CHAPTER - 1

INTRODUCTION

1 - 26
1.1 It has been unequivocally established that, in many cases, application of mathematical principle is mandatory for solving equations. Contextually, question arises whether a particular equation has any solution. The so called \textit{existence theorems} provide credence to the concept that solutions do exist.

Let $f$ be a function of the real variable $x$, continuous in the closed interval $[a,b]$ and assuming the values of different signs at its end points. Then the equation

\[(1.1.1) \quad f(x) = 0\]

has at least one solution inside the interval.

Existence theorems are often expressed in the form of "\textit{fixed point}" principles. For example, let us view the equation (1.1.1) in the following way. Write (1.1.1) in the form $\lambda f(x) + x = x$, where $\lambda$ is a
positive parameter. Denoting $\lambda f(x) + x$ by $F(x)$ we get the equation

$$F(x) = x.$$  \hspace{1cm} (1.1.2)

Therefore, in geometrical terms, a theorem ensuring the existence of a solution for equation (1.1.2) is formulated as **fixed point Theorem**: 

If $F$ is a continuous function mapping a closed interval into itself, then the function has at least one fixed point.

Fixed point theorems have numerous applications in Mathematics. Most of the theorems ensuring the existence of solutions for differential, integral, operator, or other equations can be reduced to fixed point theorems. They are also used in new areas of Mathematical applications, e.g., in Mathematical economics, game theory, etc.

The first is the famous French Mathematician, H. Poincare (1854-1912) the founder of the fixed
point approach, who had deep insight into its future importance for problem of Mathematical analysis and celestial mechanics, and contributed to its development.

The study of the fixed point theory began in 1910 with the proof by L.E.J. Brouwer [12]-[13] that any continuous mapping of an n-simplex into itself has a fixed point. This result was extended by Schauder [102] to compact convex sets in Banach space in 1927. Later in 1930, he [103] generalized his own result by using the condition of compactness.

In 1922, S. Banach [9] proved an interesting result known as "Banach Fixed point theorem" which shows that if \((X,d)\) is a complete metric space and \(T\) is a contraction mapping from \(X\) into itself (there exists \(0 \leq k < 1\) such that \(d(Tx,Ty) \leq k \cdot d(x,y)\) for all \(x, y \in X\)), then there exists a unique point \(x_0 \in X\) such that
(1.1.3) \( x_0 = T x_0 \)

and for fixed \( x \in X \), the sequence \( \{T^n(x)\} \) converges to \( x_0 \).

It is known that contraction mapping is continuous, but the converse is not necessarily true.

For example, a translation map \( T : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[ T(x) = x + p \quad p > 0 \]

is continuous, but not a contraction.

**Contraction Feature in Fixed Point Results**

1.2 There have been numerous generalizations of Banach's fixed point theorem by weakening its hypothesis while retaining the convergence property of successive iterates to the unique fixed point of mapping.

In 1930, R. Caccioppoli observed that it is possible to replace the contraction property by the assumption of convergence. More precisely the result of Caccioppoli is as follows:
If \((X,d)\) is a complete metric space and \(T\) be a mapping from \(X\) into itself has the property that

\[(1.2.1) \quad d(Tx,Ty) \leq \|T\| d(x,y)\]

and if \(T^n(x) = T(T^{n-1}(x))\), \(T^n(y) = T(T^{n-1}(y))\), then the sequence \(\{T^n(z)\}\) converges to a fixed point \(z_0 = T(z_0)\)

if \(\sum_{n=1}^{\infty} \|T^n\| < \infty\), where

\[(1.2.2) \quad d(T^n(x),T^n(y)) \leq \|T^n\| d(x,y).\]

Another interesting and useful generalization of contraction mappings was proposed by E. Rakotch \[93\] as follows:

**THEOREM R.** A mapping \(T\) from a complete metric space \((X,d)\) into itself is called Rakotch contraction if there exists a decreasing function \(\alpha(t)\) with \(\alpha(t) < 1\) and such that for all \(x,y \in X\),

\[(1.2.3) \quad d(Tx,Ty) \leq \alpha(d(x,y)) d(x,y).\]

In 1965, Chu and Diaz \[22\] showed that even if \(T\) is not contraction, it is possible that \(T^n\) is contraction, where \(n\) is an integer. This result was
generalized by Sehgal [104].

In 1968, Kannan [69] proved the following result:

**THEOREM K.** Let $T$ be a mapping from a complete metric space $(X,d)$ into itself satisfying the condition:

\[(1.2.4) \quad d(Tx,Ty) \leq k \left[ d(x,Tx) + d(y,Tx) \right] \]

for all $x, y \in X$, $0 \leq k < 1/2$. Then $T$ has a unique fixed point.

