In this chapter we present our studies of the general relativistic effects on a rotating perfect fluid disk around Schwarzschild black hole. The general theory of relativity plays a significant role in determining the structure and stability of the disk. This was particularly emphasized by Prasanna and Coworkers (1991), when inner edge of the disk is very close to the relevant horizon. The Newtonian treatment is inadequate for the study of structure and dynamics. The structure and stability of rotating thick disk around compact object was discussed by Chakraborty and Prasanna (1982) amongst many others such as Abramowicz et al (1978, 1980) and Jaroszynski et al (1980). Paczynski and Witta (1980) are the first to give a full fledged theory of accretion disk by using a Pseudo Newtonian potential. We have on the other hand solved complete hydrodynamical equations on a curved background geometry. We study the steady state structures by solving these hydrodynamical equations. We obtain several features such as the formation of Cusp at the inner edge of the disk and the restriction on the possible values of the parameters which determine steady state structure of the disk. In addition to above we also find that the frequency of the axisymmetric oscillation of the disk is higher in general theory.
of relativity than in the Newtonian theory of gravitation. We present set of solution of the steady state equations in which boundary of the disk is some both for the general relativity and for the Newtonian gravity and therefore have the same size. As the frequency of the oscillation is found to be a sensitive function of size of the disk we used this set as solutions to arrive at the above mention conclusion regarding frequency.

4.1 Steady State Solution

We consider a non-accreting, non-self gravitating, perfect fluid disk rotating around Schwarzschild black hole of mass. We use momentum equation and the continuity equation as obtained from the conservation of energy momentum tensor. The Schwarzschild background geometry representing the gravitational fluid may be obtained from Kerr metric represented by (2.1.1) by putting $a=0$ we have

$$ds^2 = -(1 - \frac{2m}{r})c^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

(4.1.1)

where in $m = \frac{Mc}{c^2}$, $M$ being the mass of black hole and other symbols having usual meaning. The general set of equations governing the dynamics of such disk are obtained from (2.4.10) to (2.4.16) by putting $a=0$. 
We have the momentum equations

\[
\begin{align*}
\left( \psi_i - \frac{\psi_i}{c^2} \right) \left[ 1 - \frac{2m}{\gamma} \right] \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) - \left( 1 - \frac{2m}{\gamma} \right) \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) - \frac{1}{\gamma^2} \left( \frac{\partial}{\partial x^i} \right) \left[ \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) \frac{\partial \psi_j}{\partial x^j} \right] - \left( 1 - \frac{2m}{\gamma} \right) \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) \\
\end{align*}
\]

\[\text{(4.1.2)}\]

the energy conservation equation

\[
\begin{align*}
\left( \psi_i - \frac{\psi_i}{c^2} \right) \left[ 1 - \frac{2m}{\gamma} \right] \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) - \left( 1 - \frac{2m}{\gamma} \right) \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) - \frac{1}{\gamma^2} \left( \frac{\partial}{\partial x^i} \right) \left[ \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) \frac{\partial \psi_j}{\partial x^j} \right] - \left( 1 - \frac{2m}{\gamma} \right) \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) \\
\end{align*}
\]

\[\text{(4.1.3)}\]

and the baryon conservation equation.

\[
\begin{align*}
\left( \frac{c}{m} \right)^2 \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) - \left( 1 - \frac{2m}{\gamma} \right) \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) - \frac{1}{\gamma^2} \left( \frac{\partial}{\partial x^i} \right) \left[ \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) \frac{\partial \psi_j}{\partial x^j} \right] - \left( 1 - \frac{2m}{\gamma} \right) \left( \frac{1}{\gamma} \frac{\partial \psi_i}{\partial t} \right) \\
\end{align*}
\]

\[\text{(4.1.5)}\]
where \( p, \rho, \) and \( \tau \) are pressure, energy density and baryon number density,

\[
\begin{align*}
\rho &= \rho_0^0 \left( \frac{\tau}{\tau_0^0} \right)^{\gamma}, \\
\rho &= \rho_0^0 \left( \frac{\tau}{\tau_0^0} \right)^{\gamma} + \rho_0^1 \left( \frac{\tau}{\tau_0^1} \right)^{\gamma} + \rho_0^2 \left( \frac{\tau}{\tau_0^2} \right)^{\gamma},
\end{align*}
\]

(4.1.7)

and

\[
\rho = \rho_0^0 + \rho_0^1 + \rho_0^2.
\]

(4.1.8)

To make the system of equations close, we choose the equation of state as

\[
\rho c^2 = \rho_0 c^2 + \frac{b}{\gamma - 1}
\]

(4.1.9)

where \( \rho_0 \) is the mass of each baryon and \( \gamma \) is constant which is equal to the ratio of specific heats, in case the pressure and internal energy \( \rho c^2 = \rho_0 c^2 \) are proportional to the temperature.

