Our main emphasis is on the investigations of the possible general relativistic effects on the disk structure and dynamics. To this end we first study the disks in Newtonian gravitational fluid. We develop some steady state solutions and compute the frequencies of oscillations, both by global and local analysis. We also investigate the dependence of the frequency on various factors. The results of this part of our study are then used extensively in the subsequent chapters to find the differences produced by replacing the gravitational field by general relativity and thereby filtering out the general relativistic effects.

Investigation's on disk dynamics in Newtonian gravitational field was carried out by Chakraborty and Prasanna (1981) (Here after referred to as (CP.82) amongst many others. We find some more steady state solutions. Besides we develop an equation for the frequency of the local oscillations which is more general than that of (CP.82). Our equation is applicable for any steady state solutions. Further, we also investigate the influence of various factors on the frequency of the global axi-symmetric oscillations. This forms a kind of ground work for the work reported in subsequent chapters for extracting out the general relativistic effects.
3.1 Steady State Solutions :

We consider a non-self gravitating perfect fluid disk rotating around a compact object of mass . The gravitational field of the compact object is described by Newtonian formulation. The equations governing the dynamics of the disk are obtained from equation (2.4.10) to (2.4.15) by putting \( \omega = 0 \) and \( \epsilon = \omega \). We then have the momentum equations.

\[
\rho \left[ \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right] = - \nabla P + \frac{1}{2} \left( \frac{\partial \mathbf{V}}{\partial t} \right)^2 + \mathbf{F} - \frac{2}{3} \frac{GM}{r^2} \mathbf{V}
\]

(3.1.1)

\[
\rho \left[ \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right] = - \nabla P + \frac{1}{2} \left( \frac{\partial \mathbf{V}}{\partial t} \right)^2 + \mathbf{F} - \frac{2}{3} \frac{GM}{r^2} \mathbf{V} - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \mathbf{V}}{\partial \theta} \right) - \frac{1}{\sin \theta \cos \phi} \frac{\partial}{\partial \phi} \left( \sin \theta \cos \phi \frac{\partial \mathbf{V}}{\partial \phi} \right)
\]

(3.1.2)

(3.1.3)

and the continuity equation

\[
\rho \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 V^h \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta V^\theta \right) + \frac{V^\phi}{r \sin \theta} \frac{\partial}{\partial \phi} \right] + \frac{\partial}{\partial t} = 0
\]

(3.1.4)

where in \( P, \rho, \mathbf{V} \) \((r, \theta, \phi)\) denote pressure, density and the component of velocity in spherical co-ordinates and the rate of change operator is

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla = \frac{\partial}{\partial t} + \mathbf{V} \cdot \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}
\]

(3.1.5)
To make the system of equations close, we chose either an equation of state as expressed by the adiabatic law \( (2.1.20) \)

\[
\frac{\partial}{\partial t} \left( \rho \mathbf{v} \right) = 0
\]  

(3.1.6)

or an isentropic equation

\[
\rho = \frac{c^2}{\gamma}
\]  

(3.1.7)

and study the resultant steady state structure.

\( \gamma \) is the adiabatic index.

We assume that in steady state the flow is purely rotational and axisymmetric and put \( \mathbf{v}_r = 0, \mathbf{v}_\theta = 0 \) and \( \mathbf{v}_z = \mathbf{v}_0 \).

For the dynamical variables are independent of \( \phi \) and \( t \). We have thus the following steady state equations for axisymmetric case, for the adiabatic case.

\[
\gamma \left[ \frac{m c^2}{\rho^2} - \frac{\mathbf{v}_r^2}{\rho} \right] = - \frac{\partial p_i}{\partial \rho}
\]  

(3.1.8)

and

\[
\frac{f_c \mathbf{v}_r^2}{\gamma} \cot \theta = \frac{\partial p_i}{\partial \theta}
\]  

(3.1.9)

all the other equations are identically zero.