In 1971, Reich [95] unified the mappings of Banach and Kannan, and studied the mapping $T$ from a complete metric space $(X,d)$ into itself satisfying the condition:

\[(1.2.5) \quad d(Tx,Ty) \leq a \cdot d(x,y) + b \cdot d(x,Tx) + c \cdot d(y,Ty) \]

for all $x, y \in X$, where $0 \leq a + b + c < 1$.

Reich's theorem is stronger than Banach's and Kannan's theorem. It can be seen from the following example.

Let $X = [0, 1]$ and $T : X \to X$ be defined by

\[Tx = x/3, \quad 0 \leq x < 1, \]
Clearly \( T \) is not continuous and hence does not satisfy the contraction condition. Kannan's condition (1.2.4) is also not satisfied because if \( x = 0, y = 1/3 \), then

\[
d(T_0, T_{1/3}) = 1/2[d(0, T_0) + d(1/3, T_{1/3})].
\]

However, Reich's condition (1.2.5) is satisfied, by putting \( a = 1/6, b = 1/9 \) and \( c = 1/3 \).

In 1972, Chatterjee [20] studied a mapping \( T \) from a complete metric space \((X, d)\) into itself satisfying the condition:

\[
(1.2.6) \quad d(Tx, Ty) \leq k [d(x, Ty) + d(y, Tx)]
\]

for all \( x, y \in X \), where \( 0 \leq k < 1/2 \), and obtained a unique fixed point of \( T \).

In 1973, a more general contractive condition was considered by Hardy and Roger [48] which is as follows:

A mapping \( T \) from a complete metric space \((X, d)\) into itself is said to be a generalized contraction if
\begin{align}
\tag{1.2.7} d(Tx,Ty) & \leq a_1 d(x,Tx) + a_2 d(y,Ty) \\
& \quad + a_3 d(x,Ty) + a_4 d(y,Tx) \\
& \quad + a_5 d(x,y)
\end{align}

for all \(x, y \in X\) with \(a_i \geq 0\) (\(i=1,2,\ldots,5\)) and \(\sum_{i=1}^{5} a_i < 1\).

In 1974, Ciric [23] considered a generalized contraction condition defined as follows:

Let \(T\) be a mapping from a metric space \((X, d)\) into itself satisfying the condition:

\[
d(Tx,Ty) \leq q \max \{d(x, y), d(x, Tx), d(y, Ty), d(x,Ty), d(y,Tx)\}
\]

for every \(x, y \in X\), where \(q \in [0,1)\).

In 1975, Hussain and Sehgal [52] proved a fixed point theorem which generalized the Kannan [69], Reich [95] and Ciric [23] generalized contraction mapping theorems as follows:

**Theorem H.** Let \(T\) be a mapping from a complete metric space \((X, d)\) into itself satisfying the condition:

\[
\tag{1.2.8} d(Tx,Ty) \leq \phi(d(x,Tx), d(y,Ty), d(x,Ty)),
\]
\[d(y, Tx), d(x, y)\]

for all \(x, y \in X\). where \(\phi : [0, \infty]^5 \to [0, \infty]\) be continuous and non-decreasing in each coordinate variable and

\[\phi(t, t, a_1t, a_2t, t) < t\] for each \(t > 0\) and

\[a_i \in \{0, 1\}\] with \(a_1 + a_2 = 2\), then there exists a unique point \(u\) in \(X\) such that \(Tu = u\).

In 1975, Fisher [33] established the following expansion condition of mapping:

THEOREM F1. Let \(T\) be a mapping from a metric space \((X, d)\) into itself satisfying

(1.2.9) \(d(Tx, Ty) \leq 1/2[d(x, Tx) + d(y, Ty)]\)

for all \(x, y \in X\). Then \(T\) is identity mapping.

Fisher used expansion mapping condition in his following result:

THEOREM F2. Let \(X\) be a compact metric space, \(T: X \to X\)

and satisfying (1.2.9) for each distinct pair \(x, y \in X\).

Then \(T^r\) has a fixed point for some positive integer \(r\), and \(T\) is invertible.
Recently, Wang and others [125] obtained some fixed point theorems for expansion mappings in complete metric space $(X,d)$ in a different fusion and proved:

**Theorem W1.** Let $T$ be a surjective mapping from a complete metric space $(X,d)$ into itself satisfying the condition:

\[(1.2.10) \quad d(Tx,Ty) \geq h \cdot d(x,y) \quad \text{for all } x, y \in X, \text{ where } h > 1. \]

Then $T$ has a unique fixed point in $X$.