Using (4.1.9) and (4.1.6) we obtain

\[
\frac{\partial}{\partial t} \left( \frac{b}{\gamma - 1} \right) = 0
\]

(4.1.10)

When equation (4.1.10) combined with the law of thermodynamics implies that the motion is adiabatic, keeping the entropy per baryon constant along flow lines. Equation (4.1.2) to (4.1.6) and (4.1.10) are the basic equations which governs dynamics of the fluid disk.
We consider an axisymmetric flow in the steady state as obtained by putting $\psi^{(n)} = \psi^{(n)} = \cdots = \psi^{(n)}$ and by taking all dynamical variables to be independent of $\theta$ and $\tau$. We are thus lead to following equations

\[
\left( \kappa \frac{\partial}{\partial \tau} \right) \left( \frac{mc^4}{\tau^2} \right) \left( 1 - \frac{H}{c^2} \right) \psi^{2} - \left( 1 - \frac{m}{c^2} \right) \frac{\partial \nu^{2}}{\partial \theta} - \left( 1 - \frac{m}{c^2} \right) \frac{\partial \nu^{2}}{\partial \theta} \frac{\partial \theta}{\partial \theta} \frac{\partial \theta}{\partial \theta} \frac{\partial \theta}{\partial \theta} = 0
\]

(4.1.11)

and

\[
\left( \kappa \frac{\partial}{\partial \tau} \right) \left( \frac{mc^4}{\tau^2} \right) \left( 1 - \frac{H}{c^2} \right) \psi^{2} - \left( 1 - \frac{m}{c^2} \right) \frac{\partial \nu^{2}}{\partial \theta} - \left( 1 - \frac{m}{c^2} \right) \frac{\partial \nu^{2}}{\partial \theta} \frac{\partial \theta}{\partial \theta} \frac{\partial \theta}{\partial \theta} \frac{\partial \theta}{\partial \theta} = 0
\]

(4.1.12)

All the other equations are identically zero.

We present the various classes of solutions for the equation (4.1.11) and (4.1.12)

We assumed that $\psi$, as a function of $\theta$ and $\tau$ is known and solve (4.1.11) and (4.1.12) algebraically for $\psi$ and $\psi^{2}$. We obtain

\[
\psi = \frac{m \left( 1 - \frac{2m}{c^2} \right) \psi \psi - \psi \left( 1 - \frac{2m}{c^2} \right) \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial \theta}}{m \left( 1 - \frac{2m}{c^2} \right) \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial \theta}} \psi \psi
\]

(4.1.13)

and

\[
\psi^{2} = \frac{m \left( 1 - \frac{2m}{c^2} \right) \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial \theta}}{\left( 1 - \frac{2m}{c^2} \right) \left( 1 - \frac{m}{c^2} \right) \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial \theta} - \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial \theta}}
\]

(4.1.14)

as we are interested in the disk which extends across plane the Cot term in the denominator of (4.1.13) should be
eliminated. A functional form which is considerably general and which eliminates $\text{Cot}\theta$ term from (4.1.13), may be the following

$$
\frac{\Lambda \int \int V f(y, z) + f y + h \sin \theta \, d y \, d z}{g h' \sin \theta - \pi (Bf' + g h)}
$$

(4.1.15)

where \( f, g, h \) and \( f \) are functions of their respective arguments as shown and \( \Lambda, \psi, \epsilon, \sigma \) and \( K \) are constant. We thus obtain

$$
\psi = \frac{g h' \sin \theta}{g h' \sin \theta - \pi (Bf' + g h)}
$$

(4.1.16)

and

$$
\phi = \frac{g h'}{m r^2} \left[ f (1 - \frac{m}{r}) (g h' \sin \theta - \pi (Bf' + g h))^2 \sqrt{F^2 - \frac{r}{c^2}}
$$

(4.1.17)

wherein the primes denote the differentiation with respect to their respective arguments. We choose \( f, g, h \) and \( f \) such that physically acceptable solutions are obtained.