**Solution 1:** We look for the solutions of the form

\[
p_i \sim f_1(\gamma) \sin k \theta + f_2(\gamma) \sin k \theta, \quad \mathbf{v}_r \sim f_1(\gamma) \sin k \theta, \quad \mathbf{v}_z \sim f_2(\gamma)
\]

and choose the form of the radial dependencies to obtain physically meaningful solutions of (3.1.8) and (3.1.9).
We obtain the following solutions

\[ \tilde{p}_n = \tilde{\alpha} \left[ \frac{1}{\gamma_2^{\tilde{n}} \tilde{\epsilon}_2^{\tilde{n}+1}} \right] \]

(3.1.10)

\[ \tilde{p}_n = \left[ \frac{\gamma_2^{\tilde{n}} \tilde{\epsilon}_2^{\tilde{n}+1}}{\tilde{\alpha}} \right] \frac{1}{\gamma_2^{\tilde{n}} \tilde{\epsilon}_2^{\tilde{n}+1}} \tilde{\gamma}_2^{\tilde{n}} \theta \left( 1 - \frac{\tilde{n}}{\tilde{k}} + \frac{\tilde{\gamma}_2^{\tilde{n}}}{\tilde{\gamma}_2^{\tilde{n}+1}} \right) \]

(3.1.11)

and

\[ \tilde{\psi}_n = \frac{\tilde{M}_n}{\tilde{\tau}_n} \left( 1 - \frac{\tilde{n}}{\tilde{k}} + \frac{\tilde{\gamma}_2^{\tilde{n}}}{\tilde{\gamma}_2^{\tilde{n}+1}} \right)^{-1} \]

(3.1.12)

We choose \( \gamma_{20} \) which insures that \( \gamma_{0} \) is the inner edge of the disk at \( \theta = \gamma/2 \) plane. Also we choose \( \gamma \) to ensure \( f_{\theta} \) and \( \tilde{\psi}_{n} \) to be positive within the disk. The parameter \( \gamma_{20} \) determines the outer edge of \( \gamma_{b} \) of disk at \( \theta = \gamma/2 \) plane. \( \tau_{n} \) is a dimensional constant we choose \( \gamma = 1 \) and rewrite the above equation from (3.1.10) to (3.1.12) in the dimensionless form. We have

\[ \tilde{p}_{\theta} = \tilde{R} \tilde{a} \tilde{e} - \tilde{R} \tilde{a} / \tilde{R} \tilde{b} \]

(3.1.13)

\[ \tilde{p}_{\theta} = k \tilde{R} \tilde{a} \tilde{e} - \tilde{R} \tilde{a} / \tilde{R} \tilde{b} \tilde{\gamma} \theta \left( 1 - \frac{\tilde{n}}{\tilde{k}} + \frac{\tilde{\gamma}_2^{\tilde{n}}}{\tilde{\gamma}_2^{\tilde{n}+1}} \right) \]

(3.1.14)

\[ \tilde{\psi}_{\theta} = \frac{1}{\tilde{R}} \left( 1 - \frac{\tilde{n}}{\tilde{k}} + \frac{\tilde{\gamma}_2^{\tilde{n}}}{\tilde{\gamma}_2^{\tilde{n}+1}} \right)^{-1} \]

(3.1.15)

where

\[ \tilde{R} = \pi / \tilde{m}, \quad \tilde{R}_{c} = \pi / \tilde{m}, \quad \tilde{R}_{e} = \pi / \tilde{m}, \]

\[ \tilde{\tau}_{\theta} = \tilde{\tau} / \tilde{\epsilon} \]
Solution II:

Another set of solutions of the equations (3.1.8) and (3.1.9) was presented by (Cp.82) in which \( f_0 \) was regarded as a function of \( x \) alone. The solutions in the dimensionless form are reproduced below.

\[
\begin{align*}
\frac{p}{p_{\infty}} &= \frac{A_1}{\kappa_1} \frac{x_0}{x_{\infty}}^{K_1} - \frac{x}{x_{\infty}}^{K_1} + \phi_1 \\
\psi &= \frac{x}{x_{\infty}}^{K_1} \\
\frac{\psi}{x_{\infty}^2} &= \frac{A_1}{\kappa_1} \frac{x_0}{x_{\infty}}^{K_1} \frac{x}{x_{\infty}}^{K_1}
\end{align*}
\]

(3.1.16) (3.1.17) (3.1.18)

where

\[
A_1 = \frac{\kappa_1 - \kappa_2}{\kappa_1 - \kappa_2} \frac{R_b^{x-1} - R_a^{x-1}}{R_b^K - R_a^K}
\]

(3.1.19)

and

\[
B_1 = \frac{1}{\kappa_1 - \kappa_2} \frac{R_a^{x-1} R_b^K - R_a^K R_b^{x-1}}{R_b^K - R_a^K}
\]

(3.1.20)

where \( R_b = \frac{m}{n+1} \)

Solution III:

For an isentropic flow we have to use equation (3.1.7) as an equation of state along with the equation (3.1.8) and (3.1.9). Such a set of equations have been used by Papaloizou and Pringle (1987) in their study of non-axisymmetric perturbations using
(3.1.7) equations (3.1.8) and (3.1.9) can be reduced to

\[ \frac{\partial C_0}{\partial \eta} + \frac{\partial C_1}{\partial \xi} + \frac{\partial C_2}{\partial \eta} \left( \frac{1}{f_{\eta}} \right) \]

(3.1.21)

and

\[ \frac{\partial C_0}{\partial \xi} + \frac{\partial C_1}{\partial \eta} \left( \frac{1}{f_{\eta}} \right) \]

(3.1.22)

The solutions of (3.1.21) and (3.1.22) are

\[ \frac{f_{\eta}}{f_0} = \frac{\eta^2}{\gamma} \left[ \frac{\partial C_0}{\partial \eta} + \frac{\partial C_2}{\partial \eta} \right] \]

(3.1.23)

and

\[ \frac{\partial C_0}{\partial \xi} = \frac{\eta^2}{\gamma} \frac{\partial C_1}{\partial \eta} \]

(3.1.24)

we can rewrite (3.1.23) and (3.1.24) in dimensionless form as

\[ \frac{\partial \tilde{C}_0}{\partial \eta} = \frac{\eta^2}{\gamma} \left( \frac{1}{R} + \frac{A_2}{L} R^3 \sin^2 \theta + B_2 \right) \]

(3.1.25)

and

\[ \frac{\partial \tilde{C}_1}{\partial \eta} = \frac{\eta^2}{\gamma} \frac{A_2}{L} R^3 \sin^2 \theta \]

(3.1.26)

To calculate \( A_2 \) and \( B_2 \), we use the condition that \( \frac{\partial \tilde{C}_0}{\partial \eta} \) for

\( R_a < R < R_b \) at \( \theta = \pi/2 \).

\[ A_2 = \frac{1}{R_a} - \frac{1}{R_b} \]

(3.1.27)

\[ B_2 = -\frac{1}{2} \left[ \left( \frac{1}{R_a} + \frac{1}{R_b} \right) + \frac{A_2}{L} \left( R_a^4 + R_b^4 \right) \right] \]

(3.1.28)
Above $A_i$ to $N$ ensures positive pressure within the inner and outer boundaries $K_i$ and $Ki$ on $C$ provided $i < N$. Thus the disk is physically plausible one.

We have used all these three classes of steady state equations for investigating the frequency of axisymmetric oscillations.

3.2 Axisymmetric Oscillation:

The equation governing linear axisymmetric perturbation may be obtained from equation (2.2.20) to (2.2.23) under Newtonian limit $c \rightarrow \infty$ and $a \rightarrow 0$

Thus we have

\[ \frac{\partial}{\partial t} \left[ \frac{2}{\pi} \delta \nu^\eta - \frac{2}{\pi} \nu^\eta \delta \nu^\eta \right] + \delta \nu^\eta \left[ \frac{m \delta^2 \eta}{\gamma^2} - \frac{\nu^2}{\gamma} \right] = - \frac{2}{\gamma^2} \delta \nu^\eta, \]

\[ \frac{\partial}{\partial t} \left[ \frac{2}{\pi} \delta \nu^\theta - \frac{2}{\pi} \cot \theta \nu^\eta \delta \nu^\eta \right] + \delta \nu^\eta \frac{\nu^2}{\pi} \cot \theta = - \frac{1}{\gamma} \frac{2}{\gamma^2} \delta \nu^\eta, \] (3.2.1)

\[ \frac{\partial}{\partial t} \delta \nu^\eta + \frac{1}{\pi} \left( \frac{\nu^2}{\partial \theta} + \nu^\theta \cot \theta \right) \delta \nu^\eta + \left( \frac{\nu^2}{\gamma^2} + \frac{\nu^\eta}{\gamma} \right) \delta \nu^\eta = 0 \] (3.2.2)

and

\[ \frac{\partial}{\partial t} \left[ \frac{1}{\pi^2} \frac{2}{\partial \eta} \left( \delta \nu^\eta \right) + \frac{1}{\pi^2} \frac{\partial \eta \nu^\eta}{\partial \theta} \right] \]

\[ + \frac{\partial}{\partial t} \delta \nu^\eta + \delta \nu^\eta \frac{\nu^2}{\gamma^2} + \frac{\delta \nu^\eta \nu^\theta}{\gamma^2} \approx 0 \] (3.2.4)
where as condition of adiabaticity gives

\begin{equation}
\frac{\partial \eta}{\partial \tau} + \gamma \frac{\partial \phi}{\partial \tau} = 0
\end{equation}

and after some rearrangement of terms we obtain the following equations which govern the linear axisymmetric perturbation.

