CHAPTER - 6
SOME FIXED POINT THEOREMS IN UNIFORM SPACE
111-127
6.1 In this chapter, we prove some fixed point theorems for more general contractive type condition than those shown by Acharya [1] and our results generalize several known results in uniform space.

6.2 Many authors have investigated fixed point results in uniform space. Angelov [5], used \( \phi \)-contraction Mishra and Singh [84-85] used multivalued mappings, Ganguli [44] used three self mappings and Bhakta and Chakrabarty [8] used contractive semi-group of mappings in uniform space generated by family of pseudo-metrics to obtain fixed point. One different from such numerous generalizations of the Banach contraction principle is that of Acharya [1], according to which a self-mapping \( S \) on a sequentially complete (Hausdorff) uniform space \( (X, \mathcal{U}) \) has a unique fixed point if it satisfies the following condition:
for each \( x, y \in X \)

\((*)\) \((x, y) \in V \) implies \((Sx, Sy) \in K V \).

where \( 0 < k < 1 \), and \( V \) is an arbitrary member of a base
generated by an associated family of pseudo-metrics of
the uniformity \( u \).

Some papers such as Acharya [1-2], Kubiak and
Cho [78], Rhoades [97] have appeared which establish
fixed point theorems in Uniform space satisfying more
general condition than the condition \((*)\), but we
propose to give most general condition in the following
section.

6.3 Throughout this chapter, let \((X, u)\) be a
sequentially complete uniform space. A sequence \( \{x_n\} \) is
said to converge to a point \( x \) of \( X \), if for each members
\( U \) of \( u \), there exists a positive integer \( N \) such that
\( (x_n, x) \in U \) for all \( n \geq N \). For any pseudo-metric \( p \) on \( X 
\) and any \( r > 0 \), let

\[ V(p, r) = \{(x, y) : x, y \in X \quad \text{and} \quad p(x, y) < r\} \]
Let \( P \) be the family of pseudo-metrics on \( X \) generating the uniformity \( u \) and let \( v \) denote the family of the form \( \bigcap_{i=1}^{n} V(p_i, r_i) \), where \( p_i \in P, r_i > 0, i = 1, 2, 3, \ldots, n \) (\( n \) is not fixed). The collection \( v \) forms a base for the uniformity \( u \). Other properties of uniformity may be observed from \([1,2]\).

Let \( \alpha(x,y), \beta(x,y), \gamma(x,y), \eta(x,y), \xi(x,y), \mu(x,y), \phi(x,y), \psi(x,y), \Delta(x,y) \) be non-negative functions of \( x \) and \( y \) satisfying

\[
\sup \left\{ \alpha(x,y) + \beta(x,y) + \gamma(x,y) + \eta(x,y) + \xi(x,y) + \mu(x,y) + \phi(x,y) + \psi(x,y) + \Delta(x,y) \right\} = \lambda < 1.
\]

Let \( T \) be a self-mapping of \( X \) satisfying:

for any \( V_i \in V, i = 1, 2, 3, \ldots, 9 \), and \( x, y \in X \):

\[
(x,y) \in V_1, (x,Tx) \in V_2, (y,Ty) \in V_3, (x,Ty) \in V_4, \ldots
\]
(v, Tx) ∈ V_5, (v, T^2 x) ∈ V_6.

(x, T^2 x) ∈ V_7, (Tx, T^2 x) ∈ V_8.

(1v, T^2 x) ∈ V_9.

implies

(6.3.1) (Tx, Ty) ∈ α(x, y) V_10 β(x, y) V_20 λ(x, y)

V_30 ρ(x, y) V_40 ξ(x, y) V_5 μ(x, y)

V_60 φ(x, y) V_70 ψ(x, y) V_80 Δ(x, y) V_9.

Now we establish the following result for more generalized contractive type condition than the conditions used by Acharya [1-2], Rhoades [97]:

THEOREM 1 [x]. Let \{T_n\} be a sequence of mappings, with fixed points z_n, such that each T_n : X → X which satisfies (6.3.1). Then T has as unique fixed point z and z_n → z.

[x] SHARMA, B.K. AND DEWANGAN, C.L.. Some fixed point theorems in uniform space, Ult. Sci. 6(1) (1994), 138-140.
PROOF. First, we shall prove $T$ has a unique fixed point in $X$.

