Chapter 2

Periodic points of toral automorphisms

2.1 General introduction

The automorphisms of the two-dimensional torus are rich mathematical objects possessing interesting geometric, algebraic, topological and measure theoretic properties.

The torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is here viewed as the topological group $[0,1) \times [0,1)$ with coordinate-wise addition modulo 1.

Let $GL(2,\mathbb{Z})$ be the set of all $2 \times 2$ matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$ and $\text{Det}(A) = ad - bc = \pm 1$.

Each such matrix $A$ gives a linear map on $\mathbb{R}^2$ by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

We define an automorphism on the torus $T_A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ by $T_A(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)(\text{mod}1)$.

Now we have,
Proposition 2.1.1. [29] Every automorphism $T_A$ (as defined above) on the torus is a homeomorphism.

Proof. The map $T$ is clearly continuous, since if $|x_1-y_1|, |x_2-y_2| < \epsilon$ then $|T_A(x_1, x_2)_1 - (T_A(y_1, y_2))_1| < (|a| + |b|)\epsilon$ and $|(T_A(x_1, x_2))_2 - (T_A(y_1, y_2))_2| < (|c| + |d|)\epsilon$. (Here suffix $1$ refers to the first coordinate and $2$ refers to the second coordinate.)

To see that $T_A$ is invertible we note that if we write the inverse matrix $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ then since $ad - bc = \pm 1$ we see that $A^{-1} \in GL(2, \mathbb{Z})$. The inverse to $T_A$ is then the toral automorphism associated to $A^{-1}$, i.e. $T_{A^{-1}}$.

On the other hand, in the following proposition we prove that every continuous automorphism on the torus is induced by a matrix from $GL(2, \mathbb{Z})$. Let $\text{Aut}(\mathbb{T}^2)$ denote the set of all continuous automorphisms on the torus.

Proposition 2.1.2. The above map $A \mapsto T_A$ from $GL(2, \mathbb{Z})$ to $\text{Aut}(\mathbb{T}^2)$ is surjective.

Proof. Let $\phi : \mathbb{T}^2 \to \mathbb{T}^2$ be any continuous toral automorphism. Since $\phi$ is continuous at $(0, 0)$ there exist $\delta > 0$ such that $\phi(([0, \delta] \times [0, \delta])) \subset [0, \frac{1}{2}] \times [0, \frac{1}{2})$ and such that $\phi(X + Y) = \phi(X) + \phi(Y)$ for all $X, Y \in [0, \delta] \times [0, \delta)$, where $+$ denotes the usual addition in $\mathbb{R}^2$.

Now, observe that $\phi(\lambda X) = \lambda \phi(X)$ for all $\lambda \in (0, 1)$ and for all $X, Y \in [0, \delta] \times [0, \delta)$, because the set $\{ \lambda \in (0, 1) | \phi(\lambda X) = \lambda \phi(X) \}$ is a closed set contains all dyadic rationals (Since the set contains $\lambda = \frac{1}{2}$, using the additivity of $\phi$, it contains all numbers of form $\lambda = \frac{m}{2^n}$). The set of all dyadic rationals is dense in $[0, 1]$ and then by continuity the set contains all $\lambda \in (0, 1)$.

Hence $\phi|_{[0,\delta] \times [0,\delta)} = L|_{[0,\delta] \times [0,\delta)}$ for some linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$. This linear transformation induces an integer matrix $A$ with determinant $\pm 1$ such that...
Ax = φ(x) for all $x \in \mathbb{T}^2$ [ The kernel of an endomorphism (different from the zero map), on a connected topological group cannot have nonempty interior ]. Hence the proof.

\[ \square \]

**Remark 2.1.3.** In fact the 1-1 correspondence in the above proposition is a group isomorphism.

For each \( A \in GL(2, \mathbb{Z}) \), let \( P(T_A) \) denote the set of all periodic points of \( T_A \).