A particular choice of \( \psi \) of the form

$$
\psi = \Lambda \left[ (1 - \frac{2 \pi}{m}) \left( - \frac{B}{m^2 \sin \theta} \right) \right]^{m/2} + D
$$

(4.1.18)

makes \( \phi \) constant.

This model was examined by Chakraborty and Prasanna (1982).

In another form we consider

$$
\phi = A \sqrt{A} + B \sqrt{B} \sin \theta \left( 1 - \frac{2 m}{r} \right) \frac{k}{2} \, ^2 + D
$$

(4.1.19)
which leads to disks in which is function of $\tau$ only. Solutions of this type in their non-relativistic limit were examined by Chakraborty and Prasanna (1981). But the solution (4.1.18) and (4.1.19) have different functional form in their non-relativistic limit obtained by taking $C = \infty$. This is due to the appearance of factor $(1 - \eta_0^2)$ in these expressions which as $C \to \infty$.

Solutions A:

We consider the pressure of the form

$$p = \frac{\Lambda}{C^2} \int \frac{\eta^2 \sin K}{\frac{\eta}{m} - \frac{2}{m} - \frac{2 \eta_0}{m} + \frac{1}{m}}$$

(4.1.20)

which have the same as (3.1.10) of the Newtonian calculations.

Requiring that $\lambda > \eta$ we find that $\eta_c = \eta_0$ and $\theta > \eta_c$ is the inner edge of the disk, the parameter $\eta_c > \eta_0$ fixes the outer edge $\eta_1$ at $\eta_{1/2}$. Using the form of we obtain

$$\eta = \frac{\Lambda}{C^2} \int \frac{\eta^2}{\frac{\eta}{m} - \frac{2}{m} - \frac{2 \eta_0}{m} + \frac{1}{m}} \sin K \left( \frac{\eta}{m} + \frac{\eta_0}{m} - 2 \right) - k - 1$$

(4.1.21)

and

$$\Omega^2 = \frac{\Lambda \eta}{\eta_0} \left( 1 - \frac{2 \eta}{\eta_0} \right)^{-1} \frac{k}{K - \frac{2}{m} - \frac{2 \eta_0}{m}}$$

(4.1.22)

Requiring that $\Omega^2$ is positive. Finally requiring that $\Omega$ should be positive for all $\eta$, $\eta_0 \leq \eta \leq \eta_1$ at $\theta < \eta_{1/2}$. We obtain

$$\frac{\eta}{m} \to \frac{3 \eta}{K - \frac{2}{m} - \frac{2 \eta_0}{m}}$$

(4.1.23)
Which for limiting case of yields

\[ (4.1.24) \]

The general proof of is given in Appendix I i.e the inner edge of the disk can go as class to as the black hole as the photon orbit this is in turns justifies the application of relativistic theory for the disk dynamics.

Solution B :-

Apart from the solution presented above, other solutions are obtained by Chakraborty and Prasanna (1982), are the following

\[ \frac{B}{c^2} = \Phi \left\{ A \left[ \left( 1 - \frac{2m}{r} \right)^{1/2} - \frac{\beta}{\sqrt{2m^2 \sin^2 \theta}} \right]^{1/2} - 1 \right\} \]

\[ (4.1.25) \]

\( \Phi \) constant

\[ (4.1.26) \]

and

\[ \nu^2 = \frac{Bc^2}{\gamma^2 \sqrt{\sqrt{n} \Phi^2 \frac{1}{2}}} \left( 1 - \frac{2m}{r} \right) \]

\[ (4.1.27) \]

where in \( A \) and \( B \) are constants as given by

\[ A = \frac{2 \alpha b^2}{(a+b)(b^2 - 2)} \]

\[ (4.1.28) \]

and

\[ B = \left[ \frac{(\nu^2 - a^2)(b^2 - 2)(a - 2)}{b^3(a - 2) - a^3(b - 2)} \right]^{1/2} \]

\[ (4.1.29) \]

where

\[ a = \frac{\gamma a}{m}, \quad b = \frac{\gamma b}{m} \]

\[ (4.1.30) \]
The solutions are obtained above are physically acceptable if $\frac{1}{n} \leq \frac{1}{m}$ throughout the interior at the disk and if it goes over to zero at the boundary. The former condition leads to the constants that inner edge cannot lie within $4$ and further

$$\frac{1}{n} \leq \alpha < 6$$

There is no restriction on the outer edge if $\alpha > 6$.

