In this chapter we present our studies of the trajectories of charged particle in an electromagnetic field superimposed over a back ground geometry. Earlier work by Prasanna and Coworkers, (Prasanna, 1980) have shown that in the presence of electromagnetic field, stable orbits can exist even very near to the event horizon such orbits being extremely close to the black holes are effected to show the general relativistic effects very prominently. In fact, as it was shown by Prasanna and Coworkers the very existence of a potential well near the event horizon is itself because of general relativistic modification of the electromagnetic field. In Newtonian description such effects will be totally absent. The dynamics of charged particle in an electromagnetic field have been discussed by Prasanna and Verma (1977) for Schwarzschild black hole and Prasanna and Vishweshwara (1978) for a Kerr black hole. The solution for stationary axisymmetric electromagnetic field have been obtained by (Ginzburg and Osoro 1965, Petterson 1974, Bicak and Dvorak 1977) around a Schwarzschild black hole and by (Chitra and Vishweshwara 1975, Petterson 1975, King et al 1975), around a Kerr black hole, Prasanna and Vishweshwara (1978), have found that the particle gyrates only when it is completely outside the ergosphere,
an effects which attributes to the initial frame dragging Prasanna and Chakraborty (1980) have shown that in a locally non-rotating frame (L.N.R.F) the effect of frame dragging is cancel out and the particle even when it is inside the ergosphere shows gyration when viewed in LNRF.

In this Chapter, we report our studies of the objects of charged particle in an uniform magnetic field superimposed on a Kerr geometry. The rotation axis of the Kerr hole, makes an angle $\gamma$ with the asymptotic magnetic field. This is a generalisation of the studies of Prasanna and Vishweshwara (1978) who considered the situation $\gamma = 0$. The electromagnetic field has been calculated by King (1976), King & Lasota (1977) by using the tetrad formalism and solving the Maxwell's equation modified by the curve background geometry. In solution (6.1) we present some aspects of the tetrad formalism while in (6.2) we present the expression of the electromagnetic field components when the angle $\gamma = 0$. Using the electromagnetic potential of Wald (1974) as some pre-requisites for what follows. In section (6.3) we present the basic equation describing the electromagnetic field as given by King (1976). Section (6.4) is devoted for the calculations of electromagnetic field tensor $F_{ij}$ while in (6.5) we discuss the orbits in such field.
For many applications of general theory of relativity, it is advantageous to choosing a suitable tetrad basis of four linearly independent vector fields. Projecting the relevant quantities on this chosen basis one considers the equations satisfied by them. This is the tetrad formalism. In the tetrad formalism the choice of the tetrad basis depends on the underlying symmetries of the space-time. At each point of space-time one considers a basis of four contravariant vectors.

\[ e^i_{(a)} \quad (a = 1, 2, 3, 4) \]  

(6.1.1)

where indices within enclosures distinguishes the tetrad indices from the tensor indices.

The alphabets such as \( a, b, \text{ etc.} \) are used for the tetrad indices and alphabets such as \( i, j, \text{ etc.} \) are for the tensor indices. The covariant vectors are given by

\[ e_{(a)i} \equiv g^{jK} e_{(a)K} \]  

(6.1.2)

where \( g^{jK} \) denotes the metric tensor. We also define inverse \( e_{(b)i} \) of the matrix \( [e_{(a)i}] \) so that

\[ e_{(a)i}e_{(c)u} = \delta^i_j \quad \text{and} \quad e_{(a)i}e_{(b)i} = \delta_{(a)(b)} \]  

(6.1.3)

we also assume that

\[ e_{(a)i} - (b)i = \theta_{(a)(b)} \]  

(6.1.4)
where $\eta_{(a)(b)}$ is a constant symmetric matrix.

If $\eta_{(a)(b)}$ represent a diagonal matrix with the diagonal elements $+1, -1, -1, -1$ then the basis vectors are $\ell_{(a)}$ are orthonormal. A tetrad formalism with special choice of basis vector $\ell_1, \ell, \eta, \eta$ and $\ell_{(a)}, \eta_{(a)}$, in which $\ell$ and $\eta$ are real and $\eta$ and $\eta$ complex conjugates of one another is called Newman formalism.

