar field \( \phi \) undergoes quasi-periodic evolution, with a mode decomposition

$$\phi^2 = \bar{\phi}^2 + \sum_{n=1}^{\infty} \zeta_n \cos(n \omega t),$$  \hspace{1cm} (3.28)

where, \( \phi_n \) and \( \zeta_n \) are slowly varying amplitudes with time and \( \bar{\phi}^2 \approx \frac{\phi_0^2}{2} \). The evolution of a particular mode \( \chi_k \) of the quantum scalar field \( \chi \), in the presence of the rapidly oscillating classical scalar field \( \phi \) is given by

$$\ddot{\chi}_k + 3H \dot{\chi}_k + (k^2 + m^2_{\chi} + 2\sigma \phi + 2h \phi^2) \chi_k = 0,$$  \hspace{1cm} (3.29)
where $k = k/a$ is comoving wave number. Performing the transformation $\chi_k = \frac{Y_k}{a}$, the above equation becomes

$$\ddot{Y}_k + (\omega_k^2 + g(\omega t))Y_k = 0,$$  \hspace{1cm} (3.30)

where,

$$\omega_k^2 = k^2 + m_\chi^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} + 2h\bar{\phi}^2$$  \hspace{1cm} (3.31)

and

$$g(\omega t) = 2\sigma\phi + 2h(\phi^2 - \bar{\phi}^2).$$  \hspace{1cm} (3.32)

g(\omega t) is (to a good approximation), a periodic function of time $t$.

$$g(x) = \sum_{n=-\infty}^{\infty} g_n e^{inx},$$  \hspace{1cm} (3.33)

where, $g_n$ is the amplitude satisfying $g_n^* = g_{-n}$. It is convenient to introduce the following quantities $g_0 = |g_n| e^{i\alpha_n}$ and $\omega_0 = \left(\frac{p}{q}\omega\right)^2 + \epsilon\Delta$, where $\alpha_n$ is the phase and $p$ and $q$ are integers.

Then 3.29 is the equation of motion for a time dependent harmonic oscillator and can be approximated by the Mathieu’s equation. This equation has parametric resonance for certain values of $\omega_k - \left(\frac{p}{q}\omega\right)^2 \equiv \Delta_n < |g_n|$. This is valid in the case when the frequencies of $\omega_k$ and $\omega$ change slowly with time. From this condition it follows that $\omega_k^2 - k^2 \equiv m_\chi^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} + 2h\bar{\phi}^2 \ll \omega^2$ and $k^2 \gg m_\chi^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H}$ for the resonance values of $k^2$.

Using the time dependence of the coefficients in the equations (3.29) and (3.30) and the Bogoliubov transformation (squeezed states), leading to $\chi$ particle production, ref [18] found the mean occupation number for the generated $\chi$ particles, per mode to be

$$N_k = |\beta_k|^2 \approx sinh^2\left(\int \left(\frac{1}{n\omega}\sqrt{|g_n|^2 - \Delta_n}dt\right)\right).$$  \hspace{1cm} (3.34)

As long as $\Delta_n < |g_n|$, the growth is exponential and a large population of particles can be produced by the oscillating coherent source.

To see the connection of squeezed states to particle production through the parametric resonance phenomenon in the presence of the background time dependent oscillator, will be considered in the next section.
3.4 Squeezed States and Particle Production.

To generalize the connection between particle production and squeezed states with relevance to this thesis, we consider the following simple action (which admits spontaneous symmetry breaking) as an example [54],

\[ S = \int d^4x \frac{1}{2} [\partial_\mu \phi^a \partial_\nu \phi^a - \frac{1}{4} \lambda (\phi^a \phi^a - v^2)^2], \]  

(3.35)

For simplicity and illustrative purposes, we use the Hartree-Fock approximation and \( \phi^a \phi^a \) term is replaced by its spatial average \( \langle \phi \rangle \), to get terms up to quadratic order in \( \phi^a \).