**Proposition 2.1.4.** [21]

*For any \( A \in GL(2, \mathbb{Z}) \) the set \( P(T_A) \) is dense in \([0, 1) \times [0, 1)\).*

**Proof.** Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}) \).

We prove that \( P(T_A) \supset \mathbb{Q}_1 \times \mathbb{Q}_1 \), where \( \mathbb{Q}_1 = \mathbb{Q} \cap [0, 1) \). A general element in \( \mathbb{Q}_1 \times \mathbb{Q}_1 \) is of the form \( x = \left( \frac{p_1}{q}, \frac{p_2}{q} \right) \) where \( p_1, p_2, q \in \mathbb{Z} \) with \( 0 \leq p_1, p_2 < q \). We note that \( T_A(X) = (\text{fractional part of } \frac{ap_1}{q} + \frac{bp_2}{q}, \text{fractional part of } \frac{cp_1}{q} + \frac{dp_2}{q}) \) is an element of the form \( \left( \frac{m}{q}, \frac{n}{q} \right) \) where \( 0 \leq m, n < q \). Note that, for a fixed \( q \in \mathbb{N} \), the set \( \left\{ \left( \frac{m}{q}, \frac{n}{q} \right) / 0 \leq m, n < q; m, n \in \mathbb{N} \right\} \) is invariant and finite. Hence the orbit of \( x \) is finite and therefore eventually periodic. Now, the result follows from the fact that for invertible maps the eventually periodic points are periodic points.

\[ \square \]

Note that, for a toral automorphism \( T_A \), the periodic points with period \( n \) are solutions of the congruent equation \( A^n x = x (mod 1) \). The following proposition is in this direction.

**Lemma 2.1.5.** [28] If \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is an isomorphism then for every Riemann mea-
surable set (having Jordan content) $S \subset \mathbb{R}^2$, $T(S)$ is Riemann measurable and

$$\text{Area}(T(S)) = |\text{Det}(T)|\text{Area}(S)$$

Proposition 2.1.6. [21]

Let $A \in \text{GL}(2, \mathbb{Z})$. Then

1. The number of solutions of $A^n x = x$ in $[0, 1) \times [0, 1)$, is $|\text{Det}(A^n - I)|$, provided $\text{Det}(A^n - I) \neq 0$.

2. If $\text{Det}(A^n - I) = 0$ then $A^n x = x$ has infinitely many solutions in $[0, 1) \times [0, 1)$.

Proof. (1) Suppose $\text{Det}(A^n - I) \neq 0$. Then note that the number of solutions of the equation, $A^n x = x$ in $[0, 1) \times [0, 1)$ is equal to the number of integer points in the image of $[0, 1) \times [0, 1)$ under $A^n - I$, treated as a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Note also that the image of $[0, 1) \times [0, 1)$ under $A^n - I$ is a parallelogram and hence the number of integer points in it, is equal to its area, which is equal to $|\text{Det}(A^n - I)|$, by previous lemma.

(2) Note that, when $\text{Det}(A^n - I) = 0$, the system $(A^n - I) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ itself, has infinitely many solutions in $[0, 1) \times [0, 1)$. \hfill \square

Observe that for any continuous toral automorphism $T_A$ the set $P(T_A)$ is a subgroup of the torus. We now ask:

Which subgroups of $[0, 1) \times [0, 1)$ arise in this way?

2.2 Automorphisms with determinant 1 and trace 2

We start by listing all the matrices from $\text{GL}(2, \mathbb{Z})$ with determinant 1 and trace 2.
Definition 2.2.1. For \( m, n \in \mathbb{Z} \) we define,

\[
A_{m,n} = \begin{cases} 
\begin{pmatrix} m & n \\ -(m-1)^2 & 2-m \end{pmatrix} & \text{if } n \neq 0 \\
\begin{pmatrix} 1 & 0 \\ m-1 & 1 \end{pmatrix} & \text{if } n = 0
\end{cases}
\]

Note that \( \text{Det}(A_{m,n}) = 1 \) and \( \text{Tr}(A_{m,n}) = 2 \) for all \( m, n \in \mathbb{Z} \).

Proposition 2.2.2. If \( A \in \text{GL}(2, \mathbb{Z}) \) is such that \( \text{Det}(A) = 1 \) and \( \text{Trace}(A) = 2 \) then \( A = A_{m,n} \) for some \( m, n \in \mathbb{Z} \) such that \( n \) divides \( (m-1)^2 \).

Proof. Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) \) be such that \( \text{Det}(A) = 1 \) and \( \text{Tr}(A) = 2 \). Then we have \( a + d = 2 \) and \( ad - bc = 1 \). Hence \( bc = -(a-1)^2 \)

If \( b \neq 0 \) then \( c = \frac{-(a-1)^2}{b} \) an integer and therefore \( A = A_{a,b} \). If \( b = 0 \) then \( a = d = 1 \)
and \( c \) can be any integer. Hence \( A = A_{c+1,0} \).