Solution C:

Disk with polytropic equation of state:

We have obtained another sets of solutions of (4.1.11) and (4.1.12) by using equation of state as

$$P = k \rho^\gamma$$

(4.1.31)

such a relativistic polytropic equation of state was considered by Tooper (1964).

Using (4.1.31) along with the equation (4.1.11) and (4.1.12) we have the following solutions for the steady state configuration.

$$\log \left(1 + \frac{P}{\rho^\gamma} \right) = \frac{1}{\beta} \left[ \log \left(1 + \frac{\rho}{\rho_0} \right)^{-\gamma/2} + \beta R^2 \sin^2 \phi \left(1 + \frac{\rho}{\rho_0} \right)^{-\gamma/2} \right] + E$$

(4.1.32)

and

$$\frac{1/2}{\rho_0^2} = \frac{\ell \beta R^2 \left(1 + \frac{\rho}{\rho_0} \right)^{-\gamma/2} \sin^2 \phi}{1 + \ell \beta R^2 \left(1 + \frac{\rho}{\rho_0} \right)^{-\gamma/2} \sin^2 \phi}$$

(4.1.33)

where $\beta$ and $E$ are dimensionless constants and $R = m/n$. 
\[
\frac{\log (1 - \frac{\beta}{\alpha})^{\frac{1}{2}}}{a - b (1 - \frac{\beta}{\alpha})^{\frac{1}{2}}} - \frac{\log (1 - \frac{\beta}{\alpha})^{\frac{1}{2}}}{b - \frac{\beta}{\alpha}}
\]

(4.1.34)

and

\[
E = \frac{a^k (1 - \frac{\beta}{\alpha})^{-\frac{k}{2}} \log (1 - \frac{\beta}{\alpha})^{\frac{k}{2}}}{b^k (1 - \frac{\beta}{\alpha})^{-\frac{k}{2}} - a^k (1 - \frac{\beta}{\alpha})^{-\frac{k}{2}}}
\]

(4.1.35)

In order to have \( P \geq 0 \) for \( a < R < b \), \( k \) should be chosen such that

\[
\log X_b \{ (X_b/\alpha)^{\frac{k}{2}} - (X_b/b)^{\frac{k}{2}} \} + \log X_a \{ (X_a/\alpha)^{\frac{k}{2}} - (X_a/b)^{\frac{k}{2}} \}
\]

\[
\log X \{ (X/\alpha)^{\frac{k}{2}} - (X/b)^{\frac{k}{2}} \}
\]

(4.1.36)

where \( X = (1 - \frac{\beta}{\alpha})^{\frac{1}{2}} \), \( X_a \) and \( X_b \) are value of \( X \) at \( R = a \) and \( R = b \) respectively.
4.2 Axi-symmetric Oscillations

The equations governing the linear axi-symmetric perturbation are obtained from the general set of perturbation equations as reported in Chapter 2.

\[
\begin{align*}
\sigma_{\nu} \frac{\partial w}{\partial \nu} & = -\frac{\nu}{c_s^2} \frac{\partial^2 \rho}{\partial \nu^2} + \frac{\nu V}{c_s^2} \left( \frac{\partial \rho}{\partial \nu} + \frac{\nu}{c_s^2} V \right) + \left( \frac{\nu}{c_s^2} \frac{\partial \rho}{\partial \nu} + \frac{\nu}{c_s^2} \left( 1 - \frac{3 m}{2 \pi} \right) V \right) \frac{\partial \eta}{\partial \nu} \\
\rho \frac{\partial \rho}{\partial \nu} & = -\rho \frac{\partial \eta}{\partial \nu} - \nu \frac{\partial \rho}{\partial \nu} \\
\frac{\partial \rho}{\partial \tau} & = -\frac{\partial \rho}{\partial \tau} - \frac{\partial \rho}{\partial \tau} + \frac{\partial \rho}{\partial \tau} \\
\frac{\partial \rho}{\partial \tau} & = -\frac{\partial \rho}{\partial \tau} - \frac{\partial \rho}{\partial \tau} + \frac{\partial \rho}{\partial \tau} \\
\end{align*}
\]