In the Newman Penrose formalism a pair of null real vectors $\lambda$ and $\eta$ and pair of complex conjugate are required to satisfy the orthogonally conditions

$$\lambda \eta = \eta \lambda = 0 \tag{6.1.5}$$

besides the requirements

$$\lambda \ell \eta = \eta \lambda \ell = 0 \tag{6.1.6}$$

The basis vectors in normalization conditions

$$\lambda \eta = 1 \quad \text{and} \quad \eta \lambda = -1 \tag{6.1.7}$$

In the Newman - Penrose formalism (N.P) the metric is expressed as

$$g^{\mu \nu} = \ell^\mu \eta^\nu + \eta^\mu \lambda^\nu - \eta^\mu \overline{\eta}^\nu - \overline{\eta}^\mu \eta^\nu \tag{6.1.8}$$
where \( \lambda^\mu, \eta^\mu, \overline{\eta}^\mu \) are Kinnersley's complex null tetrads

\[
\lambda^\mu = \frac{1}{\Delta} \left[ (\Delta, \Delta, 0, a) \right]
\]

\[
\eta^\mu = \frac{1}{2\Delta} \left[ (\Delta, -\Delta, 0, a) \right]
\]

\[
\overline{\eta}^\mu = \frac{1}{\sqrt{2} (\eta^\alpha \overline{\eta}^\alpha)} \left[ i \alpha \sin \theta, 0, 1, i \alpha \sin \theta \right]
\]

(6.1.9)

where

\[
\Delta = \sqrt{n^2 + \alpha^2 \cos^2 \theta}
\]

\[
\eta^\pm = \frac{1}{\sqrt{2}} (\eta^\alpha \overline{\eta}^\alpha)^{\pm 1/2}
\]

(6.1.10)

(We use the units \( c = G = 1 \), throughout this chapter.)

In terms of the above complex null tetrad, the electromagnetic field \( F_{ij} \) is described by three complex quantities \( \phi_0, \phi_1, \) and \( \phi_2 \) which are related to through

\[
\phi_0 = F_{ij} \lambda^i \overline{m}^j
\]

\[
\phi_1 = \frac{i}{2} F_{ij} \left( \overline{\lambda}^i \overline{\eta}^j + \overline{m}^i \overline{m}^j \right)
\]

\[
\phi_2 = \frac{i}{2} F_{ij} \overline{m}^i \eta^j
\]

(6.1.1)

The tetrad formalism has a remarkable property that the electromagnetic field equations become separable. We consider the solution for \( \phi_0, \phi_1, \) and \( \phi_2 \) as presented by Kin\( \text{g} \)
(1976), and calculate the field components \( f_{ij} \) using (6.1.9) and (6.1.11).

6.2 ELECTROMAGNETIC FIELD COMPONENTS FOR AXI-SYMMETRIC CASE

Prasanna and Vishweshwara (1978) have studied the charged particle trajectories in an uniform magnetic field superimposed over Kerr geometry, such that the direction of the magnetic field at large distance from the black hole is aligned with the rotation axis of the hole. The axisymmetric electromagnetic field is described by four potential \( A_i = (A_t, \phi, \theta, A_\varphi) \) with

\[
A_t = \alpha B \left[ \left( 1 - \frac{\mu \sin^2 \theta}{r^2} \right) \right]
\]

(6.2.1)

\[
A_\varphi = -\frac{\beta \sin^2 \theta}{r^2} \left[ \left( \gamma^2 + \alpha^2 \right)^2 - \Delta \alpha^2 \sin^2 \theta - 4 \mu \alpha^2 \right]
\]

(6.2.2)

(Wald, 1974) being the uniform magnetic field.

Defining the electromagnetic field tensor \( f_{ij} \) by

\[
f_{ij} = A_{j,i} - A_{i,j}
\]

(6.2.3)
We find that Wald's field is also presented by the following non-zero components of

\[ f_{01}(\omega) = \frac{\alpha \beta^2 \gamma}{\varepsilon} \left( \frac{1}{s^2} - \frac{a^2}{s^4} \right) (2 - \sin^2 \theta) \]  

(6.2.4)

\[ f_{02}(\omega) = \frac{\alpha^2 \beta \gamma}{\varepsilon} \sin \theta \cos \theta \left[ 1 - \frac{a^2}{\varepsilon^2} (2 - \sin^2 \theta) \right] \]  

(6.2.5)

\[ f_{23}(\omega) = \frac{B \sin \theta \cos \theta}{\varepsilon} \left[ \frac{\alpha^4 \beta^2}{\varepsilon^2} \left( \cos^2 \theta + \sin^2 \theta \right) - \Delta a^2 \sin^2 \theta - 4 m^2 \right] \]  

(6.2.6)

\[ f_{13}(\omega) = \frac{B \sin^2 \theta}{\varepsilon} \left[ \frac{\alpha^2}{\varepsilon^2} \left( \cos^2 \theta + \sin^2 \theta \right) - \Delta a^2 \sin^2 \theta - \frac{m^2 \gamma^2 a^2 \cos^2 \theta}{\varepsilon^2} \right] (2 - \sin \theta) \]  

(6.2.7)

where subscript \((\text{W})\) reminds us that these are Wald's field components.