\( \square \)

Remark 2.2.3. Note that the characteristic polynomial of any matrix of type \( A_{m,n} \)
is \( (x - 1)^2 \) and hence 1 is an eigen value. Therefore no matrix of type \( A_{m,n} \) can be hyperbolic.

We now calculate the set of all periodic points of the continuous toral automorphisms induced by the matrices of form \( A_{m,n} \). By induction we can prove that, for any \( k \in \mathbb{N} \), \( A_{m,n}^k = A_{km-k+1,kn} \) for all \( m, n \in \mathbb{Z} \).

Let \( A \in \text{GL}(2, \mathbb{Z}) \). Then \( X \in \mathbb{T}^2 \) is a periodic point of \( T_A \) if and only if it is a solution of \( A^kX = X \) for some \( k \in \mathbb{N} \). Thus,

\[
P(T_A) = \bigcup_{k=1}^{\infty} \{ X \in \mathbb{T}^2 : A^kX = X \}.
\]

Notation 2.2.4. Let \( \mathbb{Q}_1 \) be the set of all rational points in \([0,1)\). Given \( r \in \mathbb{Q}_1 \), we
write $S_r = \{(x,y) \in \mathbb{T}^2 \text{ such that } rx + y \text{ is rational }\}$ and let $S_\infty = \mathbb{Q}_1 \times [0,1)$. Note that $S_0 = [0,1) \times \mathbb{Q}_1$.

**Theorem 2.2.5.** For $m, n \in \mathbb{Z}$, the set of all periodic points of the continuous toral automorphism $T_{A_{m,n}}$ is either the set $S_r$ for some $r \in \mathbb{Q} \cup \{\infty\}$ or $\mathbb{T}^2$.

**Proof.** Case: 1

When $m \neq 1$ and $n \neq 0$.

The periodic points with period $k$ can be obtained by solving the equation $A_{m,n}^k X = X (\text{mod} 1)$, which is equivalent to the system of linear equations,

\[
\begin{align*}
(km - k + 1)x_1 + knx_2 &= x_1 + m_1 \\
-k(m-1)^2/n x_1 + (1 - km + k)x_2 &= x_2 + m_2
\end{align*}
\]

for some $m_1, m_2 \in \mathbb{Z}$.

That is, $(x_1, x_2) \in \mathbb{T}^2$ satisfies the equation $A^n X = X$ if and only if

\[
\begin{align*}
(km - k + 1)x_1 + knx_2 &= m_1 \in \mathbb{Z} \\
-k(m-1)^2/n x_1 + (1 - km + k)x_2 &= m_2 \in \mathbb{Z}
\end{align*}
\]

if and only if

\[
\begin{align*}
(km - k + 1)x_1 + knx_2 &= m_1 \in \mathbb{Z} \\
-k(m-1)^2/n x_1 + (1 - km + k)x_2 &= m_2 \in \mathbb{Z}
\end{align*}
\]

which implies that,

\[
\begin{align*}
(m - 1)x_1 + nx_2 &= \mathbb{Q} \\
-(m-1)^2/n x_1 + (1 - m)x_2 &= \mathbb{Q}
\end{align*}
\]
which reduces to solving the single equation,
\[
\frac{(m - 1)}{n} x_1 + x_2 \in \mathbb{Q}.
\]
Conversely, if
\[
\frac{(m - 1)}{n} x_1 + x_2 \in \mathbb{Q}
\]
holds. Then \((x_1, x_2)\) satisfies the equation (2.2). Then we can prove that \((x_1, x_2)\) satisfies the equation (2.1) for some suitable \(k\). \(P(T_{A_{m,n}}) = \{(x_1, x_2)\mid \frac{(m - 1)}{n} x_1 + x_2 \in \mathbb{Q}\} = S_{\frac{(m - 1)}{n}}\) as desired.

[ From the equations (2.1) and (2.2), it is observed that:

\((x_1, x_2)\) is a fixed point of \(T_{A_{m,n}}\) if and only if \((\frac{x_1}{k}, \frac{x_2}{k})\) is a fixed point of \(T_{A_{m,n}}^k\).]

**Case: 2**

When \(m = 1\) and \(n = 0\). We get \(A_{1,0}\) is the identity matrix and hence \(P(T_{A_{m,n}}) = Fix(T_{A_{m,n}}) = \mathbb{T}^2\).