**Theorem W2.** Let $T$ be a surjective mapping from a complete metric space $(X,d)$ into itself. If there exist non-negative real numbers $a, b, c$ with $a+b+c > 1$ and $c < 1$ such that

\[(1.2.11) \quad d(Tx,Ty) \geq a \cdot d(x,Tx) + b \cdot d(y,Ty) + c \cdot (x,y) \quad \text{for all } x, y \in X. \]

Then $T$ has a unique fixed point in $X$.

All the above results rigorously demonstrate a trend of perpetual change in contraction conditions, notwithstanding the last four results which deals with expansion conditions.
Let $A$ and $B$ be the mappings from a metric space $(X,d)$ into itself. Then

(i) $A$ and $B$ are said to commute if $ABx = BAx$ for all $x \in X$, and

(ii) a point $t \in X$ is said to be the common fixed point of $A$ and $B$ if $At = Bt = t$.

It has been convincingly demonstrated that the scope of using commuting condition is limited to only those fixed point results which include more than one self-mappings. The reason is, results containing more than one self-mapping normally gives coincidence points. In fact, commuting condition facilitates obtaining fixed point from such coincidence points which are common to those self-mappings. This aspect has provided ample stimulation for the authors to use and investigate commuting condition for their fixed point results. Consequently, commuting condition was
further weakened by weak commutativity [105], compatibility [60], and compatibility of type (A) [86] to obtain fixed point.


In 1979, Yeh [128] extended the result of Jungck [59] and obtained a unique common fixed point of three continuous self mappings of a complete metric space. Further, Barada [11], Chang [18], Conserva [25], Das and Naik [26], Ding [30], Fisher [34-35] and Singh and Singh [120], proved some more results on these lines.


S. Sessa [105], weakening the notion of commutativity for point to point mappings, established the idea of weak commutativity for two self mappings f and
of a metric space \((X,d)\), i.e.,
\[ d(fgx,gfx) \leq d(fx,gx) \]
for all \(x\) in \(X\). According to this concept, he extended the result of Jungck [59]. He and others gave some common fixed point theorems for weakly commuting mappings in metric and 2-metric spaces, (see, e.g., Bhaskaran and Subrahmanyam [10], Fisher and Sessa [36], Naidu and Prazad [88], Rhoades and Sessa [100], Sessa [105], Sessa, Mukherjee and Som [106], Sessa and Fisher [107], and Singh, Ha and Cho [121]).

Recently, Jungck [60] proposed further weak version of the concepts of commuting mappings and weakly commuting mappings, which is called compatible mappings. He and others proved nice common fixed point theorems using this concept. They are Jungck [60-62], Kaneko and Sessa [66], Kang, Cho and Jungck [68], Sessa, Rhoades and Khan [108], and Rhoades, Park and Moon [99] etc.
More recently, Murthy, Chang, Cho and Sharma [86] introduced the concept of compatible mappings of type (A) in 2-metric spaces and gave some fixed point theorems for these mappings. These concepts further extended in several spaces (see, e.g., [63, 87]).

**Altman Contraction for Fixed Point Results**

1.4 In 1975, M. Altman [3] generalized Banach fixed point theorem characterized by novel conditions that stimulated several authors specially because Cauchy sequence could be obtained more conveniently by way of Altman conditions.

Altman contraction was as below:

**THEOREM A**. Let \( X \) be a complete metric space and \( f : X \rightarrow X \) a generalized contraction, i.e.,

\[
d(fx, fy) \leq Q(d(x, y))
\]

for all \( x, y \in X \), where \( Q \) satisfies the following:

(a) \( 0 < Q(t) < t \) for all \( t \in (0, t_1] \),

(b) \( g(t) = t/(t-Q(t)) \) is non-increasing.
\[ (c) \quad \int_0^{t_1} q(t) \, dt < \infty \]

and

\[ (d) \quad Q \text{ is non-decreasing.} \]

Then \( f \) has a unique fixed point in \( X \).

In 1986, Watson and others [126], Carbone and Singh [15], and Carbone and others [16] (using the concept of weak commutativity [105]) proved fixed point theorems in Altman sense [3] of self-mappings on a complete metric space.

**A Concept Of Compatibility Of Type \((S)\)**:

1.5 In section 1.3, the nature and development of commuting condition for fixed point results have been documented. Deriving inspiration from such phenomena we propose to define a concept known as compatibility of type \((S)\) in \( 2 \)-metric space.