\begin{align}
- \rho \varepsilon e^2 \frac{\partial \phi}{\partial t} &= \frac{2 \rho \varepsilon}{\eta} \frac{\partial \psi}{\partial t} + \left( \frac{\eta \rho}{\rho_0} - \frac{\rho_0}{\eta} \right) \frac{\partial \psi}{\partial t} - \frac{2}{\eta} \frac{\partial \phi}{\partial t}, \\
- \rho_0 \varepsilon \frac{\partial \phi^0}{\partial t} &= \frac{2 \rho_0 \varepsilon}{\eta} \frac{\partial \psi^0}{\partial t} + \frac{\rho_0}{\eta} \cosh \delta \frac{\partial \phi^0}{\partial t} - \frac{1}{\eta} \frac{2}{\eta} \frac{\partial \phi^0}{\partial t},
\end{align}

\begin{align}
\frac{\partial \psi}{\partial t} &= - \frac{1}{\phi_0} \left( \frac{\partial \psi_0}{\partial t} + \cos \phi \frac{\partial \phi_0}{\partial t} \right) - \left( \frac{\partial \phi_0}{\partial t} + \frac{\partial \phi_0}{\partial \phi} \right), \\
\frac{\partial \phi^0}{\partial t} &= - \frac{1}{\phi_0} \left[ \frac{2}{\phi_0} \frac{\partial \phi_0}{\partial \phi} ( \phi_0^2 \frac{\partial^2 \phi^0}{\partial \phi^2} ) + \frac{1}{\phi_0} \sin \phi \frac{\partial \phi_0}{\partial \phi} ( \sin \phi \frac{\partial \phi^0}{\partial \phi} ) \right] - \frac{\phi_0}{\phi_0} \frac{\partial \phi_0}{\partial \phi} - \frac{\phi}{\phi} \frac{\partial \phi_0}{\partial \phi}.
\end{align}
equation (3.2.7) and (3.2.8) are pulsation equation obtained by putting the form (3.2.6) for the time dependence of the Langrangian displacement. Equations (3.2.7) and (3.2.8) have to be solved as an eigen value problem along with the equations (3.2.9) to (3.2.11) with proper boundary conditions. Equations (3.2.9) to (3.2.11) are the initial value equations, obtained after integrating once with respect to time. Equation (3.2.11), which is derived from the equation of state, remained unaltered whether we use (3.1.6) or (3.1.7).

Using equation (3.2.7) to (3.2.11), a formula for the frequency $\sigma^2$ can be developed by using the procedure of Chandrasekhar and Friedmann (1972 a,b). We choose trial displacement $\xi^0_j$ and $\xi^0_i$ which may be completely arbitrary except that they satisfy the same boundary conditions as the true displacement $\xi^0_j$ and $\xi^0_i$. We multiply the equation (3.2.7) by $\xi^0_j$ and (3.2.8) by $\xi^0_i$ add them and integrate over the entire region of $\xi$ and $\phi$. We use the equations (3.2.10) to (3.2.11) and reduce it in a form which symmetrical in $\xi^0_j$ and $\xi^0_i$. This is done by integrating various terms by parts and using steady state equations (3.1.8) and (3.1.9).
We then bring the resultant equation into a symmetrical form

\[
\epsilon^2 \int \int f_0 \left( \frac{\partial^2 \xi}{\partial t^2} + \frac{\partial^2 \eta}{\partial t^2} \right) \, d\alpha \, d\beta =
\]

\[
\int \int \left( \varepsilon_1 \varepsilon_2 \cos \theta \left( \frac{\partial^2 \eta}{\partial \alpha^2} + \frac{\partial^2 \eta}{\partial \beta^2} \right) - \sin \theta \frac{\partial \eta}{\partial \alpha} \left( \frac{1}{\sin \theta} \frac{\partial \eta}{\partial \theta} \right) + \varepsilon_1 \varepsilon_2 \cos \theta \frac{\partial^2 \eta}{\partial \alpha^2} \right) \, d\alpha \, d\beta
\]

\[
\gamma \int \int \left( \frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \xi}{\partial \beta^2} \right) \left( \frac{\partial^2 \eta}{\partial \alpha^2} + \frac{\partial^2 \eta}{\partial \beta^2} \right) \, d\alpha \, d\beta
\]