Pick $p$ to be the Minkowski pseudo-metric corresponding to $V$. Write

\[ p(x,y) = r_1, \]
\[ p(x,Tx) = r_2, \]
\[ p(y,Ty) = r_3, \]
\[ p(x,Ty) = r_4, \]
\[ p(y,Tx) = r_5, \]
\[ p(y,T^2x) = r_6, \]
\[ p(x,T^2x) = r_7, \]
\[ p(Tx,T^2x) = r_8, \]
\[ p(Ty,T^2x) = r_9. \]

Fix $\epsilon > 0$. Then

\[ (x,y) \in (r_1+\epsilon)V_1, \quad (x,Tx) \in (r_2+\epsilon)V_2, \]
\[ (y,Ty) \in (r_3+\epsilon)V_3, \quad (x,Ty) \in (r_4+\epsilon)V_4, \]
\[ (y,Tx) \in (r_5+\epsilon)V_5, \quad (y,T^2x) \in (r_6+\epsilon)V_6, \]
\[ (x,T^2x) \in (r_7+\epsilon)V_7, \quad (Tx,T^2x) \in (r_8+\epsilon)V_8. \]
\[(x, y) \in (r_9 + \varepsilon) V_9.\]

From (6.3.1) with \(a = a(x, y), \beta = \beta(x, y), \text{ etc}.\)

\[(x, y) \in a(r_1 + \varepsilon) V_10 \beta(r_2 + \varepsilon)\]

\[v_20 \lambda(r_3 + \varepsilon) v_30 \eta(r_4 + \varepsilon)\]

\[v_40 \varepsilon(r_1 + \varepsilon) v_50 \mu(r_6 + \varepsilon)\]

\[v_60 \phi(r_7 + \varepsilon) v_70 \psi(r_8 + \varepsilon)\]

\[v_80 \Delta(r_1 + \varepsilon) V_9,\]

which implies

\[p(Tx, Ty) \leq a(r_1 + \varepsilon) + \beta(r_2 + \varepsilon)\]

\[+ \lambda(r_3 + \varepsilon) + \eta(r_4 + \varepsilon)\]

\[+ \varepsilon(r_1 + \varepsilon) + \mu(r_6 + \varepsilon)\]

\[+ \phi(r_7 + \varepsilon) + \psi(r_8 + \varepsilon)\]

\[+ \Delta(r_1 + \varepsilon) V_9,\]

since \(\varepsilon\) is a arbitrary

(6.3.2) \[p(Tx, Ty) \leq \alpha(x, y) + \beta(x, Tx)\]

\[+ \gamma(x, Tx) + \gamma(x, Ty)\]

\[+ \delta(y, Tx) + \delta(y, T^2 x)\]

\[+ \phi(x, T^2 x) + \psi(Tx, T^2 x)\]

116
\[ \Delta \Delta p(Tv, T^2x) \]
\[ \leq \max \{p(x,y), p(x,Tx), \]
\[ p(y,Tv), p(x,Ty), p(y,Tx), \]
\[ p(y,T^2x), p(Tx, T^2x), p(Ty, T^2x) \} \]
It follows that $p(x, T^k x) = \delta_p[O(x, n+1)]$ for every integer $k \leq n$. Let $h$ be any positive integer, then there exists $T^k x \in O(x, h)$ ($1 \leq k \leq n$) such that

$$p(x, T^k x) = \delta_p[O(x, n+1)].$$

Applying triangle inequality, we obtain

$$p(x, T^k x) \leq p(x, T x) + p(T x, T^k x) \leq p(x, T x) + \lambda \cdot \delta_p[O(x, n+1)]$$

$$\leq p(x, T x) + \lambda \cdot p(x, T^k x).$$

Therefore,

$$\delta_p[O(x, n+1)] = p(x, T^k x) \leq \frac{1}{1-\lambda} p(x, T x).$$

We shall show that the sequence $\{x_n\}$ in a Cauchy sequence. Let $n$ and $m$ $(n < m)$ be any positive integers. Since $T$ is satisfying condition (6.3.2) then we have

$$p(T^n x, T^m x) \leq p(T T^{n-1} x, T^{m-n+1} x)$$

$$\leq \lambda \cdot \delta_p[O(T^{n-1} x, m-m+2)].$$

It follows, there exists an integer $k$, $1 \leq k \leq m-n+1$ such that
\[ \delta_0[0(T^{m-n}x, m-n+2)] = p(T^{m-n}x, T^{n+1}x). \]

