Chapter 2

Periodic points of toral automorphisms

2.1 General introduction

The automorphisms of the two-dimensional torus are rich mathematical objects possessing interesting geometric, algebraic, topological and measure theoretic properties.

The torus $\mathbb{T}^2 = \frac{\mathbb{R}^2}{\mathbb{Z}^2}$ *is here viewed as the topological group* $[0,1) \times [0,1)$ *with coordinate-wise addition modulo 1.*

Let $GL(2,\mathbb{Z})$ be the set of all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a,b,c,d \in \mathbb{Z}$ and $Det(A) = ad - bc = \pm 1$.

Each such matrix A gives a linear map on $\mathbb{R}^2$ by
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
We define an automorphism on the torus $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ by $T_A(x_1,x_2) = (ax_1 + bx_2, cx_1 + dx_2)(mod 1)$.

Now we have,

Proposition 2.1.1. *[29] Every automorphism T_A (as defined above) on the torus is a homeomorphism.*

Proof. The map T is clearly continuous, since if $|x_1-y_1|, |x_2-y_2| < \epsilon$ then $|(T_A(x_1,x_2))_1-(T_A(y_1,y_2))_1| < (|a|+|b|)\epsilon$ and $|(T_A(x_1,x_2))_2-(T_A(y_1,y_2))_2| < (|c|+|d|)\epsilon$.(Here suffix 1 refer to the first coordinate and 2 refers to the second coordinate.)

To see that T_A is invertible we note that if we write the inverse matrix $A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$ then since $ad-bc = \pm 1$ we see that $A^{-1} \in GL(2,\mathbb{Z})$. The inverse to T_A is then the toral automorphism associated to A^{-1}, i.e. $T_{A^{-1}}$.

□

On the other hand, in the following proposition we prove that every continuous automorphism on the torus is induced by a matrix from $GL(2,\mathbb{Z})$. Let $Aut(\mathbb{T}^2)$ to denote the set of all continuous automorphisms on the torus.

Proposition 2.1.2. *The above map $A \mapsto T_A$ from $GL(2,\mathbb{Z})$ to $Aut(\mathbb{T}^2)$ is surjective.*

Proof. let $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ be any continuous toral automorphism. Since ϕ is continuous at $(0,0)$ there exist $\delta > 0$ such that $\phi(([0,\delta) \times [0,\delta))) \subset [0,\frac{1}{2}) \times [0,\frac{1}{2})$ and such that $\phi(X+Y) = \phi(X) + \phi(Y)$ for all $X, Y \in [0,\delta) \times [0,\delta)$, where $+$ denotes the usual addition in $\mathbb{R}^2$.

Now, observe that $\phi(\lambda X) = \lambda\phi(X)$ for all $\lambda \in (0,1)$ and for all $X, Y \in [0,\delta)\times[0,\delta)$, because the set $\{\lambda \in (0,1) |\, \phi(\lambda X) = \lambda\phi(X)\}$ is a closed set contains all dyadic rationals (Since the set contains $\lambda = \frac{1}{2}$, using the additivity of ϕ, it contains all numbers of form $\lambda = \frac{m}{2^n}$). The set of all dyadic rationals is dense in $[0,1]$ and then by continuity the set contains all $\lambda \in (0,1)$.

Hence $\phi|_{[0,\delta)\times[0,\delta)} = L|_{[0,\delta)\times[0,\delta)}$ for some linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$. This linear transformation induces an integer matrix A with determinant ± 1 such that

$Ax = \phi(x)$ for all $x \in \mathbb{T}^2$ [The kernel of an endomorphism (different from the zero map), on a connected topological group cannot have nonempty interior]. Hence the proof.

$\square$

Remark 2.1.3. In fact the 1-1 correspondence in the above proposition is a group isomorphism.

For each $A \in GL(2,\mathbb{Z})$, let $P(T_A)$ denote the set of all periodic points of T_A.

Proposition 2.1.4. *[21]*

For any $A \in GL(2,\mathbb{Z})$ the set $P(T_A)$ is dense in $[0,1) \times [0,1)$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$.