**Case: 3** When \(m \neq 1\) and \(n = 0\).

We have \(A_{m,0} = \begin{pmatrix} 1 & 0 \\ m - 1 & 1 \end{pmatrix}\). Note that, for any \(k \in \mathbb{N}\) we have \(A_{m,0}^k = A_{km-k+1,0}\)

Now, solving the congruence equation \(A_{m,0}^k X = X (mod 1)\) is equivalent to solving the single condition \(k(m - 1)x_1 \in \mathbb{Z}\). This implies that \(x_1 \in \mathbb{Q}_1\).

Conversely, as before if \(x_1 \in \mathbb{Q}_1\) then we can find \(k \in \mathbb{Z}\) such that \(k(m - 1)x_1 \in \mathbb{Z}\).

Hence \(P(T_A) = \mathbb{Q}_1 \times [0, 1) = S_{\infty}\).

**Case: 4** When \(m = 1\) and \(n \neq 0\). In this case \(A_{1,n} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\).

Then for any \(k \in \mathbb{N}\) we have \(A_{1,n}^k = A_{1,kn}\) Now, solving the congruence equation \(A_{1,n}^k X = X (mod 1)\) is equivalent to solving the single condition \(knx_2 \in \mathbb{Z}\). This implies that \(x_2 \in \mathbb{Q}_1\).
Conversely, if $x_2 \in \mathbb{Q}_1$, then we can find some $k \in \mathbb{Z}$ such that $k(m - 1)x_2 \in \mathbb{Z}$.

Hence $P(T_A) = [0, 1) \times \mathbb{Q}_1 = S_0$. \hfill $\square$

**Remark 2.2.6.** Let $A \in GL(2, \mathbb{Z})$ be of $A_{m,n}$ type. Then from the relation, $A_{m,n}^k = A_{km-k+1, kn}$ (for all $k \in \mathbb{N}$), it is clear that $A$ and its all powers pertain to the same case among the four cases discussed in the previous proposition. Hence they have the same set of periodic points.

**Remark 2.2.7.** The set $S_r$ can be thought of as the points on the line through the origin with slope $-r$ and its rational translates in $[0, 1) \times [0, 1)$. From above proposition, $r = \frac{(m-1)}{n}$ when $m \neq 1$. On the other hand given any rational $r = \frac{p}{q} \in \mathbb{Q}$, we can find $m, n \in \mathbb{Z}$ with $n|(m-1)^2$ such that $\frac{p}{q} = \frac{m-1}{n}$. For, choose $m = pq + 1, n = q^2$. Hence every $S_r$ arises as $P(A_{m,n})$ for some $m, n \in \mathbb{Z}$.

**Proposition 2.2.8.** The following are equivalent for a subset of the torus.

1. It is $P(T_{A_{m,n}})$ for some $A_{m,n} \in GL(2, \mathbb{Z})$.
2. It is $S_r$ for some $r \in \mathbb{Q} \cup \{\infty\}$.

**Proof.** Follows from the above remark. \hfill $\square$

### 2.3 Main theorem

**Definition 2.3.1.** A continuous toral automorphism $T_A \in GL(2, \mathbb{Z})$ is said to be *hyperbolic* if $A$ has no eigen values with absolute value 1.

**Example 2.3.2.** The matrices of the type $A_{m,n}$ are not hyperbolic, because 1 is an eigen value.
It is already known [21] that for a hyperbolic continuous toral automorphism, the periodic points are precisely the rational points. In this chapter, we calculate the set of periodic points for other continuous toral automorphisms; this happens to be the subgroup of \( T^2 \) generated by \( \mathbb{Q}_1 \times \mathbb{Q}_1 \cup \) (a line with rational slope). In fact, for all non-hyperbolic continuous toral automorphism, there are uncountably many periodic points.

**Lemma 2.3.3.** If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}) \) is hyperbolic then \( \text{Det}(A - I) = (a - 1)(d - 1) - bc \neq 0 \).

**Proof.** Let \( A \) be hyperbolic. Now, suppose \( (a - 1)(d - 1) - bc = 0 = (ad - bc) - (a + d) + 1 \)

**Case: 1** If \( ad - bc = 1 \), then \( a + d = 2 \). Which implies by proposition 2.2.2 that \( A = A_{m,n} \) for some \( m, n \). Which is a contradiction, by remark 2.2.3.