\[
= \int \int \left[ n \cos \theta \left( \frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \eta}{\partial \beta^2} \right) + n \sin \theta \left( \frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \eta}{\partial \beta^2} \right) \right] \, d\alpha \, d\beta
\]

\[
+ \gamma \int \int \left[ \frac{\partial \psi}{\partial \alpha} \left( \frac{\partial \psi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta} \right) + \sin \theta \cos \theta \left( \frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \eta}{\partial \beta^2} \right) \right] \, d\alpha \, d\beta
\]

\[
+ \gamma \int \int \left[ \frac{\partial \psi}{\partial \alpha} \left( \frac{\partial \psi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta} \right) + \sin \theta \cos \theta \left( \frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \eta}{\partial \beta^2} \right) \right] \, d\alpha \, d\beta
\]

\[
+ \gamma \int \int \left[ \frac{\partial \psi}{\partial \alpha} \left( \frac{\partial \psi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta} \right) + \sin \theta \cos \theta \left( \frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \eta}{\partial \beta^2} \right) \right] \, d\alpha \, d\beta
\]

\[
+ \gamma \int \int \left[ \frac{\partial \psi}{\partial \alpha} \left( \frac{\partial \psi}{\partial \alpha} + \frac{\partial \psi}{\partial \beta} \right) + \sin \theta \cos \theta \left( \frac{\partial^2 \xi}{\partial \alpha^2} + \frac{\partial^2 \eta}{\partial \beta^2} \right) \right] \, d\alpha \, d\beta
\]

\[
(3.2.12)
\]
As shown by Chandrasekhar and Friedmann (1972), the symmetrical form of the equation implies a variational principle, in the following sense.

We identify \( \frac{\partial}{\partial \tau} \) with \( \frac{\partial}{\partial \tau} \) and write an equation for \( \epsilon^2 \)

\[
\epsilon^2 \int \left[ \frac{1}{2} \nabla^2 \frac{\partial^2}{\partial \theta^2} + \left( \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \eta \partial \theta} \right) \right] \frac{1}{r^2} \sin \theta \, d\Omega \\
+ \int \left[ \frac{1}{2} \nabla^2 \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \eta} \right] \frac{1}{r^2} \sin \theta \, d\Omega
\]

Now if one evaluates (3.2.13) by two trial displacements \( \xi \) and \( \eta \), where \( \xi \) is an arbitrary variations in \( \xi \) such that the latter introduces a variation \( \delta \sin \theta \) and traces back the calculations that lead to (3.2.12) starting from (3.2.7) to (3.2.11) one obtains

\[
(3.2.13)
\]
It is clear from (3.2.14) that demanding \( \Delta c^2 = 0 \), amounts to solving the original eigenvalue equation (3.2.7) and (3.2.8) along with the initial value equation (3.2.9) to (3.2.10). Thus if \( \Delta c^2 = 0 \) for all variations \( \eta_1^*, \eta_2^* \) is a solution of the original eigenvalue equations and therefore \( c^2 \) is true eigenvalue (Chandrasekhar and Friedmann 1972, Chakraborty and Prasanna 1981), Rewriting the equation (3.2.13) in the dimensionless form we have

\[
\Delta c^2 \int \int \left[ 2 R \phi \psi \sin \Theta \left( \frac{\partial \psi}{\partial \Theta} + \frac{\partial \phi}{\partial \Phi} \right) \right] d\Phi d\Theta = \int \int \left[ 2 R \phi \psi \sin \Theta \left( \frac{\partial \psi}{\partial \Theta} + \frac{\partial \phi}{\partial \Phi} \right) \right] d\Phi d\Theta - \frac{2 R \phi \psi \sin \Theta}{\partial \Phi} \left( \frac{\partial \phi}{\partial \Theta} \right) + \frac{\partial \phi}{\partial \Theta} \left( \frac{\partial \phi}{\partial \Theta} \right) - \frac{\partial \phi}{\partial \Theta} \left( \frac{\partial \phi}{\partial \Theta} \right) + \frac{\partial \phi}{\partial \Theta} \left( \frac{\partial \phi}{\partial \Theta} \right)
\]