The electromagnetic field described by the Wald's field as presented above are axisymmetric and stationary. The lines of Electric field and magnetic field for this field have been plotted by Chakraborty and Prasanna (1982) Fig(6.1).

6.3 ELECTROMAGNETIC FIELD SCALARS FOR NON-AXISYMMETRIC CASE :-

King (1976) has given a general non-axisymmetric but stationary solution of source free Maxwell equation in terms of scalars \(\psi^I\)'s in the form of infinite series of multipoles, labeled by positive integers \(\lambda\) and \(m\) with \(\lambda \geq 1\) and \(m \geq |m|\). For
each \( \ell, m \) there is a multipole decaying at infinity but having unphysical behavior on the black hole's horizon \( \mathcal{H} = \mathcal{H}_c \) (class I solution) and another multiple which well behaved at horizon but not decaying at infinity (class II solution).

The stationary solution for \( (\ell, m) \) uncharged multipoles are reproduced below.

\[ \Phi^{(\ell, m)}_0 : \Phi_{\ell m}(\theta, \phi) \, \gamma_{\ell m} \]  

\[ \Phi^{(\ell, m) >}_{\ell} = S_{\ell m}(n) \, \gamma_{\ell m} \]  

\[ \Phi^{(\ell, m) >}_{\ell} = \frac{1}{2} a \frac{\sin \theta}{\sin \theta} \, \gamma_{\ell m} \int_{\mathcal{H}} \frac{S_{\ell m} \, \text{d}n}{\Delta} \]

\[ - \frac{3}{2} \Delta (\ell + 1) \gamma_{\ell m} \int_{\mathcal{H}} \frac{S_{\ell m} \, \text{d}n}{\Delta} \]

where

\[ S = 2 \sqrt{m^2 - \alpha^2}, \quad q = i \alpha m / \delta \]

\[ \delta = \sqrt{\left( \partial_{\theta} + i \cos \theta \partial_{\phi} \right) \theta \, \gamma_{\ell m} = - \left( \partial_{\theta} - i \cos \theta \partial_{\phi} \right) \gamma_{\ell m}} \]

For solutions regular at the horizon one has

\[ K_{\ell m} = \frac{\gamma_{\ell m}}{2 \ell + 1} \frac{2\ell + 1}{2\ell + 3} \frac{2\ell + 5}{2\ell + 7} \cdots \frac{2\ell + 2\ell}{2\ell + 2\ell} \cdots \frac{2\ell + 2\ell}{2\ell + 2\ell} \cdots \]  

\[ (\ell - \ell_1)^{-1} (\ell - \ell_2)^{-1} \cdots \frac{1}{\Delta} \int_{\mathcal{H}} \left[ - \Delta - 1, \Delta - 2 \right] \]  

\[ S_{\ell m} = \frac{\gamma_{\ell m}}{2 \ell + 1} \frac{2\ell + 1}{2\ell + 3} \frac{2\ell + 5}{2\ell + 7} \cdots \frac{2\ell + 2\ell}{2\ell + 2\ell} \cdots \]  

\[ (\ell - \ell_1)^{-1} (\ell - \ell_2)^{-1} \cdots \frac{1}{\Delta} \int_{\mathcal{H}} \left[ - \Delta - 1, \Delta - 2 \right] \]  

(Solution I)
For solutions regular at infinity, \( g_{\infty} = \infty \) and

\[
R_{\infty} = R_{\infty}(x) = \left( \frac{\gamma + \eta}{\gamma - \eta} \right)^{1+q} \left( \frac{\eta - \gamma}{\eta - \delta} \right)^{-q} \left[ 1 + \frac{1 \pm i \beta}{\eta - \gamma} \right] \]

(6.3.7)

and

\[
\sum_{\ell m} \frac{i}{\ell + \frac{q}{2}} \left( \frac{\gamma}{\eta} \right)^{\ell + q} \left( \frac{\eta}{\gamma} \right)^{-q} \left[ 1 - \frac{1 \pm i \beta}{\ell + q} \right]
\]

(6.3.8)

(Solution II)

In the above equation \( F \) is the standard hypergeometric function and \( \gamma_{lm}(\theta, \varphi) \) is the standard spherical harmonic function. The field scalars \( \varphi_0, \varphi_1, \varphi_2 \) are given by (King and Lasota 1976),

\[
\varphi_0 = 2 \frac{\sqrt{2} \times i \beta}{\delta} \sum_{m=1}^{\infty} \gamma_{lm}(\gamma, \delta) \varphi_{1,m}^{(1,m)}
\]

(6.3.9)

and are related to \( F_{ij} \) by equation (6.1.11)

We need to calculate \( \varphi_{1,m}^{(1,m)} \) using (6.3.1) to (6.3.3) and either (6.3.7) and (6.3.8) for solution II or (6.3.5) and (6.3.6) for solution I. For considering particle orbits near the event horizon we use solution II.