(4.2.1)

\[
\begin{align*}
\frac{\partial^2 \rho}{\partial \tau^2} & = -s_3 \frac{\partial \rho}{\partial \tau} - s_4 \frac{\partial \rho}{\partial \tau} + \frac{s_4}{c_s^2} \frac{\partial \rho}{\partial \nu} \\
\end{align*}
\]

(4.2.2)

\[
\begin{align*}
\frac{\partial^2 \rho}{\partial \tau^2} & = -s_3 \frac{\partial \rho}{\partial \tau} - s_4 \frac{\partial \rho}{\partial \tau} + \frac{s_4}{c_s^2} \frac{\partial \rho}{\partial \nu} \\
\end{align*}
\]

(4.2.3)

\[
\begin{align*}
\frac{\partial^2 \rho}{\partial \tau^2} & = -s_3 \frac{\partial \rho}{\partial \tau} - s_4 \frac{\partial \rho}{\partial \tau} + \frac{s_4}{c_s^2} \frac{\partial \rho}{\partial \nu} \\
\end{align*}
\]

(4.2.4)

\[
\begin{align*}
\frac{\partial^2 \rho}{\partial \tau^2} & = -s_3 \frac{\partial \rho}{\partial \tau} - s_4 \frac{\partial \rho}{\partial \tau} + \frac{s_4}{c_s^2} \frac{\partial \rho}{\partial \nu} \\
\end{align*}
\]

(4.2.5)

where in

\[
\begin{align*}
s_1 & = \left( 1 - \frac{x m}{s_3} \right) \\
s_2 & = \left( 1 - \frac{y m}{s_3} \right) \\
s_3 & = \left( 1 + \frac{z m}{s_3} \right) \\
s_4 & = \left( 1 - \frac{\gamma \rho \mu^2}{s_3} \right) \\
\end{align*}
\]

\[
\begin{align*}
x_L & = \frac{s_1}{s_3^2} \frac{\partial \rho}{\partial \nu} \left( \frac{\partial \rho}{\partial \nu} \right) + \frac{1}{s_3} \frac{\partial \rho}{\partial \theta} \left( \frac{\partial \rho}{\partial \theta} \right) \\
\gamma & = \frac{\partial \nu}{\partial \theta} + \frac{\nu}{s_3} \cos \theta \\
\lambda & = \frac{s_1}{s_3^2} \frac{\partial \rho}{\partial \nu} + \frac{1}{s_3} \left( 1 - \frac{3 m}{2 \pi} \right) \frac{\partial \rho}{\partial \nu} \\
\end{align*}
\]

(4.2.6)
and $q^i = \eta^i (\psi, \mu, n)$ where $\xi^i$ is langrangian displacement whose time dependence go as $e^{\alpha t}$. The initial value equations (4.2.1) to (4.2.3) obtained by integrating with respect to time and eliminating while equations (4.2.4) and (4.2.5) are pulsation equations which contain the square of the frequency terms. In the above we have used adiabatic equation of state (2.2.12) using (4.2.1) to (4.2.5) we develop a formula for $\sigma^2$, by choosing trial displacement $\xi^i_n$ and $\xi^i_G$ by following the same procedure outlined in Chapter 3 for the Newtonian case, we obtain.