Again, we have

\[ p(T^{m-n}x, T^{n+1}x) = p(T^{n-2}x, T^{n+1}x) \]
\[ \leq \lambda \cdot \delta_0[0(T^{n-2}x, k+2)] \]
\[ \leq \lambda \cdot \delta_0[0(T^{n-2}x, m-n+3)]. \]

Therefore, we have the following system of inequalities.

\[ p(T^n x, T^m x) \leq q \cdot \delta_0[0(T^{n-1}x, m-n+2)] \]
\[ \leq q^2 \cdot \delta_0[0(T^{n-2}x, m-n+3)]. \]

Proceeding in this manner, we obtain

\[ p(T^n x, T^m x) \leq \lambda \cdot \delta_0[0(T^{n-1}x, m-n+2)] \]
\[ \leq \lambda^n \cdot \delta_0[0(x, m+1)] \]
\[ \leq \frac{\lambda}{1-\lambda} p(x, Tx). \]

Therefore \( \{x_n\} \) is Cauchy. Since \( X \) is complete, \( \{x_n\} \) converges. Call the limit \( z \). From (6.3.2)

\[ p(z, Tz) \leq p(z, x_{n+1}) + p(x_{n+1}, Tz) \]
\[ \leq p(z, x_{n+1}), \lambda \max p(x_n, z), p(x_n, x_{n+1}), \]
\[ p(z, Tz), p(x_n, Tz)p(z, x_{n+1}), p(z, x_{n_2}). \]
Taking the limit at \( n \to \infty \) we obtain

\[ p(z, Tz) \leq \rho(z, Tz), \]

which implies that

\[ p(z, Tz) = 0. \]

Therefore \((z, Tz) \in V\) for every \( V \) in \( v \) and \( z = Tz \).

For uniqueness, assume \( z \) and \( w \) are two fixed point of \( T \). Let \( V \in v \) and let \( p \) denote the corresponding pseudo-metric. From (6.3.2)

\[ p(z, w) \leq \rho(z, w), \]

which implies that

\[ p(z, w) = 0, \]

i.e.,

\[ (z, w) \in V. \]

Since \( V \) is arbitrary,

\[ z = w. \]

Now we show that \( z_n \to z \).

Let \( V \in v \) with corresponding pseudo-metric \( \rho \).
\[ A = \mathbf{z}_{1,\mathbf{z}} \ldots. \]

\[ p(z_{n}, \mathbf{z}) \]

\[ = p(T_{n}\mathbf{z}_{n}, \mathbf{z}) \]

\[ \leq p(T_{n}\mathbf{z}_{n}, \mathbf{z}) + p(T_{n}\mathbf{z}, \mathbf{z}) - p(T_{n}\mathbf{z}_{n}, \mathbf{z}) \leq p(T_{n}\mathbf{z}, \mathbf{z}). \]

Since \( T_{n} \to \mathbf{T} \) uniformly, there exists a positive integer \( N \) such that for all \( n \geq N \), \( (T_{n}\mathbf{z}_{n}, \mathbf{z}) \in V \) for each \( z_{n} \in A \). From (6.3.2)

\[ p(T_{n}\mathbf{z}, \mathbf{z}) \leq \max \{ p(z_{n}, \mathbf{z}), p(z_{n}, \mathbf{z}_{n}) \}. \]

For each \( n \) such that the maximum is equal to \( p(z_{n}, \mathbf{z}) \), we obtain

\[ p(z_{n}, \mathbf{z}) \leq (1/(1-\lambda)) p(z_{n}, \mathbf{z}_{n}). \]

Hence for each \( n \) such that the maximum is equal to \( p(z_{n}, \mathbf{z}) \), we obtain

\[ p(z_{n}, \mathbf{z}) \leq (1+\lambda)p(z_{n}, \mathbf{z}_{n}). \]