\[
+ \int \int \left[ \frac{2}{R} \phi \psi \cos \Theta \left( \frac{\partial \phi}{\partial \Theta} + \frac{\partial \psi}{\partial \Phi} \right) \right] \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} \right) + \frac{\partial \phi}{\partial \Theta} \left( \frac{\partial \phi}{\partial \Theta} \right) - \frac{\partial \phi}{\partial \Theta} \left( \frac{\partial \phi}{\partial \Theta} \right) + \frac{\partial \phi}{\partial \Theta} \left( \frac{\partial \phi}{\partial \Theta} \right)
\]

\[
+ \int \int \left[ 4 R \phi \psi \cos \Theta - 2 \phi \psi \sin \Theta + 2 \phi \psi \sin \Theta \right] \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} \right) + \frac{\partial \phi}{\partial \Theta} \left( \frac{\partial \phi}{\partial \Theta} \right) + \frac{\partial \phi}{\partial \Theta} \left( \frac{\partial \phi}{\partial \Theta} \right)
\]

\[
- \int \int \left[ \frac{R \phi \psi}{\partial \Phi} \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} + \frac{\partial \psi}{\partial \Phi} \right) \right] d\Phi d\Theta + \int \int \left[ \frac{R \phi \psi}{\partial \Phi} \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} + \frac{\partial \psi}{\partial \Phi} \right) \right] d\Phi d\Theta + \int \int \left[ \frac{R \phi \psi}{\partial \Phi} \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} + \frac{\partial \psi}{\partial \Phi} \right) \right] d\Phi d\Theta + \int \int \left[ \frac{R \phi \psi}{\partial \Phi} \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} + \frac{\partial \psi}{\partial \Phi} \right) \right] d\Phi d\Theta
\]

\[
+ 4 R^2 \phi \psi \left( \frac{\partial \phi}{\partial \Theta} \right) + 2 \phi \psi \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} \right) + \phi \psi \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} \right)
\]

\[
+ 2 R \phi \psi \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} \right) + \phi \psi \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} \right) + \phi \psi \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} \right) + \phi \psi \sin \Theta \left( \frac{\partial \phi}{\partial \Theta} \right)
\]

(3.2.15)
It is interesting to note that (3.2.15) is valid for all the steady state solution (I),(II) and (III). We use (3.2.15) to study the frequencies of the oscillation of the disk whose steady state structure are given by solution (I),(II) or (III). We choose trial functions for $z^n$ and $\eta^5$ which satisfy the required boundary condition. The boundary condition is that the Lagrangian perturbation

$$\Delta \Phi = \delta \left( \frac{z^n}{\eta^5} \right) \frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}$$

should be zero at the boundary of the disk and it's derivative to be finite everywhere (Chandrasekhar, 1974). We choose two types of trial functions

type (I) Fixed boundary.

we set

$$z^n \eta^5 = \frac{\partial}{\partial \theta} \left( 1 + \alpha f_1 \right)$$

(3.2.16)

and

$$z^n \eta^5 = \frac{\partial}{\partial \theta} \left( 1 + \beta f_2 \right)$$

(3.2.17)

Which ensures a more restrictive boundary condition $\delta \Phi = 0$. This is the kind of boundary condition adopted by Papaloizou and Pringle (1987). In above $f_1$ and $f_2$ are two chosen functions and $\alpha$ and $\beta$ are two adjustable parameters.

type (2) Non-Stationary Boundary

We take

$$z^n \eta^5 = \frac{\cos \theta}{\Phi} \left( 1 + \alpha \frac{\partial}{\partial \theta} \right)$$

(3.2.17)

$$z^n \eta^5 = \frac{\cos \theta}{\Phi} \left( 1 + \beta \frac{\partial}{\partial \theta} \right)$$

(3.2.18)
We substitute these trial functions in (3.2.15), integrate it numerically and calculate \( \sigma^2 \) by choosing \( \xi \) and \( \zeta \) such that

\[
\xi \Gamma = \infty \quad \text{and} \quad \zeta B = \infty.
\]

3.3 Local Radial Oscillation:

To study the local radial oscillation we use the condition and \( \sigma_j \leq \sigma_j \) (Kato, 1975, Kota and Fuke 1980). We take the extreme situation when \( \gamma_j \) and \( \omega_j \). We have developed a formula for \( \sigma^2 \) by using steady state differential equation (3.1.8), (3.1.9) along with (3.2.7) to (3.2.11). The equations governing the radial perturbation are obtained from original set of equation (3.2.7) to (3.2.11) as follows.

\[
S \psi^2 = \left( \frac{d}{\delta x} + \frac{\psi}{\eta} \right) \psi^2
\]