We prove that $P(T_A) \supset \mathbb{Q}_1 \times \mathbb{Q}_1$, where $\mathbb{Q}_1 = \mathbb{Q} \cap [0,1)$. A general element in $\mathbb{Q}_1 \times \mathbb{Q}_1$ is of the form $x = (\frac{p_1}{q}, \frac{p_2}{q})$ where $p_1, p_2, q \in \mathbb{Z}$ with $0 \leq p_1, p_2 < q$. We note that $T_A(X) =$ (fractional part of $\frac{ap_1}{q} + \frac{bp_2}{q}$, fractional part of $\frac{cp_1}{q} + \frac{dp_2}{q}$) = an element of the form $(\frac{m}{q}, \frac{n}{q})$ where $0 \leq m, n < q$. Note that, for a fixed $q \in \mathbb{N}$, the set $\{(\frac{m}{q}, \frac{n}{q})/0 \leq m, n < q; m, n \in \mathbb{N}\}$ is invariant and finite. Hence the orbit of x is finite and therefore eventually periodic. Now, the result follows from the fact that for invertible maps the eventually periodic points are periodic points.

$\square$

Note that, for a toral automorphism T_A, the periodic points with period n are solutions of the congruent equation $A^n x = x(mod 1)$. The following proposition is in this direction.

Lemma 2.1.5. [28] If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism then for every Riemann measurable set (having Jordan content) $S \subset \mathbb{R}^2$, $T(S)$ is Riemann measurable and

$$Area(T(S)) = |Det(T)|Area(S)$$

Proposition 2.1.6. *[21]*

Let $A \in GL(2, \mathbb{Z})$. Then

(1) The number of solutions of $A^n x = x$ in $[0,1) \times [0,1)$, is $|Det(A^n - I)|$, provided $Det(A^n - I) \neq 0$.

(2) If $Det(A^n - I) = 0$ then $A^n x = x$ has infinitely many solutions in $[0,1) \times [0,1)$.

Proof. (1) Suppose $Det(A^n - I) \neq 0$. Then note that the number of solutions of the equation, $A^n x = x$ in $[0,1) \times [0,1)$ is equal to the number of integer points in the image of $[0,1) \times [0,1)$ under $A^n - I$, treated as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Note also that the image of $[0,1) \times [0,1)$ under $A^n - I$ is a parallelogram and hence the number of integer points in it, is equal to its area, which is equal to $|Det(A^n - I)|$, by previous lemma.

(2) Note that, when $Det(A^n - I) = 0$, the system $(A^n - I)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ itself, has infinitely many solutions in $[0,1) \times [0,1)$. □

Observe that for any continuous toral automorphism T_A the set $P(T_A)$ is a subgroup of the torus. We now ask:

Which subgroups of $[0,1) \times [0,1)$ arise in this way?

2.2 Automorphisms with determinant 1 and trace 2

We start by listing all the matrices from $GL(2, \mathbb{Z})$ with determinant 1 and trace 2.

Definition 2.2.1. For $m, n \in \mathbb{Z}$ we define,

$$A_{m,n} = \begin{cases} \begin{pmatrix} m & n \\ \frac{-(m-1)^2}{n} & 2-m \end{pmatrix} & \text{if } n \neq 0 \\ \begin{pmatrix} 1 & 0 \\ m-1 & 1 \end{pmatrix} & \text{if } n = 0 \end{cases}$$

Note that $Det(A_{m,n}) = 1$ and $Tr(A_{m,n}) = 2$ for all $m, n \in \mathbb{Z}$.

Proposition 2.2.2. *If $A \in GL(2, \mathbb{Z})$ is such that $Det(A) = 1$ and $Trace(A) = 2$ then $A = A_{m,n}$ for some $m, n \in \mathbb{Z}$ such that n divides $(m-1)^2$.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$ be such that $Det(A) = 1$ and $Tr(A) = 2$. Then we have $a + d = 2$ and $ad - bc = 1$. Hence $bc = -(a-1)^2$

If $b \neq 0$ then $c = -\frac{(a-1)^2}{b}$ an integer and therefore $A = A_{a,b}$. If $b = 0$ then $a = d = 1$ and c can be any integer. Hence $A = A_{c+1,0}$.