Quantizing the field \( \phi^a \) \((a\) are internal indices, which we shall not consider in the discussion) by carrying out a mode expansion

\[ \phi^a(x,t) = \int \sqrt{\frac{1}{2}} \frac{d^3k}{2\omega(k)(2\pi)^3} (a_k e^{ik\cdot x} + a_k^\dagger e^{-ik\cdot x}), \]  

(3.36)

we get the Hamiltonian

\[ H = \int d^3k \omega(k) \left( \frac{\Omega^2(k,t)}{\omega(k)} (a_k^\dagger a_k + a_k a_k^\dagger) - \frac{\omega^2(k) - \Omega^2(k,t)}{4\omega(k)} [a_k^\dagger a_{-k}^\dagger + a_{-k} a_k] \right), \]  

(3.37)

where, \( \Omega^2(k,t) = k^2 + \lambda \langle \phi \rangle^2 (t) - v^2 \), \( \omega^2(k) = \Omega(k,t \to \infty) = \sqrt{k^2 + m^2} \) and \( m^2 = \frac{1}{2} \lambda v^2 \).

The above Hamiltonian can be diagonalized using squeezing transformations given by

\[ A_k(t) = \mu a_k(t) + e^{i\theta} \nu a_k^\dagger(t) = \cosh(r)a_k(t) + e^{i\theta} \sinh(r)a_k^\dagger(t) = S^\dagger(\xi,t)a_k S(\xi,t), \]  

\[ A_k^\dagger(t) = e^{-i\theta} \mu a_k^\dagger(t) + \nu a_k(t) = e^{-i\theta} \cosh(r)a_k^\dagger(t) + \sinh(r)a_k(t) = S^\dagger(\xi,t)a_k S(\xi,t). \]  

(3.38)

(3.39)

where the unitary transformation for squeezing \( S(\xi,t) \) is given by

\[ S(\xi,t) = \exp \left[ \frac{1}{\sqrt{2}} \int d^3k [\xi(a_k^\dagger)^2 - \xi^*(a_k)^2] \right]. \]  

(3.40)

and \( \xi = re^{i\theta} \) is a squeezing parameter [55]. The diagonalized Hamiltonian is

\[ H = \int d^3k [\Omega(k,t)(A_k^\dagger(t)A_k(t)) + \frac{1}{2}]. \]  

(3.41)
For each mode $a_k(t)$, the initial vacuum is given by $a_k(t)|0,0>=0$ and for each mode $A_k(t)$, the final vacuum is given by $A_k(t)|0,t>=0$. The initial vacuum is related to the final vacuum at a later time $t$ by

$$|0,t> = \exp[\frac{1}{\sqrt{2}} \int d^3k [\xi(a_k^\dagger)^2 - \xi^*(a_k)^2]]|0,0>.$$ (3.42)

This shows us that the final vacuum is populated by the following number of free quanta. 

The average number of quanta for each mode is realted to the squeezing parameter by

$$N_k = \langle 0,0|A_k^\dagger(t)A_k(t)|0,0\rangle = |\nu(t)|^2,$$ (3.43)

the vacuum is populated at later time by physical particles.

To calculate the squeezing parameter, we go over to the “co-ordinate representation” by defining the operators

$$A_k(t) = \frac{1}{2\sqrt{\Omega(k,t)}}(\Omega(t)\pi(k,t) + ip(k,t)),$$ (3.44)

$$A_k^\dagger(t) = \frac{1}{2\sqrt{\Omega(k,t)}}(\Omega(t)\pi(k,t) - ip(k,t)),$$ (3.45)

where $\pi$ and $p$ represents co-ordinate and momentum. Putting eq (3.45) in (3.41), the diagonalised Hamiltonian reduces to the following

$$H(t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \left[ (\Omega(k,t))^2 \pi^2(k,t) + p^2(k,t) \right] = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} H_k(t),$$ (3.46)

which resembles a time dependent harmonic oscillator. In the mean field approximation this Hamiltonian operator acts on a wave function $\psi$, which is mode decomposed into

$$|\psi > = \prod_k |\psi >_k.$$ (3.47)

by

$$-i\frac{\partial}{\partial t} |\psi >_k = H_k(t)|\psi >_k = \frac{1}{2} \left[ (\Omega(k,t))^2 \pi^2(k,t) + p^2(k,t) \right] |\psi >_k.$$ (3.48)