(3.3.1)

\[
S \psi = - \frac{p}{x} \frac{\psi}{\eta} \left( \psi^2 + \xi \right) - \frac{\psi}{\eta} \frac{d^2 \psi}{d \eta^2}
\]

(3.3.2)

\[
S \psi = - \frac{p}{x} \frac{\psi}{\eta} \left( \psi^2 + \xi \right) - \frac{\psi}{\eta} \frac{d^2 \psi}{d \eta^2}
\]

(3.3.3)

\[
2 \frac{p}{x} \psi + \frac{d \psi}{d \eta} + \frac{d \psi}{d \eta} \frac{d \psi}{d \eta} - \frac{1}{x} \frac{d \psi}{d \eta} \frac{d \psi}{d \eta} = 0
\]

(3.3.4)

\[
\psi^2 \frac{d \psi}{d \eta} = \frac{2}{x} \frac{p}{x} \psi \frac{d \psi}{d \eta} - \left( \frac{p}{x^2} + \frac{\psi}{\eta} \right) \frac{d \psi}{d \eta} - \frac{\psi}{\eta} \psi \frac{d \psi}{d \eta}
\]

(3.3.5)
Using initial value equation (3.3.1) to (3.3.3) in (3.2.4) and assuming $\xi$ to be a function of $\eta$ only, we get differential equation

$$f(1 + \frac{\partial}{\partial \eta}) \left( \frac{\partial^2 \xi}{\partial \eta^2} \right) + \frac{\partial \xi}{\partial \eta} f' \xi = 0$$

(3.3.6)

for $\xi$, which has the solution

$$\xi_{\eta} = \eta \psi(\eta, \xi)$$

(3.3.7)

where $\eta$ is constant of integration.

Using the steady state equations (3.1.8) and (3.1.9) and the perturbation equations (3.3.1) to (3.3.4) in (3.3.5), we obtain

$$e^2 = \frac{2}{(\gamma - 1) \gamma^2} \left[ \frac{\gamma - 1}{\eta^2} \frac{d}{d \eta} \left( \frac{d P}{d \eta} \right) + \frac{\eta^2 v^2}{2} + \frac{2 - \gamma}{2} \frac{d \eta^2}{d \eta} \right]$$

(3.3.8)

as the local frequency at each $(\eta, \xi)$.

This $e^2$ is valid for any steady state solutions, giving $P_{s}, \xi_{s}, \Omega_{s}$. It is also interesting to find that the various terms of (3.3.8) represent energy densities of various forms.

Result and Discussion:

Table (3.I) and Fig. (3.I - 3.IV) shows the profiles of $P_{s}, f_{s}, \xi_{s}^2$ along $\Theta = \lambda/2$ and the boundary $\partial B$ for the solution (1) with $\gamma / \sigma \beta, F_{\alpha} > 8$ and $\sigma = 2\alpha$. We find that the outer boundary is approximate at 40.3.

We find that along $\Theta = \lambda/2$ plane, as $R$ increases beyond $Ra$, the pressure first increases, reaches a maximum value and then decreases to zero at $R_{s}$; the density increases while
decreases. In fact we see that velocity is roughly Keplerian. The thickness of the disk is also quite large.

The profiles of pressure and velocity for solution (II) and (III) are qualitatively similar to those of the solution (I) but the profile of density is different in these solutions. In solution (I) density increases with R while in solution (II) if either increases or decreases depending on whether we take as positive or negative. In solution (III) density first increases with R and then decreases similar to the variation in pressure.

The values of $\sigma^2$ (in unit of $m^2 s^{-1}$) are presented in Table(3.II) to(3.IV). In Table(3.II) we present the frequency for the fixed boundary and solution (I) of the steady state. We present the result with choices $f_1 = R$, $f_2 = \cos \varTheta$ and $f_1 = \frac{1}{R}$, $f_2 = \cos^2 \varTheta$. We have taken different values of the parameters $R_0$, $R_1$, $K$ and $\ell$ and have calculated $\sigma^2$. It will be seen from the table that there is a systematic variations in $\sigma^2$ as we change the parameters, and these variations are, at least qualitatively similar when different choice of $f_1$ and $f_2$ are made. To calculate the $\sigma^2$ we have developed the formula for $\sigma^2$ and used trial functions which are not completely arbitrary. Therefore the value of $\sigma^2$ which we have obtained not the true eigen frequency. However the general agreement between the result, as we choose different $f_1$ and $f_2$ seems to indicate that the $\sigma^2$ which have computed are very close to the true eigenvalue.
Table III presents the frequency for non-stationary boundary and solution (I) while Table (III) shows the frequency for solution (II) and (III).