Using the \( \varphi_{1,m}^{(1,m)} \) we can calculate scalars \( \varphi_0 \) and then calculate \( F_{ij} \) using (6.1.11). The calculation of \( \varphi_0^{(1,m)} \) and therefore \( \varphi_0 \) poses no difficulty and was carried out precisely.
However the exact calculation of $\Phi_{1}^{(1,m)}$ seems to be quite formidable as it involves terms like $\left[ / n, m \right]/ \left( / n, m \right)^{-1}$ in the integrands.

We therefore use the $\Phi_{1}^{(1,m)}$ as given by

$$
\Phi_{1}^{(1,m)} \approx \frac{\gamma_{r}^{-m} \cdot \gamma_{r}^{d} \cdot f_{m}(\theta, \phi)}{6.3.10}$$

which holds in asymptotic limit calculate $\Phi_{1}$ and multiply it by $f_{w}$, $X_{\gamma}$ and $\gamma$ of $(n, b, \varphi)$. By comparing resultant fields $F_{ij}$ with Wald's field as described in section (6.2) we find out the function $X$ and $\gamma$. This is presented in section (6.4).

6.4 ELECTROMAGNETIC FIELD COMPONENT ($F_{ij}$) FOR NON-AXISYMMETRIC CASE:

First we present the calculations of $\Phi_{1}$ and $\Phi_{2}$ for a general situation when $\gamma \neq c$ for class II solution. Here $\gamma$ is the angle between the direction of rotating axis of the hole and the magnetic field at infinity.

To calculate $\Phi_{1}$ we use the hyper geometric function

$$
{}_2F_1[-2,1;2n,\frac{2n+2}{\delta}] for \ \delta \leq 1 which may be written as (Gradshteyn and Ryzhik, 1965)

$$
{}_2F_1[-2,1;2n,\frac{2n+2}{\delta}] = 1 + \frac{5-2n}{\delta} + \frac{(2n-5)^2}{2(2n+1)\delta^2} \quad (6.4.1)
$$

using above, we obtain

$$
{}_{11}F_1 - \frac{1}{\delta^2} \frac{\gamma_{r}^{-m} \cdot \gamma_{r}^{d} \cdot f_{m}(\theta, \phi)}{6.4.2}
$$
\[ R_{10} = \frac{1}{\Delta} \left[ \sin \left( 2\pi n - \frac{1}{2\pi} \right) \right] \]  
(6.4.3)

and
\[ R_{1-1} = \frac{1}{\Delta} \left[ \frac{1}{2} \left( 2 \cos \left( 2\pi n - \frac{1}{2\pi} \right) + 2 \cos \left( 2\pi n - \frac{1}{2\pi} \right) \right) \right] \]  
(6.4.4)

where in \( z = \left( \frac{\pi - \theta}{\pi}, \frac{\pi - \theta}{\pi} \right) \) and \( \alpha = \alpha \).

Likewise, the values of relevant \( \gamma_{2m} \) are
\[ \gamma_{11} = -\frac{1}{\Delta} \sin \theta \]  
\[ \gamma_{10} = \frac{1}{\Delta} \cos \theta \]  
\[ \gamma_{11} = \frac{1}{\Delta} \sin \theta \]  
(6.4.5)

Using (6.4.2)-(6.4.5) in (6.3.1) and (6.3.9) we obtain
\[ \Phi = \frac{E}{\Delta} \left[ \sin \left( \beta \sin T + q \cos T \right) \right. \]  
\[ + \hat{e} \sin \left( \beta \cos T - q \sin T \right) \]  
\[ - i \Delta \cos \theta \sin \theta \]  
(6.4.6)
where in

\[ p = -2a^2 + \Lambda \]

\[ q = a\beta + 2\alpha (\gamma_1 - \gamma_2) \]

\[ \frac{d}{dt} = q + \frac{u}{\gamma} \lambda \gamma \]

\[ q = \frac{u}{\gamma} \lambda \gamma \quad (6.4.7) \]

In order to calculate \( q^2 \), we expand the hypergeometric function

\[ \int 
\]

which for \( \lambda = 1 \) yields unity.