It is indeed remarkable that every commuting pair of self-mappings are compatible of type \((S)\), however the converse is not necessarily true.
Before we introduce the concept of compatibility of type (S) in 2-metric space, a brief discussion on 2-metric space is needed.

In a series of papers [37]-[40], Gahler introduced a notion of 2-metric space as below:

A 2-metric space is a set $X$ with a real valued function $d$ on $X \times X \times X$ satisfying the following conditions:

$(m_1)$ for distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.

$(m_2)$ $d(x,y,z)=0$ if at least two of $x, y, z$ are equal.

$(m_3)$ $d(x,y,z) = d(x,z,y) = d(y,z,x)$.

$(m_4)$ $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $x,y,z,u \in X$.

While investigating the above concept Gahler [37]-[40] considered 2-metric, a real valued function of a point triple on a set $X$, whose abstract properties were suggested by the area function for a triangle.
determined by a triple in Euclidean space. According to Gahler [37] – [40] a given 2-metric was a natural topology associated with area function.

For the first time Sharma and others [114] have introduced the contraction type mappings in 2-metric space. Iseki [54] generalized Cirić’s result in 2-metric space. During the last three decades, several authors (e.g., [21], [54]-[55], [86], [88], [94], [96], [109-112], [115-119]) have studied the aspects of the fixed point theory in the setting of 2-metric spaces. They have been motivated by various concepts already known for metric spaces and have thus introduced analogous of such concepts in the framework of the 2-metric spaces. Khan [70], Murthy-Chang-Cho-Sharma [86] and Naidu-Prasad [88] introduced the concepts of weakly commuting pair of self mappings, compatible pairs of self mappings of type (A) in a metric space and the weak continuity of a 2-metric
Now we define compatible mappings of type (S) in 2-metric space as below:

**DEFINITION 1.** Let $S$ and $T$ be mappings from a 2-metric space $(X,d)$ into itself. The mappings $S$ and $T$ are said to be compatible of type (S) if

$$
\lim_{n \to \infty} d(SSTx_n, STSx_n, a) = 0 \quad \text{and} \quad \lim_{n \to \infty} d(TSTx_n, TTSx_n, a) = 0,
$$

for all $a \in X$, whenever $\{x_n\}$ is a sequence in $X$ such that

$$
\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \quad \text{for some } t \in X.
$$

Following example shows that every commuting pair of self-mappings are compatible of type (S) but converse is not necessary true.

**EXAMPLE 1.** Let $X = \{1, 2, 3, 4\}$ and define a 2-metric $d : X \times X \times X \to \mathbb{R}$ as follows:

$$
d(x, y, z) =
\begin{cases}
0, & \text{if } x = y \text{ or } y = z \text{ or } z = x \\
1/2, & \text{if } x, y, z \text{ are distinct and } \{x, y, z\} \neq \{1, 2, 3\}.
\end{cases}
$$
Define two mappings from 2-metric space $(X, d)$ into itself by

$$S(1) = S(2) = S(3) = 2, \ S(4) = 1$$

and

$$T(1) = T(2) = T(3) = 3, \ T(4) = 1.$$  

Since

$$ST(4) \neq TS(4).$$

It means that the pair $(S, T)$ is not commuting on $X$. But

$$d(STT(4), STS(4), a) = 0$$

and

$$d(TST(4), TTS(4), a) = 0$$

for all $a \in X$, where $S(4) = T(4) = 1$.

Clearly that the pair $(S, T)$ is compatible of type $(S)$, but it is not commuting on $X$.

**Fuzzy Concept For Fixed Point**

1.6 The notion of fuzzy set, introduced by Zadeh [129] in 1965, became very interesting for both pure and applied Mathematicians. Later, Kaleva and Sheikkala
have proposed the concept of fuzzy metric space. It has also raised enthusiasm among engineers, biologists, psychologists, economists and others.

Among the branches of pure Mathematics, General topology was the first branch to which fuzzy sets have been applied systematically first by Chang [17] in 1968.

Still, there are many viewpoint regarding the notion of a metric in fuzzy topology. Those can be divided into two separate lines. The first line is formed on which fuzzy metric on a set X is treated as a map $d : X \times X \rightarrow \mathbb{R}^+$ where $X \subseteq \mathcal{P}(X)$.

Along the above line, investigations were done by Deng [28], Erceg [32], and Hu [51] satisfying some collection of axioms that are analogous to the ordinary metric space. In such an approach numerical distance are set up between fuzzy objects. The authors of this approach are interested in fuzzy
uniformity ([57]), fuzzy topology ([32], [28], [51]),
and separation properties in metric space ([7], [28],
[32], [51]).