The frequency of axisymmetric oscillation of a non-self gravitating perfect fluid disk, rotating around a compact object seems to be \( \omega_{1} \) when \( \omega_{1} \) for the size of the disk 40m.

From the equation (3.4.8) for the local frequency it is clear that for \( \gamma > 5/3 \) is positive at all \( \gamma, \Theta \) and is given by \( \epsilon^2 \sim (\Omega \epsilon/\gamma)^{2} \).

For other values of \( \gamma \), \( \epsilon^2 \) may be negative which leads local instability. This may lead to large values of turbulent viscosity (Kato 1978). In case of the atomicity of the gas molecules which determine the value \( r \) and gas disks it seems to indicate an interesting rotating between the turbulent viscosity. This point require further investigations.
Caption for Figures and Tables

1. Fig. (3.1) Profile of pressure at $e$, $n/2$ plane for disk having $Ra = 8.1$, and $Rb = 114.3$

2. Fig. (3.2) Profile of density at $e$, $n/2$ plane for disk size $Ra = 8.1$, and $Rb = 114.3$

3. Fig. (3.3) Profile of velocity at $e$, $n/2$ for disk having $Ra = 8.1$, and $Rb = 114.3$

4. Fig. (3.4) Boundary $C$, for the disk having $Ra = 8.1$, and $Rb = 114.3$

Table (3.1) Profiles of $p$, $\rho$, $V_\theta$ along $e$, $n/2$ and thickness in degrees for solution (I) for $k = 2$, $l = 1$, $Ra = 8$

Table (3.2) Frequency for the stationary boundary
- Choice I boundary $f_1$, $f_2$, $f_3$, $\omega_0$
- Choice II $f_1$, $f_2$, $f_3$, $(\omega_0^2 \delta$, $Ra = 8$, $\kappa = 1$

Table (3.3) Frequency for the Non-stationary boundary
$Ra = 8$, $Rb = 40$, $Rb = 114.3$, $l = 1$, $r = 1.33$

Table (3.4) Frequency for the solution (II) and (III) for stationary boundary $Ra = 8$, $Rb = 40$, $r = 1.33$
<table>
<thead>
<tr>
<th>$k$</th>
<th>$\frac{1}{n}$</th>
<th>$\chi^2_{\nu_1,\nu_2}$</th>
<th>$\psi$</th>
<th>$M_i - 5l_i$</th>
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</thead>
<tbody>
<tr>
<td>8.10</td>
<td>0.04</td>
<td>0.06</td>
<td>0.42</td>
<td>4.9</td>
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<td>0.13</td>
<td>0.32</td>
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<td>1.98</td>
<td>0.32</td>
<td>0.22</td>
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<tr>
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<td>0.17</td>
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<tr>
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<td>0.59</td>
<td>0.59</td>
<td>0.14</td>
<td>18.4</td>
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<tr>
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<td>0.63</td>
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<tr>
<td>40.2</td>
<td>0.04</td>
<td>0.65</td>
<td>0.13</td>
<td>3.0</td>
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TABLE 3.2

<table>
<thead>
<tr>
<th>$i''$</th>
<th>$k_{kk}$</th>
<th>$k$</th>
<th>Choice I</th>
<th>Choice II</th>
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<tr>
<td>20</td>
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<td>1.18</td>
<td>1.35</td>
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<td>40.3</td>
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<td>1.11</td>
<td>1.32</td>
</tr>
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<td>2.5</td>
<td>1.07</td>
<td>1.30</td>
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<td>3.0</td>
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<tr>
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<td>0.79</td>
<td>0.96</td>
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### TABLE 3.3

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<td>2</td>
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<tr>
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</tr>
<tr>
<td>4</td>
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</table>

### TABLE 3.4

**Solution II**

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<tbody>
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<td>3</td>
<td>0.12</td>
</tr>
<tr>
<td>1.5</td>
<td>3</td>
<td>0.10</td>
</tr>
<tr>
<td>1.9</td>
<td>3</td>
<td>0.14</td>
</tr>
</tbody>
</table>

**Solution III**

<table>
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<th>( \varepsilon \times 10^3 )</th>
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<td>0.25</td>
</tr>
<tr>
<td>-1.30</td>
<td>0.26</td>
</tr>
<tr>
<td>-1.50</td>
<td>0.27</td>
</tr>
<tr>
<td>-2.05</td>
<td>0.28</td>
</tr>
</tbody>
</table>