Using this and (6.4.2)-(6.4.5) in (6.3.2) and (6.3.9) we obtain

\[ q_2 = \frac{-A\Delta B}{\sqrt{2} e^2} \left[ \sin \left( u \sin \theta + u \cos \theta \cos \varphi \right) \right. 
\]

\[ + i \sin \left( u \sin \theta + u \cos \theta \cos \varphi \right) 
\]

\[ + \cos \left( i u \sin \theta - u \sin \theta \right) \left. \right] \]

\[ (6.4.8) \]

where in

\[ u = \frac{a^2 - a_2 \cos \theta}{\gamma - \alpha \cos \varphi} \]

\[ \Lambda = \gamma_2 - \gamma \cos \varphi \quad (6.4.9) \]
Having calculated \( i \) and \( j \) we now proceed to find \( k \). As indicated at the end of sec. 6.3 it is almost certain that the exact calculation of \( k \) is almost impossible. We, therefore, use the form (6.3.11) for \( k \) valid for asymptotic limit and use (6.3.9) to obtain

\[
\frac{j_i}{\eta} = \frac{1}{i} \left( \frac{i}{i + j} \right) \left( \frac{i}{i + j} \right)^{\frac{1}{2}} \left( \frac{j}{i} \right) \left( \frac{j}{i} \right)^{\frac{1}{2}} \tag{6.4.10}\]

We now multiply above by multiplying by functions \( X \), \( Y \), \( \eta \), and \( j \) and purpose that

\[
\frac{i}{i} \left( \frac{i}{i + j} \right) \left( \frac{i}{i + j} \right)^{\frac{1}{2}} \left( \frac{j}{i} \right) \left( \frac{j}{i} \right)^{\frac{1}{2}} \tag{6.4.11}\]

as the form of \( i \), which may be valid at smaller values of \( k \).

We use the above forms of \( \xi, \eta, \) and \( j \) in (6.4.11) and calculate \( k \). Taking the limit \( \gamma \rightarrow 0 \) we expect to recover the antisymmetric fields (6.2.4) - (6.2.7).

Hence, by comparing our calculation with (6.2.4) - (6.2.7) we can find \( X \) and \( Y \). However \( \gamma \) can not be calculated by this method as in the limit \( \gamma \rightarrow 0 \), all \( \gamma \)-dependent terms from \( j \), \( \eta \), and \( j \) drop out.
We rewrite (6.1.11) in the form

\[ -F_{01} \cos \theta \Delta + F_{02} (\gamma^2 + \alpha^2) + \frac{i F_{03}}{\sin \theta} \]

\[ + F_{12} \alpha - F_{23} \alpha - F_{13} \cos \gamma \sqrt{\Delta} \]

\[ = \sqrt{2} \Psi_0 (\gamma + i \alpha \cos \theta) \quad (6.4.12) \]

\[ F_{01} \cos \theta \Delta - F_{02} (\gamma^2 + \alpha^2) + \frac{i F_{03}}{\sin \theta} \]

\[ + F_{12} \alpha + F_{23} \beta \cos \gamma \Delta + F_{23} \alpha \]

\[ = 2 \sqrt{2} \Psi_0 (\gamma - i \alpha \cos \theta) \quad (6.4.13) \]

and

\[ -F_{11} (\gamma^2 + \alpha^2) - F_{12} \cos \theta + \frac{i F_{23}}{\sin \theta} - F_{31} \alpha = 2 \Psi_0 \]

\[ (6.4.14) \]

Substituting \( \Psi_0, \Psi_1 \), and \( \Psi_2 \) in above, and separately out the real and remaining parts, we construct 6 equations which can be solved simultaneously for 6 components of which is the limit yield.

\[ F_{22} = B \left[(\gamma^2 + \alpha^2) \chi - \frac{a^2 \Delta \sin^2 \theta}{\chi^2} \right] \sin \chi \cos \theta \]

\[ F_{21} = \frac{a}{\chi^2 + \alpha^2} \left[F_{22} - B \Delta \sin \theta \cos \theta \chi \right] \]

\[ F_{13} = \frac{B}{\chi^2} \left[\gamma (\gamma^2 + \alpha^2) - a^2 \chi \right] \sin^2 \theta \]

\[ F_{21} = \frac{a}{\chi^2 + \alpha^2} \left[F_{13} - B \gamma \chi \right] \]

\[ f_{03} = 1_{12} > 0 \quad (6.4.15) \]
Now in the limit $\gamma \to 0$, we must recover the Wald's field, described in sec. (6.2). Hence comparing Wald's field with (6.4.15), we immediately obtain

$$X = \frac{(\gamma^2 + \alpha^2)^2}{\varepsilon^2} \Delta \alpha^2 \sin^2 \theta - 4 \eta \alpha^2,$$

and

$$Y = \chi \sin^2 \theta - \frac{m (\gamma^2 - \alpha^2 \eta \varepsilon^2)}{\varepsilon} (2 - \sin^2 \theta)$$  \hspace{1cm} (6.4.16)