$$\sigma^2 \int \int \left[ \frac{\xi^i_n}{\xi^2_n} \left( \xi^i \right)^2 + \frac{\xi^i_G}{\xi^2_G} \right] e^{\alpha t} d\nu d\phi =$$

$$\int \int \left[ \frac{2 \xi^i_n}{\xi^2_n} \frac{\xi^i n}{\xi^2_n} \frac{\partial \eta^i}{\partial \nu} + \frac{\partial \xi^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu} + \gamma \frac{\partial \xi^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu} + \gamma \frac{\partial \xi^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu} \right] \xi^i d\nu d\phi$$

$$- \frac{5 \xi_n}{\xi^2_n c^2} \left( \frac{\partial \eta^i_n}{\partial \nu} \right)^2 + 2 \gamma \frac{\partial \xi^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu} - \frac{\partial \xi^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu}$$

$$+ \frac{2 \gamma \xi_n}{\xi^2_n} \xi_n \left( \frac{\partial \eta^i_n}{\partial \nu} \right)^2 + \gamma \frac{\partial \xi^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu}$$

$$+ \int \int \left[ \left( \frac{\partial \xi^i_n}{\partial \nu} \right) \frac{\partial \eta^i_n}{\partial \nu} + \frac{2 \gamma \xi_n}{\xi^2_n} \right] \frac{\partial \eta^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu} + \gamma \frac{\partial \xi^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu}$$

$$+ \frac{\xi^2_n}{\gamma} \left[ \left( \frac{\partial \xi^i_n}{\partial \nu} \right) \frac{\partial \eta^i_n}{\partial \nu} + \frac{2 \gamma \xi_n}{\xi^2_n} \right] \frac{\partial \eta^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu}$$

$$+ \frac{\xi^2_n}{\gamma} \left[ \left( \frac{\partial \xi^i_n}{\partial \nu} \right) \frac{\partial \eta^i_n}{\partial \nu} + \frac{2 \gamma \xi_n}{\xi^2_n} \right] \frac{\partial \eta^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu}$$

$$+ \frac{\xi^2_n}{\gamma} \left[ \left( \frac{\partial \xi^i_n}{\partial \nu} \right) \frac{\partial \eta^i_n}{\partial \nu} + \frac{2 \gamma \xi_n}{\xi^2_n} \right] \frac{\partial \eta^i_n}{\partial \nu} \frac{\partial \eta^i_n}{\partial \nu}$$

(4.2.7)
In case we used a polytropic equation of state the equation (4.2.2), (4.2.3) and (4.2.6) gets modified using such modified equation and following the same procedure as mentioned, above we obtain

\[
\left( \frac{\partial}{\partial \xi} \right)^2 - \frac{2y}{c^2 s_3 f} \delta^2 (s_1 p) - \frac{1}{s_3 f} \delta \frac{d^2}{d \xi^2}
\]

\[
+ \frac{2y}{c^2 s_3 f} \frac{1}{s_3 f} \delta^2 (s_1 p)^2
\]

\[
+ 2 \frac{2y}{c^2 s_3 f} \delta \left( \frac{1}{c^2 s_3 f} \frac{\partial p}{\partial \xi} + \frac{1}{s_3 f} \frac{\partial \phi}{\partial \xi} \right)
\]

\[
+ \frac{2y}{c^2 s_3 f} \delta \left( \frac{1}{c^2 s_3 f} \frac{\partial \phi}{\partial \xi} \right)
\]

\[
+ \frac{2y}{c^2 s_3 f} \delta \left( \frac{1}{c^2 s_3 f} \frac{\partial \phi}{\partial \xi} \right)
\]

\[
+ \left( \frac{2y}{c^2 s_3 f} \delta \right)^2 \left( \frac{1}{c^2 s_3 f} \frac{\partial \phi}{\partial \xi} \right)
\]

\[
(4.2.8)
\]

In obtaining the above we use the differential equations (4.1.11) to (4.1.12) themselves and not their solutions. The result is therefore applicable for any steady-state solutions. This is a point of difference from the earlier approach (Chakraborty and Prasanna (1982)). Where in they have used the steady-state solution explicitly for this reduction. As more than one class of steady-state solution are possible. The present approach of calculation, has a distinct merit over an earlier approach. In obtaining above we have also used the boundary condition at the boundary.
Equations (4.2.7) and (4.2.8) imply a variational principle in the following sense. Let us evaluate either (4.2.7) or (4.2.8) by using two trial displacements \( \gamma_1^{\alpha} \) and \( \gamma_1^{\beta} \). The latter is obtained by an arbitrary variation, in \( \gamma_1^{\alpha} \) which induces a variation \( S \epsilon^{-2} \) in \( \epsilon^{-2} \). Then if \( S \epsilon^{-2} \leq 0 \) for all variations in \( \gamma_1^{\alpha}, \gamma_1^{\alpha} \) is a solution of the original eigenvalue (Chandrasekhar and Friedmann, 1972)