□

Remark 2.2.3. Note that the characteristic polynomial of any matrix of type $A_{m,n}$ is $(x-1)^2$ and hence 1 is an eigen value. Therefore no matrix of type $A_{m,n}$ can be hyperbolic.

We now calculate the set of all periodic points of the continuous toral automorphisms induced by the matrices of form $A_{m,n}$. By induction we can prove that, for any $k \in \mathbb{N}$, $A^k_{m,n} = A_{km-k+1,kn}$ for all $m, n \in \mathbb{Z}$.

Let $A \in GL(2, \mathbb{Z})$. Then $X \in \mathbb{T}^2$ is a periodic point of T_A if and only if it is a solution of $A^k X = X$ for some $k \in \mathbb{N}$. Thus,

$P(T_A) = \cup_{k=1}^{\infty} \{X \in \mathbb{T}^2 : \ A^k X = X\}$.

Notation 2.2.4. Let $\mathbb{Q}_1$ be the set of all rational points in $[0, 1)$. Given $r \in \mathbb{Q}$, we write $S_r = \{(x, y) \in \mathbb{T}^2$ such that $rx + y$ is rational $\}$ and let $S_\infty = \mathbb{Q}_1 \times [0, 1)$. Note that $S_0 = [0, 1) \times \mathbb{Q}_1$.

Theorem 2.2.5. *For $m, n \in \mathbb{Z}$, the set of all periodic points of the continuous toral automorphism $T_{A_{m,n}}$ is either the set S_r for some $r \in \mathbb{Q} \cup \{\infty\}$ or $\mathbb{T}^2$.*

Proof. **Case: 1**

When $m \neq 1$ and $n \neq 0$.

The periodic points with period k can be obtained by solving the equation $A^k_{m,n}X = X(mod 1)$, which is equivalent to the system of linear equations,

$$(km - k + 1)x_1 + knx_2 = x_1 + m_1$$

$$-\frac{k(m-1)^2}{n}x_1 + (1 - km + k)x_2 = x_2 + m_2$$

for some $m_1, m_2 \in \mathbb{Z}$

That is, $(x_1, x_2) \in \mathbb{T}^2$ satisfies the equation $A^n X = X$ if and only if

$$(km - k + 1)x_1 + knx_2 = m_1 \in \mathbb{Z}$$

$$-\frac{k(m-1)^2}{n}x_1 + (1 - km + k)x_2 = m_2 \in \mathbb{Z}$$

if and only if

$$\left.\begin{aligned} (km - k + 1)x_1 + knx_2 = m_1 \in \mathbb{Z} \\ -\tfrac{k(m-1)^2}{n}x_1 + (1 - km + k)x_2 = m_2 \in \mathbb{Z} \end{aligned}\right\} \tag{2.1}$$

which implies that,

$$\left.\begin{aligned} (m - 1)x_1 + nx_2 \in \mathbb{Q} \\ -\tfrac{(m-1)^2}{n}x_1 + (1 - m)x_2 \in \mathbb{Q} \end{aligned}\right\} \tag{2.2}$$

which reduces to solving the single equation,

$$\frac{(m-1)}{n}x_1 + x_2 \in \mathbb{Q}.$$

Conversely, if

$$\frac{(m-1)}{n}x_1 + x_2 \in \mathbb{Q}$$

holds. Then (x_1, x_2) satisfies the equation (2.2). Then we can prove that (x_1, x_2) satisfies the equation (2.1) for some suitable k. $P(T_{A_{m,n}}) = \{(x_1, x_2) | \frac{(m-1)}{n}x_1 + x_2 \in \mathbb{Q}\} = S_{\frac{(m-1)}{n}}$ as desired.

[From the equations (2.1) and (2.2), it is observed that:

(x_1, x_2) is a fixed point of $T_{A_{m,n}}$ if and only if $(\frac{x_1}{k}, \frac{x_2}{k})$ is a fixed point of $T^k_{A_{m,n}}$.]

Case: 2

When $m = 1$ and $n = 0$. We get $A_{1,0}$ is the identity matrix and hence $P(T_{A_{m,n}}) = Fix(T_{A_{m,n}}) = \mathbb{T}^2$.