Investigations along second line deals with the
distance between objects is fuzzy, the objects
themselves may be fuzzy or not. Such problems were
studied by Eklund and Gahler [31], Kaleva [64], and
Kaleva and Sheikshala [65].

In a paper Kramosil and Michalek [77] introduced
the concept of fuzzy metric space. Using the idea of
Kramosil and Michalek [77] and along the lines of 2-
metric space, Sharma [113] defined fuzzy 2-metric space
as follows:

A fuzzy 2-metric space is 3-tuple \((X, M, *)\), where
\(X\) is a nonempty set, \(*\) is a continuous norm and \(M\) is a
fuzzy set \(X^3 \times [0, \infty)\) satisfying:

\[
(M_1) \quad M(x, y, z, t) = 0
\]

for all \(x, y, z \in X\) and \(t \in [0, 1]\).
(M2) \( M(x, y, z, t) = 1 \) for each \( t > 0 \), when at least two of the three points are equal.

(M3) \( M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t) \).

(M4) \( M(x, y, z, t_1 + t_2 + t_3) \geq M(x, y, w, t_1) \).
   
   * \( M(x, w, z, t_2) \)
   
   * \( M(w, y, z, t_3) \)

for all \( x, y, z \in X \) and \( t_1, t_2, t_3 \in [0,1] \).

The function value \( M(x, y, z, t) \) may be interpreted as the probability that the area of triangle is less than \( t \) and also the function \( M(x, y, z, \cdot) : [0, \infty) \to [0,1] \) is left-continuous.

He [49], Heilpern [50], Jong and others [58] have studied some fixed point results in fuzzy metric space.

**Fixed Point Results In Convex Metric Spaces**

1.7 In 1970, Takahashi [122] has introduced the concept of convexity in metric space as below:

Let \( X \) be a metric space and \( I \) be the closed
unit interval. A mapping \( W : X \times X \times I \rightarrow X \) is said to be a convex structure on \( X \) if for all \( x, y \in X, \lambda \in I \),
\[
d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, y),
\]
for all \( u \in X \).

Metric space \((X, d)\) together with a convex structure is called Takahashi convex metric space.

He generalized some fixed point theorems previously proved for Banach space. Subsequently, Guay and Singh [45], Hadzic and Gajic [47], Machado [81], Naimpally and Singh [89], Talman [123] are among others who have obtained fixed point results in convex metric space in the sense of Takahashi.

**Fixed Point Results In Uniform Spaces**

1.8 In the past several years there have been extensive debate on the existence of the fixed points of mappings that satisfy contractive-type conditions in Uniform space. Various authors tried to utilize different conditions. Angelov

In 1974, Acharya [1] proved the following theorem as generalization of Banach contraction principle in Uniform space:

**THEOREM A’** Let \((X,\upsilon)\) be a sequentially complete (Hausdorff) Uniform space and \(S\) be a mapping from \(X\) into itself such that for each \(x, y \in X\)

(*) \((x,y) \in \upsilon\) implies \((Sx, Sy) \in k\upsilon\),

where \(0 < k < 1\), and \(\upsilon\) is an arbitrary member of a base generated by an associated family of pseudo-metrics of the uniformity \(\upsilon\). Then \(S\) has a unique fixed point in \(X\).

A number of papers (cf. Acharya [2], Kubiak and Cho [78], Rhoades [97]) have appeared which established
1.9 In Chapter-2, firstly the difference between compatible mappings and compatible mappings of type (A) has been outlined with suitable examples. Consequently some common fixed point theorems for Altman type contraction condition under the compatible condition of type (A) have been established.

The primary objective of introducing the notion of compatible mappings of type (S) in 2-metric space has been highlighted in Chapter - 3. Initially, some coincidence point theorems are proved and then fixed point theorems for compatible mappings of type (S) have been proved in 2-metric spaces with the help of such coincidence point theorem.

Chapter - 4 deals with fixed point theorems for multi-valued mappings and single valued mappings under more generalized contraction conditions in convex
metric spaces.

The concept of compatible of type (A) in fuzzy 2-metric space has been introduced in Chapter - 5 and attempts have been made to prove some fixed point theorems for nonlinear contraction mappings and expansion mappings under the compatible condition of type (A).

The proof of some fixed point theorems in Uniform space for more general contraction type condition than those shown by Acharya has been included in Chapter-6. The generalization of Rhoades's results has also been depicted.

Last but not the least the thesis contains adequate and pertinent bibliography and a list of our research publications.

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