In either case, \( p(z_{n}, \mathbf{z}) \leq (p(z_{n}, \mathbf{z}_{n})/(1-\lambda)), \) so that

\[ (z_{n}, \mathbf{z}) \in (1-\lambda) V \]

for all \( n \geq N \). Since \( V \) is arbitrary, \( z_{n} \to \mathbf{z} \). This completes the proof of the theorem.
Further, we establish following common fixed point theorem for a pair of mappings satisfying general contractive type condition as considered in Theorem 1:

**THEOREM 2.** Let $T_1$, $T_2$ be self mappings of $X$ satisfying:

$(x,y) \in V_1$, $(x, T_1x) \in V_2$, 
$(y, T_2y) \in V_3$, $(x, T_2y) \in V_4$, 
$(y, T_1x) \in V_5$, $(y, T_1^2x) \in V_6$, 
$(x, T_1^2x) \in V_7$, $(T_1x, T_1^2x) \in V_8$, 
$(T_2y, T_1^2x) \in V_9$.

implies

$(6.3.3)$

$(T_1x, T_2y) \in \alpha(x,y)$ $V_{10}$ $\beta(x,y)$ $V_{20}$ $\lambda(x,y)$

$V_{30} \eta(x,y)$ $V_{40}$ $\xi(x,y)$ $V_{5}$ $\mu(x,y)$

$V_{60} \phi(x,y)$ $V_{70}$ $\psi(x,y)$ $V_{80}$ $\Delta(x,y)$ $V$

for any $V_i \in V$, $i = 1,2,3,...,9$ and $x, y \in X$. Then $T_1$ and $T_2$ have a unique common fixed point $z$.

**PROOF.** Let $x_0 \in X$. Define $\{x_n\}$ by $x_0, x_1 = T_1x_0, x_2 = T_2x_1, ..., x_{2n} = T_2x_{2n-1}, x_{2n+1} = T_1x_{2n}, ...$. Let $V \in V$

with $p$ the corresponding pseudo-metric. Write

122
\[ p(x, y) = r_1. \]
\[ p(x, T_1 x) = r_2. \]
\[ p(y, T_2 y) = r_3. \]
\[ p(x, T_2 y) = r_4. \]
\[ p(y, T_1 x) = r_5. \]
\[ p(y, T_1^2 x) = r_6. \]
\[ p(y, T_1^2 x) = r_7. \]
\[ p(T_1 x, T_1^2 x) = r_8. \]
\[ (T_2 y, T_1^2 x) = r_9. \]

Fix \( \varepsilon > 0 \). Then
\[ (x, y) \in (r_1 + \varepsilon)V_1. \]
\[ (x, T_1 x) \in (r_2 + \varepsilon)V_2. \]
\[ (y, T_2 y) \in (r_3 + \varepsilon)V_3. \]
\[ (x, T_2 y) \in (r_4 + \varepsilon)V_4. \]
\[ (y, T_1 x) \in (r_5 + \varepsilon)V_5. \]
\[ (y, T_1^2 x) \in (r_6 + \varepsilon)V_6. \]
\[ (x, T_1^2 x) \in (r_7 + \varepsilon)V_7. \]
\[ (T_1 x, T_1^2 x) \in (r_8 + \varepsilon)V_8. \]
\[ (T_2 y, T_1^2 x) \in (r_9 + \varepsilon)V_9. \]
From (6.3.3), with \( \alpha = \alpha(x,y) \), \( \beta = \beta(x,y) \), etc...

\[(r_1x, T_2y) \in \alpha(r_1 + \epsilon) \cup \beta(r_2 + \epsilon) \]

\[\nu_2 \alpha(r_3 + \epsilon) \cup \nu_3 \mu(r_4 + \epsilon)\]

\[\nu_4 \psi(r_5 + \epsilon) \cup \nu_5 \mu(r_6 + \epsilon)\]

\[\nu_6 \phi(r_7 + \epsilon) \cup \nu_7 \psi(r_8 + \epsilon)\]

\[\nu_8 \Delta(r_9 + \epsilon) \]

which implies

\[p(r_1x, T_2y) \leq \alpha(r_1 + \epsilon) + \beta(r_2 + \epsilon)\]

\[+ \lambda(r_3 + \epsilon) + \eta(r_4 + \epsilon)\]

\[+ \xi(r_5 + \epsilon) + \mu(r_6 + \epsilon)\]

\[+ \phi(r_7 + \epsilon) + \psi(r_8 + \epsilon)\]

\[+ \Delta(r_9 + \epsilon).\]