Case: 3 When $m \neq 1$ and $n = 0$.

We have $A_{m,0} = \begin{pmatrix} 1 & 0 \\ m-1 & 1 \end{pmatrix}$. Note that, for any $k \in \mathbb{N}$ we have $A^k_{m,0} = A_{km-k+1,0}$

Now, solving the congruence equation $A^k_{m,0}X = X(mod1)$ is equivalent to solving the single condition $k(m-1)x_1 \in \mathbb{Z}$. This is implies that $x_1 \in \mathbb{Q}_1$.

Conversely, as before if $x_1 \in \mathbb{Q}_1$ then we can find $k \in \mathbb{Z}$ such that $k(m-1)x_1 \in \mathbb{Z}$.

Hene $P(T_A) = \mathbb{Q}_1 \times [0, 1) = S_\infty$.

Case: 4 When $m = 1$ and $n \neq 0$. In this case $A_{1,n} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

Then for any $k \in \mathbb{N}$ we have $A^k_{1,n} = A_{1,kn}$ Now, solving the congruent equation $A^k_{1,n}X = X(mod1)$ is equivalent to solving the single condition $knx_2 \in \mathbb{Z}$. This implies that $x_2 \in \mathbb{Q}_1$.

Conversely, if $x_2 \in \mathbb{Q}_1$ then we can find some $k \in \mathbb{Z}$ such that $k(m-1)x_2 \in \mathbb{Z}$. Hence $P(T_A) = [0,1) \times \mathbb{Q}_1 = S_0$.

□

Remark 2.2.6. Let $A \in GL(2,\mathbb{Z})$ be of $A_{m,n}$ type. Then from the relation, $A^k_{m,n} = A_{km-k+1,kn}$(for all $k \in \mathbb{N}$), it is clear that A and its all powers pertain to the same case among the four cases discussed in the previous proposition. Hence they have the same set of periodic points.

Remark 2.2.7. The set S_r can be thought of as the points on the line through the origin with slope $-r$ and its rational translates in $[0,1)\times[0,1)$. From above proposition, $r = \frac{(m-1)}{n}$when $m \neq 1$. On the other hand given any rational $r = \frac{p}{q} \in \mathbb{Q}$, we can find $m, n \in \mathbb{Z}$ with $n|(m-1)^2$ such that $\frac{p}{q} = \frac{m-1}{n}$. For, choose $m = pq+1, n = q^2$. Hence every S_r arises as $P(A_{m,n})$ for some $m, n \in \mathbb{Z}$.

Proposition 2.2.8. *The following are equivalent for a subset of the torus.*

(1) It is $P(T_{A_{m,n}})$ for some $A_{m,n} \in GL(2,\mathbb{Z})$.

(2) It is S_r for some $r \in \mathbb{Q} \cup \{\infty\}$.

Proof. Follows from the above remark. □

2.3 Main theorem

Definition 2.3.1. A continuous toral automorphism $T_A \in GL(2,\mathbb{Z})$ is said to be *hyperbolic* if A has no eigen values with absolute value 1.

Example 2.3.2. The matrices of the type $A_{m,n}$ are not hyperbolic, because 1 is an eigen value.

It is already known [21] that for a hyperbolic continuous toral automorphism, the periodic points are precisely the rational points. In this chapter, we calculate the set of periodic points for other continuous toral automorphisms; this happens to be the subgroup of $\mathbb{T}^2$ generated by $\mathbb{Q}_1 \times \mathbb{Q}_1 \cup$ (a line with rational slope). In fact, for all non-hyperbolic continuous toral automorphism , there are uncountably many periodic points.

Lemma 2.3.3. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$ is hyperbolic then $Det(A - I) = (a - 1)(d - 1) - bc \neq 0$.

Proof. Let A be hyperbolic. Now, suppose $(a-1)(d-1)-bc = 0 = (ad-bc)-(a+d)+1$

Case: 1 If $ad - bc = 1$, then $a + d = 2$. Which implies by proposition 2.2.2 that $A = A_{m,n}$ for some m, n. Which is a contradiction, by remark 2.2.3.