**Case: 2** If \( ad - bc = -1 \), then \( a + d = 0 \). Therefore the characteristic polynomial is \( x^2 - 1 \). Hence the eigen values are \( \pm 1 \). Which contradicts hypothesis. \( \square \)

**Remark 2.3.4.** If \( A \) is hyperbolic then so is \( A^n \) for all \( n \in \mathbb{N} \). (If \( \lambda \) is an eigen value of \( A \), then \( \lambda^n \) is eigen value of \( A^n \)). Then \( \text{Det}(A^n - I) \neq 0 \) for all \( n \in \mathbb{N} \). Hence the above lemma will apply to all the positive powers of \( A \).

**Proposition 2.3.5.** If \( T_A \) is hyperbolic then \( P(T_A) = \mathbb{Q}_1 \times \mathbb{Q}_1 \).

**Proof.** The fixed points of \( A^n \) are given by the congruence equation \( A^n X \equiv X \ (mod\ 1) \).

If we let \( A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \) then we get, \( (a_n - 1)x_1 - b_nx_2 = k_1 \in \mathbb{Z} \) and \( c_nx_1 - (d_n - 1)x_2 = k_2 \in \mathbb{Z} \).

Now the condition \( c_n b_n - (a_n - 1)(d_n - 1) \neq 0 \) guarantees that the above system of linear equations is consistent and its solutions are having rational coordinates (by Cramer’s rule). Hence \( P(\bar{A}) = \mathbb{Q}_1 \times \mathbb{Q}_1 \).
Even though we assumed \( A \) to be hyperbolic in the above proposition, we have not used its full strength. What we needed only is that the determinant \( \det(A^n - I) = c_n b_n - (a_n - 1)(d_n - 1) \) is nonzero for all \( n \in \mathbb{N} \). Thus, in view of proposition 2.1.6 the previous proposition 2.3.5 can be improved/restated as

**Proposition 2.3.6.** If \( T_A \) is a toral automorphism such that for each \( n \in \mathbb{N} \) there are only finitely many periodic points with period \( n \), then \( P(T_A) = \mathbb{Q}_1 \times \mathbb{Q}_1 \).

We are now in a position to prove the main theorem.

**Theorem 2.3.7.** For any continuous toral automorphism \( T_A \), the set \( P(T_A) \) of periodic points of \( T_A \) is one of the following:

1. \( \mathbb{Q}_1 \times \mathbb{Q}_1 \).
2. \( S_r \) for some \( r \in \mathbb{Q} \cup \{ \infty \} \); where \( S_r = \{(x, y) \in \mathbb{T}^2 \mid rx + y \text{ is rational} \} \).
3. \( \mathbb{T}^2 \).

**Proof.** Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}) \).

For any \( n \in \mathbb{N} \) we write \( A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \) where \( a_n, b_n, c_n, d_n \) are integers.

**Case: 1**

If \( \det(A^n - I) = c_n b_n - (a_n - 1)(d_n - 1) \neq 0 \) for all \( n \in \mathbb{N} \) then proof follows from Proposition 2.3.6.

**Case: 2**

Suppose \( \det(A^n - I) = c_n b_n - (a_n - 1)(d_n - 1) = 0 \) for some \( n \in \mathbb{N} \).

Let \( S = \{k \mid \det(A^k - I) = c_k b_k - (a_k - 1)(d_k - 1) = 0 \} \).

**Subcase: (2a)**
If $\det(A^k) = -1$ for some $k \in S$, then $\text{Tr}(A^k) = 0$. Therefore $A^k = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$

Note that $A^{2k} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Hence $P(A) = T^2 = S_0$ since $P(A) \supset P(A^k) \forall k \in \mathbb{N}$.

**Subcase: (2b)** If $\det(A^k) = 1$ for all $k \in S$ then $\text{Tr}(A^k) = 2$. Therefore $A^k = A_{m,n}$ for some $m, n \in \mathbb{Z}$

From remark 2.2.6, it follows that $A^k$ and its powers namely, $A^{2k}, A^{3k}, A^{3k}, \ldots$ share the same set of periodic points. Note that, for any $j \in \mathbb{N}$ the periodic points of $T_A$ with period $j$ are contained in $P(A^{jk})$. Hence, from theorem 2.2.5, $P(T_A) = S_r$ for some $r \in \mathbb{Q} \cup \{\infty\}$.

We conclude this chapter with the following remark.

**Remark 2.3.8.** Even though there are apparently four kinds of subsets which can appear as the set of periodic points for some continuous toral automorphism, there are only three up to homeomorphism.