Using above $X$ and $Y$ and considering the general case when $\gamma \neq 0$, we solve (6.4.12) - (6.4.14) to obtain

$$f_{01} = f_{01}(\omega) \cos \gamma + \sin [\frac{1}{2\Delta^2} (\varepsilon_1 - \varepsilon_2) (\alpha \sin \phi - y \beta \cos \phi)]$$  \hspace{1cm} (6.4.17)

$$f_{02} = f_{02}(\omega) \cos \gamma + \sin [\frac{1}{2\Delta^2} (\varepsilon_1 + \varepsilon_2) + x \beta \sin \phi]$$  \hspace{1cm} (6.4.18)

$$f_{13} = f_{13}(\omega) \cos \gamma + \sin \sin \phi [\frac{1}{2} (\varepsilon_2 - \varepsilon_4) \frac{\Delta^2 + \alpha^2}{\Delta^2} - y \Delta^2]$$  \hspace{1cm} (6.4.19)
\[ f_{23} = f_{32}(\cos \theta \cos \phi + \sin \phi \sin \phi \cos \phi + x \sin \phi \cos \phi) \left( \frac{e_1 - e_2}{\rho} \right) \left[ \frac{a \sin \phi}{\rho^2} \right] \]

\text{(6.4.20)}

\[ F_{12} = \frac{1}{2} (e_1 + e_2) \sin \phi \]

\text{(6.4.21)}

\[ F_{13} = \frac{1}{2} (e_2 + e_3) \sin \phi \sin \theta \]

\text{(6.4.22)}

where in

\[ e_1 = -B \left[ \rho \sin \theta \cos \phi + \rho \cos \phi \sin \phi + \rho \cos \phi \sin \phi \right] \]

\[ e_2 = -B \cos \theta \left[ \rho \sin \theta \cos \phi + \rho \cos \phi \sin \phi + \rho \cos \phi \sin \phi \right] \]

\[ e_3 = -\Delta \frac{B}{\rho} \left[ \rho \sin \theta \cos \phi + \rho \cos \phi \sin \phi + \rho \cos \phi \sin \phi \right] \]

\[ e_4 = \Delta \frac{B}{\rho} \left[ \rho \sin \theta \cos \phi + \rho \cos \phi \sin \phi + \rho \cos \phi \sin \phi \right] \]

\text{(6.4.23)}
The calculations of \( f_{1,2} \) as reported above, are complete except for functions \( x \) and \( y \). It is also interesting to note that the calculations of \( f_0 \) and \( f_1 \) and therefore of \( \epsilon_1 \), \( \epsilon_2 \), and finally of \( f_{1,2} \) and \( f_{0,3} \) are exact. Further \( f_{1,2} \) and \( f_{0,3} \) are the components which are non-zero only when \( \gamma \neq 0 \).

These components are zero in the axisymmetric case. Further, we also find that \( f_{0,3} \) when \( \alpha = 0 \). This is quite expected as the electric field components are generated because of the rotating of the black hole.

6.5 ORBITS OF CHARGED PARTICLES

We now discuss the orbits of charged particles in the electromagnetic field \( F_{ij} \) in Kerr geometry. The equations of motion of a charged particle of mass \( M_0 \) and charge \( e \) is given by

\[
\frac{d^2 x^i}{ds^2} + \hat{\Gamma}^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = \frac{e}{M_0 c^2} \frac{F^i}{F^j} \frac{dx^j}{ds}
\]

(6.5.1)

where \( x^i \) are four coordinates \((t, \gamma, \theta, \varphi)\), \( \hat{\Gamma}^i_{jk} \) are connection coefficients of underlying geometry. Solving (6.5.1) we can find the orbit.

We undertake, a special case of the above. We limit ourselves to the selection when \( \gamma \) is non-zero but small.
We neglect the dependent terms in the expression (6.4.17) to (6.4.20) which are small compared to the corresponding dependent terms in these. The fields are now roughly axisymmetric and so that the angular momentum can be regarded roughly as a constant. Besides, the fields are stationary and as such, the energy is a constant of motion. These are related to $U_t$ and $U_\phi$ (being the components of 4-velocity) by

$$U_t = \frac{E'}{m_0} - \frac{e}{m_0} A_t$$

(6.5.2)

and

$$U_\phi = \frac{\mathbf{l}}{m_0} - \frac{e}{m_0} A_\phi$$

(6.5.3)