we choose trial functions of the type

\[
\gamma_1^{\alpha} = p_1^{\alpha} (1 + \alpha f_1)
\]

and

\[
\gamma_1^{\beta} = p_1^{\beta} (1 + \beta f_2)
\]

where \( f_1 \) and \( f_2 \) are functions of \( \alpha \) and \( \beta \) and \( \alpha \) and \( \beta \) are adjustable parameters. Using trial functions of above type, we evaluate \( \epsilon^{-2} \) numerically by choosing \( \alpha \) and \( \beta \) such that \( \frac{\partial \epsilon^{-2}}{\partial \alpha} = \delta \) and \( \frac{\partial \epsilon^{-2}}{\partial \beta} = 0 \). We also used the same form of and as we did in Newtonian calculation.

4.3 Results and Discussion:

We use the dimensionless parameter \( \Omega = \gamma/\rho, \rho = m, \rho = m, \rho = m, \rho = m, \) and \( \alpha = \gamma/c \) and \( \beta \) and \( \gamma \) are expressed in dimensionless form. Table (4.1) and Fig. (4.1 - 4.4) shows the profiles of \( \beta, \gamma, \gamma, \) along \( \gamma = \gamma/\beta \) plane and the boundary \( \beta \) for the solution (4.1.20) to (4.1.22) for different choices of the parameters. We find that along \( \beta = \beta/\lambda \) as \( \lambda \) increases \( \beta \) first
increases, reaches a maximum value and then decreases to zero at \( t = t_1 \). The density, on the other hand, increases while decreases with the increase of \( t \). In fact for large values of \( t \) it is a constant multiple of the Keplerian velocity. It is clearly seen that thickness of the disk is quite considerable as compared to radial extension. The Newtonian counterparts of the solution (4.1.1) to (4.1.7) have been presented in Chapter 5. We find that the profile pressure, velocity and density are qualitatively the same as in the Newtonian case, however restriction (4.1.6) on the possible values of the inner edge is of pure general relativistic origin having no Newtonian counterpart.

We study the equation describing the boundary curve of a meridional section of the disk as obtained by putting \( \rho = 0 \). It is clear from Figure (4.6) that if it has a cusp at the inner edge, the boundary curve should change the sign of curvature at points such as \( \ell_1 \) and \( \ell_2 \) which are points of inflexion, where the curvature is zero. The existence of such points may be regarded as a proof of a cusp-like structure at the boundary. At such points of inflexion we have

\[
\frac{d^2 \ell}{d \tau^2} = 0, \quad \frac{d^3 \ell}{d \tau^3} \neq 0
\]

(4.3.1)

Using the solution of the study state, the points of inflexion as obtained numerically from the condition (4.3.1) for various choices of inner and outer edges are shown in Table (4.2). It is clear that for small value of \( R_0 \), such points of inflexion exist while of \( R_0 > 11.0 \) such points do not exist.
Further using the most frequently relativistic definition of angular momentum

\[ \lambda = -\mu p \mu \ell \]  

(4.3.2)

which for Schwarzschild geometry reads as

\[ \lambda = -\pi R_0 \mu (\pi \mu / M_0)^{1/2} \mu (\mu) \]  

(4.3.3)

we find that the disks of solution (B) are of constant angular momentum \( \rho_0 \), which in dimensionless form is given by

\[ \lambda_{\ell} = \sqrt{A} \]  

(4.3.4)

where \( A \) is given by (4.1.28)

The values of \( J_0 \) are presented in the Table (4.3). We obtain

\[ |\lambda_{K} (6)| < |\lambda_{\ell}| < |\lambda_{K} (4)| \]

when \( R_b \) is chosen as

\[ R_b = \frac{2R_{\alpha}}{R_{\alpha} - 4} + 0.1 \]  