The solution of the eq(3.48) in the coordinate representation is a Gaussian

$$< \pi |\psi > (t) = C(t) \exp[-B(t)\pi^2],$$ (3.49)
which is a squeezed state in the co-ordinate representation. The parameters $r$ and $\xi$ are related to $B$ by $\cosh(r) = \frac{\omega^2 + |B|^2}{4\omega ReB}$ and $\sin(2\xi) = \frac{ImB}{\sinh(r)ReB}$, this gives $B(t) = \frac{i\dot{\psi}}{\psi}$. The equation of motion of $\psi$ is given by

$$-\frac{d^2\psi}{dt^2} + (k^2 + \lambda(<\phi^2> - v^2))\psi = 0.$$  

(3.50)

This equation can be formally viewed as the stationary Schrödinger equation for the wave function $\psi(t)$ of a particle with mass $m = 1/2$, on a line $t$, having energy $\omega^2$ and moving through a potential $V(t) = \lambda(<\phi^2> - v^2)$.

$$\frac{d^2\psi}{dt^2} + (E - V(t))\psi = 0.$$  

(3.51)

Consider a non-equilibrium scenario such that $<\phi^2> = 0$ in the region $t < t_1$ (symmetry is restored), i.e $V = -\lambda v^2$ and $<\phi^2> = v^2$ in the region $t > t_1$ (symmetry is broken) i.e $V = 0$, then this equation (3.51) becomes equivalent to a potential barrier problem [56, 29].

The usual solutions to the barrier are given by

$$\psi(t)_- = e^{-i\omega_i t} + R^* e^{i\omega_i t}, \quad t < t_1$$  

(3.52)

$$\psi(t)_+ = T e^{-i\omega_f t}, \quad t > t_1.$$  

(3.53)

where $R$ and $T$ are reflection and transmission coefficients across the barrier and $\omega_i$ and $\omega_f$ are the initial and final frequencies ($\Omega$ and $\omega$), defined by

$$T = \frac{|\psi_+|^2 e^{-i\omega_f t}}{|\psi_-|^2 e^{-i\omega_i t}},$$  

(3.54)

and $|R|^2 + |T|^2 = 1$.

Now considering the particle creation problem modelled by a time dependent harmonic oscillator, the solution is

$$\psi(t)_- = \frac{e^{-i\omega_i t}}{\sqrt{2\omega_i}}, \quad t < t_1,$$  

(3.55)

$$\psi(t)_+ = \frac{\mu e^{-i\omega_f t}}{\sqrt{2\omega_f}} + \nu e^{i\omega_f t}, \quad t > t_1,$$  

(3.56)
Comparing the two solutions the transmission coefficient \( T \) can be related to squeezing parameter \( r \) through \( \sinh^2(r) = |\nu|^2 = \frac{R}{T} \). Therefore from the calculation of reflection and transmission coefficient one can get the Bogoliubov coefficients and number of particles generated. This method is a particularly method of calculating the Bogoliubov coefficients and we shall be using it in later chapters.

We emphasize that the analogy with Schrodinger wave equation of eqn(3.48) is purely mathematical. When we consider a quantum field, the modes \( \phi_k^a \) are not particles moving in real space and time \( t \). The mode functions do not represent reflected or transmitted waves. The quantum-mechanical analogy is used only to visualize the qualitative behavior of the mode functions.

The same analogy can be taken over to the expanding space times, where the usual effective frequency is given by \( \omega^2 = \frac{k^2}{a^2} + m^2 - \frac{a''}{a} \). Extension of this method to curved space-times is used to explain the Unruh effect and Hawking radiation.

One cannot define the usual vacuum for these situation, as the modes do not oscillate but behave as growing and decaying exponents, so the analogy with a harmonic oscillator breaks down. There one can only define the vacuum in the space time where the expansion is constant ans spacetime is flat in the remote past and the distant future. It can be shown that the two vacua in these regions are related by Bogoliubov transformations, giving rise to presence of particles in the later times.

In the next chapter we will explore these techniques to generate the baryon asymmetry in the context of minimally coupled gravitational baryogenesis.