Case: 2 If $ad - bc = -1$, then $a + d = 0$. Therefore the characteristic polynomial is $x^2 - 1$. Hence the eigen values are ± 1. Which contradicts hypothesis. □

Remark 2.3.4. If A is hyperbolic then so is A^n for all $n \in \mathbb{N}$. (If λ is an eigen value of A,then λ^n is eigen value of A^n). Then $Det(A^n - I) \neq 0$ for all $n \in \mathbb{N}$. Hence the above lemma will apply to all the positive powers of A.

Proposition 2.3.5. *If T_A is hyperbolic then $P(T_A) = \mathbb{Q}_1 \times \mathbb{Q}_1$.*

Proof. The fixed points of A^n are given by the congruence equation $A^n X \equiv X (mod 1)$.

If we let , $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ then we get, $(a_n - 1)x_1 - b_n x_2 = k_1 \in \mathbb{Z}$ and $c_n x_1 - (d_n - 1)x_2 = k_2 \in \mathbb{Z}$.

Now the condition $c_n b_n - (a_n - 1)(d_n - 1) \neq 0$ guarantees that the above system of linear equations is consistent and its solutions are having rational coordinates (by Cramer's rule). Hence $P(\widetilde{A}) = \mathbb{Q}_1 \times \mathbb{Q}_1$.

□

Even though we assumed A to be hyperbolic in the above proposition, we have not used its full strength. What we needed only is that the determinant $Det(A^n - I) = c_n b_n - (a_n - 1)(d_n - 1)$ is nonzero for all $n \in \mathbb{N}$. Thus, in view of proposition 2.1.6 the previous proposition 2.3.5 can be improved/restated as

Proposition 2.3.6. *If T_A is a toral automorphism such that for each $n \in \mathbb{N}$ there are only finitely many periodic points with period n, then $P(T_A) = \mathbb{Q}_1 \times \mathbb{Q}_1$.*

We are now in a position to prove the main theorem.

Theorem 2.3.7. *For any continuous toral automorphism T_A, the set $P(T_A)$ of periodic points of T_A is one of the following:*

1. $\mathbb{Q}_1 \times \mathbb{Q}_1$.
2. S_r *for some* $r \in \mathbb{Q} \cup \{\infty\}$; *where* $S_r = \{(x, y) \in \mathbb{T}^2 \mid rx + y \text{ is rational } \}$.
3. $\mathbb{T}^2$

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$.

For any $n \in \mathbb{N}$ we write $A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ where a_n, b_n, c_n, d_n are integers.

Case: 1

If $Det(A^n - I) = c_n b_n - (a_n - 1)(d_n - 1) \neq 0$ for all $n \in \mathbb{N}$ then proof follows from Proposition 2.3.6.

Case: 2

Suppose $Det(A^n - I) = c_n b_n - (a_n - 1)(d_n - 1) = 0$ for some $n \in \mathbb{N}$.

Let $S = \{k \mid Det(A^k - I) = c_k b_k - (a_k - 1)(d_k - 1) = 0\}$.

Subcase: (2a)

If $Det(A^k) = -1$ for some $k \in S$, then $Tr(A^k) = 0$. Therefore $A^k = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$

Note that $A^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Hence $P(A) = \mathbb{T}^2 = S_0$ since $P(A) \supset P(A^k)\ \forall\ k \in \mathbb{N}$.

Subcase: (2b) If $Det(A^k) = 1$ for all $k \in S$ then $Tr(A^k) = 2$. Therefore $A^k = A_{m,n}$ for some $m, n \in \mathbb{Z}$

From remark 2.2.6, it follows that A^k and its powers namely, $A^{2k}, A^{3k}, A^{3k}, ...$ *share the same set of periodic points.* Note that, for any $j \in \mathbb{N}$ the periodic points of T_A with period j are contained in $P(A^{jk})$. Hence, from theorem 2.2.5, $P(T_A) = S_r$ for some $r \in \mathbb{Q} \cup \{\infty\}$.

□

We conclude this chapter with the following remark.

Remark 2.3.8. Even though there are apparently four kinds of subsets which can appear as the set of periodic points for some continuous toral automorphism, there are only three upto homeomorphism.