where in $A_t$ and $A_\phi$ are some as (6.2.1) & (6.2.2) except for that now we multiply these by $c^2\gamma$. Such a modified $A_t$ and $A_\phi$ will give rise to $f_{01}$, $f_{02}$, $f_{03}$ and $f_{23}$ as given by (6.4.17) - (6.4.20) in the limit when $\sin \gamma$ dependent terms are neglected in these. (Prasanna & Vishweshwara, 1978). Using (6.5.1)-(6.5.3) we write the equations of motion as follows.
\[
\frac{d\varphi}{ds} = \frac{1}{\Delta s} \left[ \frac{1}{\varepsilon} \left( 1 - \frac{2m}{\varepsilon} \right) \left( \frac{\varepsilon}{m_e c^2} + \frac{e}{m_\gamma} \right) \right. \\
+ \left. \frac{1}{\varepsilon} \left( \frac{E'}{m_\gamma} - \frac{e}{m_\gamma} A_t \right) \right] \\
(6.5.4)
\]

\[
\frac{d^2 \eta}{ds^2} + \frac{m_\gamma \Delta}{\varepsilon^2} \left( \eta^2 - \alpha^2 \sin^2 \theta \right) \left( \frac{\eta}{\varepsilon} \right)^2 \\
- \frac{1}{\Delta \varepsilon^2} \left\{ m_\gamma \left( \eta^2 - \alpha^2 \cos^2 \theta \right) - \alpha^2 \eta \sin^2 \theta \right\} \left( \frac{\eta}{\varepsilon} \right)^2 \\
- \frac{m_\gamma \Delta}{\varepsilon^2} \left( \frac{d\varphi}{ds} \right)^2 - \frac{\Delta \sin^2 \varphi}{\varepsilon^2} \left( \eta^2 + 2a^2 \sin^2 \gamma c \cos \varphi - m_\gamma \alpha^2 \sin^2 \theta + m_\gamma \alpha^2 \cos^2 \gamma \right) \\
+ m_\gamma \alpha^2 \sin^2 \varphi \cos^2 \gamma \left( \frac{d\varphi}{ds} \right)^2 - \frac{2 m_\gamma \Delta}{\varepsilon^2} \sin^2 \theta \left( \eta^2 - \alpha^2 \cos^2 \theta \right) \frac{dt}{ds} \frac{dt}{ds} \\
- \frac{2}{\varepsilon} \frac{dx}{ds} \left( \frac{dx}{ds} \right) \\
- \frac{\xi}{m_\gamma^2 \varepsilon} \left\{ F_{nt} \frac{dt}{ds} + F_{n\varphi} \frac{d\varphi}{ds} + F_{n\theta} \frac{d\theta}{ds} \right\} \\
(6.5.6)
\]
Equations (6.3.4) - (6.3.7) are the same as used by Prasanna and Vishweshwara (1978) except for $F_{\eta \phi}$ dependent terms in (6.3.6) and (6.3.7) and that the fields $F_{z j}$ have multiplying factor of either $\cos \gamma$ or $\sin \gamma$. The toroidal magnetic field $F_{\eta \phi}$ will not allow the particle to remain continued to plane even when we set the initial conditions as $\Theta = \eta_L$ and $d\Theta/ds = \ldots$

For a true axisymmetric case with $F_{\eta \gamma} = 0$, $F_{t \phi} > 0$ such an initial condition keeps the particle continued to $\Theta = \eta_L$ plane (Prasanna & Vishweshwara, 1978).
We study the trajectories of particles as described by equations, (6.3.4) to (6.3.7) for \( \gamma \neq 0 \) case. We use the dimensionless variables

\[
\begin{align*}
\tau &= \frac{m}{\tilde{m}}, \quad \sigma &= \frac{S}{\tilde{m}}, \quad \tilde{t} = \frac{t}{\tilde{t}}, \quad \lambda = \frac{e \beta \tilde{m}}{m_0} \\
\alpha &= \frac{\alpha}{\tilde{m}}, \quad \bar{L} = \frac{L}{m m_b}
\end{align*}
\]

and

\[
E = E' / m_0
\]

(6.5.8)

rewrite the equations in dimensionless forms

\[
\frac{d \tau}{d \sigma} = \frac{1}{\Delta \tilde{E}} \left[ A (E, A_2) - \gamma \alpha R (L + \bar{A}_q) \right]
\]

(6.5.9)

\[
\frac{d \phi}{d \sigma} = \frac{1}{\Delta \tilde{E}} \left[ (\varepsilon - 2 R) (L + \bar{A}_q) \right. + \gamma \alpha R (E - A_2) \left. \right] / \xi_{1/2} \beta
\]