Since \( \epsilon \) is arbitrary

\[p(T_1x, T_2y) \leq \alpha p(x,y) + \beta p(x, T_1x) + p(y, T_2y)\]

\[+ \phi(x, T_1x) + \psi(T_1x, T_2x)\]

\[+ \phi(T_1x, T_2x) + \psi(T_2y, T_2x)\]

124
s \max \{p(x, y), p(x, T_1 y), p(y, T_2 y), p(x, T_2 y), p(y, T_1 x), p(x, T_1 x), p(y, T_2^2 x), p(x, T_1^2 x), p(T_2 y, T_1^2 x)\}.

remaining part of proof follows the lines of Theorem 1.

Following theorem is an extension of Theorem 1 when $T$ is in form of $T^q$:

**THEOREM 3.** Let $T : X \to X$ such that $T^q$ satisfies (6.3.1) for some fixed positive integer $q$. Then $T$ has a unique fixed point $w$ and iterates $\{T^q w\}$ converges to $z$ for each $x \in X$.

**PROOF.** Using proof of the Theorem 1 and putting $T = T^q$, we can obtain that $T^q$ has a unique fixed point $z$, i.e., $z = T^q(z)$. Therefore $T(z) = T^{q+1}(z) = T^q(T(z))$, and $T(z)$ is also a fixed point of $T^q$. Uniqueness implies $z = T(z)$.
Following theorem is an extension of Theorem 2 when \( T_1 \) and \( T_2 \) are in form of \( T_1^p \) and \( T_2^q \) respectively:

**THEOREM 4.** Let \( T_1, T_2 \) be self mappings of \( X \). Suppose there exist integers \( p \) and \( q \) such that \( T_1^p, T_2^q \) satisfies (6.3.3). Then \( T_1 \) and \( T_2 \) have a common unique fixed point \( z \).

**PROOF.** Follows from the method of Theorem 2.

Let \( \alpha(x,y), \beta(x,y), \gamma(x,y), \eta(x,y), \xi(x,y) \) be non-negative functions of \( x \) and \( y \) satisfying

\[
\sup \{ \alpha(x,y) + \beta(x,y) + \gamma(x,y) + \eta(x,y) + \xi(x,y) \} = \lambda < 1.
\]

Let \( T \) be a self-mapping of \( X \) satisfying:

for any \( V_i \in \mathcal{V}, i = 1,2,3,...,5 \), and \( x, y \in X \):

\[
(x,y) \in V_1, (x,Tx) \in V_2,
\]

\[
(y,Ty) \in V_3, (x,Ty) \in V_4,
\]

\[
(y,Tx) \in V_5,
\]

implies 

126
(6.3.5) \((x, y) \in a(x, y) \land \exists (x, y) \land \lambda (x, y) \land \chi (x, y) \land \nu (x, y) \) 

\(\nu_3 \in (x, y) \land \nu_4 \in (x, y) \land \nu_5 \in (x, y)\).

Theorem 2 of Rhoades [97] is obtained as Corollary from Theorem 1 as below:

**COROLLARY 1.** Let \(T_i : X \to X, i = 1, 2, \ldots\) be a sequence of mappings, each of which satisfies the condition (6.3.5) for the same function \(\alpha, \beta, \gamma, \eta, \xi\). If \(\{T_n\}\) converges point wise to a function \(T\), then \(T\) has a unique point \(z\) and \(\{z_n\}\) converges to \(z\), where each \(z_n\) is the unique fixed point corresponding to \(T_n\).

Theorem 1 of Rhoades [97] obtained as Corollary from Theorem 2 as below:

**COROLLARY 2.** Let \(T : X \to X\) which satisfies (6.3.5).

Then \(T\) has a unique fixed point \(z\) and \(\{T^n x\}\) converges to \(z\) for each \(x \in X\).

***** **