(4.3.5)

which is in agreement with the conclusion drawn by Abramowicz et al (1978). Here \( \lambda_{K} = \lambda_{K} (\mu) \) is the angular momentum obtained by inserting Keplerian velocity in (4.3.3) and rewriting dimensionless form. However for \( R_b = \alpha \), we have cases when

\[ |\lambda_{\ell}| > |\lambda_{K} (\mu)| \]  

and it still shows cusp. This is not allowed in models of Abramowicz et al (1978). It appears that the models as presented by equation (4.1.25) to (4.1.27) are more general, and these include the models of Abramowicz et al (1978) as a special case.
Further rewriting the equation of boundary in cartesian coordinates \( \xi \) and shifting the origin to inner edge \( \epsilon \) the equation to the boundary for the small value of \( \tau \) becomes showing that the boundary curves is parallel to the \( \gamma \) axis at the inner edge (fig 4.5). using the non relativistic counterpart part of solution (4.1.27) we find that they do not shows any cusp, indicating the general relativistic origin of the formation of the cusp.

However, not all the general relativistic solutions shows the existence of cusp. The steady state solution (4.1.20) which retain the same form in it's non-relativistic limit is an example of this type. It does not shows any cusp. We find using the definition of angular momentum \( \ell / \hbar \) that the disk as described by solution (4.1.25) are of constant angular momentum where as those described by solution (4.1.20) are not so.

We also note that the solution (4.1.20) we dose not change it's form in non-relativistic limit and as the surface \( \rho = 0 \) forms the boundary of the disk. The size of the disk remains the same both in general relativity as well as Newtonian case. To determine the general relativistic effects on the frequency of the global axisymmetric oscillation, we use solution (4.1.20) to (4.1.22) along with their Newtonian limit.

The basic reason for adopting such solution is the following. As discussed in Chapter 3 we find that the frequency depends quite sensitively on size of the disk and therefore
adopting of the same size disk would eliminate the effect of the size on frequency estimates. Besides the size the frequency also depends on the choice of trial function (Chakraborty and Mishra 1991) and therefore it seems to reasonable that the calculations of the variation of the frequency with the variation in various parameters governing the disk structure are more reliable. We find that the frequency is systematically higher in general relativistic gravitational field that in Newtonian field. Bringing in the analogy from the acoustic oscillation of a stretched membrane where higher frequency corresponds to greater tension, it seems to indicate that the gravitational field as given by general theory of relativity is stronger compared to it's Newtonian counter part.
Appendix I

From the shape of the boundary of the disk (Fig. 2) it is clear that the inner edge \( (\theta_\mu, \rho/\tau) \) is positive. Therefore equation (4.1.11) leads to

\[
\frac{\partial}{\partial \theta} \left( \begin{array}{c} \theta \\ \rho \end{array} \right)^T - \frac{\partial^{\|}}{\partial \rho} \leq c
\]

which can be rewritten as

\[
\frac{\partial}{\partial \theta} - \frac{\partial^{\|}}{\partial \rho} \leq \frac{\partial}{\partial \rho} \left( \begin{array}{c} \theta \\ \rho \end{array} \right)^T
\]

for the extreme case when \( \rho \to \infty \) we thus have

\[
\rho \to \infty
\]

(A.1.1)

(A.1.2)

(A.1.3)
Caption for Figures and Tables

Fig.(4.1) Profile of pressure at $\Theta = \pi/2$ plane for disk having $R_a = 3.1$, and $R_b = 25.1$

Fig.(4.2) Profile of density at $\Theta = \pi/2$ plane for disk size $R_a = 3.1$, and $R_b = 25.1$

Fig.(4.3) Profile of velocity at $\Theta = \pi/2$ plane for disk having $R_a = 3.1$, and $R_b = 25.1$

Fig.(4.4) Boundary $\Sigma_B$ for the disk having $R_a = 3.1$, and $R_b = 25.1$

Table(4.1) Profile of $\frac{1}{u^2},\frac{1}{v^2},\frac{1}{w^2}$ and the thickness for the disk $k = 2$, $l = 1$, $R_a = 3.7$, $R_b = 39.4$, $R_c = 15.0$ for solution (A)

Table(4.2) Profile of infection at $R_1$ and $R_2$ for various choice of $R_a$ for solution (A). $R_b$ is kept fixed at 100.

Table(4.3) Frequency of axi-symmetric oscillation on general relativity and Newtonian gravitational for $k = 2$, $l = 1$ and for various choices of $R_a$ and $R_b$ for solution A.
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