(6.5.10)
\[
\begin{align*}
\frac{\partial^2 \Theta}{\partial \epsilon_2} &= \nabla^2 \left( R^2 \alpha^2 \cos^2 \Theta \right) + \left( \frac{4}{\Delta} \right)^2 - \frac{1}{\Delta} \left[ R^2 \alpha^2 \cos^2 \Theta \right. \\
&\left. \frac{\partial^2}{\partial \epsilon_2} \right] \\
- \frac{R \Delta}{\Delta} \left( \frac{\partial B}{\partial \epsilon_2} \right)^2 - \nabla^2 \sin^2 \Theta \left( \frac{\partial \phi}{\partial \epsilon_2} \right)^2 + \frac{2}{\Delta} \left( R^2 \alpha^2 \cos^2 \Theta - R^2 \cos^2 \Theta \sin^2 \Theta - \frac{1}{\Delta} R \right) \frac{\partial^2}{\partial \epsilon_2} \\
&+ \frac{R \alpha^2 \cos^2 \Theta}{\Delta \Delta} \left( \frac{\partial \phi}{\partial \epsilon_2} \right)^2 - \frac{2}{\Delta} \left( R^2 \alpha^2 \cos^2 \Theta \right) \frac{\partial \phi}{\partial \epsilon_2} \frac{\partial^2 \phi}{\partial \epsilon_2} \\
&- \frac{2}{\Delta} \sin^2 \Theta \left( \frac{\partial R}{\partial \epsilon_2} \right) \frac{\partial \phi}{\partial \epsilon_2} \\
&- \frac{1}{\Delta} \left[ F_{\text{ext}} \frac{\partial^2}{\partial \epsilon_2} + F_{\text{ext}} \frac{\partial}{\partial \epsilon_2} + F_{\text{ext}} \frac{\partial \phi}{\partial \epsilon_2} \right] \\
\end{align*}
\]

\[\text{(6.5.11)}\]

\[
\begin{align*}
\frac{d^2 \Theta}{d \epsilon_2^2} &= \frac{2 R \alpha^2}{\Delta^2} \sin \Theta \cos \Theta \left( \frac{d \phi}{d \epsilon_2} \right)^2 + \frac{\alpha^2}{\Delta} \sin \Theta \cos \Theta \left( \frac{d R}{d \epsilon_2} \right)^2 \\
- \frac{\alpha^2}{\Delta^2} \sin \Theta \cos \Theta \left( \frac{d \phi}{d \epsilon_2} \right)^2 - \sin \Theta \cos \Theta \frac{1}{\Delta^2} \left( R^2 + \alpha^2 \right)^2 \\
&= \frac{d R}{\Delta^2} \sin \Theta \left( \frac{d \phi}{d \epsilon_2} \right)^2 \\
&+ \frac{d R}{\Delta^2} \sin \Theta \left( \frac{d \phi}{d \epsilon_2} \right)^2 \\
&+ \frac{2}{\Delta} \left( \frac{d \phi}{d \epsilon_2} \right) \\
&\frac{d R}{d \epsilon_2} \frac{d \phi}{d \epsilon_2} \\
&- \frac{1}{\Delta^2} \left[ F_{\text{ext}} \frac{d^2}{d \epsilon_2^2} + F_{\text{ext}} \frac{d}{d \epsilon_2} + F_{\text{ext}} \frac{d \phi}{d \epsilon_2} \right] \\
\end{align*}
\]

\[\text{(6.5.12)}\]
and solve them numerically by using the parameters

\[ \alpha: \theta = 0^\circ, \quad \phi = 30^\circ, \quad \psi = 50^\circ, \quad L = 3 \times 10^5 \quad (6.5.13) \]

we choose the initial conditions

\[ \theta = 90^\circ, \quad \left(\frac{d\Phi}{d\phi}\right)_0 = 0, \quad \left(\frac{dR}{d\phi}\right)_0 = 0, \quad \left(\frac{\bar{R}}{d\phi}\right)_0 = 2. \quad (6.5.14) \]

and choose \( \left(\frac{dR}{d\phi}\right)_0 \) such that the normalisation condition \( \int_{-1}^{1} g_{ij} \phi^i \) is valid. For numerical solutions of above, we use subroutines as given by Antia (1991).

The choices \( (6.5.13) \) and \( (6.5.14) \) are the same as that of Fig.6 of Chakraborty and Prasanna (1982), who discuss the trajectories of charged particle off the equational plane for a truly axisymmetric situations and with a small but non-zero \( \gamma \).

Table \( (6.1) \) and \( (6.2) \) present the trajectories for \( \gamma = 1, 1^\circ \). The sign change of \( \frac{d\phi}{d\phi} \) indicates the gyrations of the particle about the magnetic field lines. The particle is bounded in radial as well as poloidal directions. The field configuration of the field-lines also show that the particle should be continued and should show multiple reflections in direction. Comparing the Table \( 6.1 \) and \( 6.2 \), we also find that the particle remains closer to plane when \( \gamma \) is smaller. This quite expected, because in the limit \( \gamma \to 0 \) we expect the particles to remain continued to plane for the initial conditions as chosen.