Chapter 1

Periodic points, Periods and Conjugacy

1.1 General introduction to discrete dynamical systems

1.1.1 Definition and Examples

A *dynamical system* is simply a pair (X, f), where X is a topological space and $f : X \to X$ is a continuous self map of X. The space X can be thought of as the underlying set on which the motion takes place and f can be thought of as the rule according to which motion takes place. For $x \in X$ the point $f(x) \in X$ is thought of as the position to which x moves (in one unit of time). The composition of f with itself, denoted by $f \circ f$ is also a continuous self map of X. For $x \in X$, the element $f(f(x))$ is called the position to which x moves after two instants of time in the dynamical system (X, f).

If n is a positive integer, the element $f^n(x)$ [where f^n denotes the composition of f with itself $n-1$ times] is thought of as the position to which x moves after n instants of time. Thus in our study "time" is "discrete" and parametrised by the set $\mathbb{N}$ of natural numbers. (The space X need not be discrete.) This is the reason that it is called a " *discrete* " *dynamical system* .

Example 1.1.1. (Translation)

Let $f(x) = x + 5$ for all $x \in \mathbb{R}$. Then $(\mathbb{R}, f)$ is a dynamical system. This represents motion on a straight line with constant velocity. [The distance between $f^n(x)$ and $f^{n+1}(x)$ does not depend on n]. The element $\frac{1}{2}$ after two instants of time reaches the position $10\frac{1}{2}$. No element is fixed.

Example 1.1.2. (Rotation)

Let S^1 be the unit circle in the complex plane. i.e., $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Let $a \in S^1$ be fixed. Let $\rho_a : S^1 \to S^1$ be defined by $\rho(z) = az$ for all $z \in S^1$. Such maps ρ_a are called *rotations*. They describe motions on the circle with constant velocity. When $a \neq 1$, there are no fixed points.

Example 1.1.3. (Logistic function)

Let $r > 0$. The logistic function h_r on $[0, 1]$ is defined by the formula $h_r(x) = rx(1 - x)$. One can prove that when $0 < r \leq 4$, h_r takes $[0, 1]$ into $[0, 1]$ and hence $([0, 1], h_r)$ is a dynamical system. When $r = 4$, the point $\frac{1}{2}$ goes to 0 after two instants of time. There are two fixed points 0 and 1. But when $r > 4$, $([0, 1], h_r)$ is not a dynamical system.

Definition 1.1.4. Let (X, f) be a dynamical system. For each positive integer n, we define the function f^n now. It is recursively defined by $f^1 = f$ and $f^{n+1} = f \circ f^n$ for all $n \in \mathbb{N}$. We also use the convention $f^0 =$ the identity function. [It is easily seen that $f^m \circ f^n = f^{m+n}$ holds for all nonnegative integers m, n.]

1.1.2 Orbits, periodic points

Let (X, f) be a dynamical system and let $x \in X$. Then the sequence $x, f(x), f^2(x), ...$ is called the *trajectory* of x and the set of all points in the trajectory of x is called the *orbit* of x.

In dynamical systems, we are normally interested in studying the nature of the orbits of distinct points of the system.

Suppose for some $x \in X$, sequence $x, f(x), f^2(x), ...$ converges to some point say $x_0 \in X$, then we must have $f(x_0) = x_0$, because f is continuous. Such points we call as *fixed points*. In dynamics we say that the point x is attracted by the fixed point x_0. The set of all points in X *attracted* by x_0 is called the *stable set* or the *basin of attraction* of the fixed point x_0 and is denoted by $W^s(x_0, f)$ or simply $W^s(x_0)$. A fixed point x_0 is said to be *attracting* if its stable set is a neighbourhood of it.

Example 1.1.5. Consider the map $f(x) = x^2 : \mathbb{R} \to \mathbb{R}$. The fixed points are 0 and 1. We can easily see that, if $-1 < x < 1$, then the sequence $(f^n(x))_{n=1}^{\infty}$ converges to 0. Hence 0 is an attracting fixed point. Also, if $|x| > 1$ then $(f^n(x))_{n=1}^{\infty} \to \infty$. The point -1 is mapped to 1 and then fixed, such points we call eventually fixed.

Definition 1.1.6. The points which will reach a fixed point after finitely many iterations are called *eventually fixed* .

Definition 1.1.7. Let (X, f) be a dynamical system. A subset A of X is said to be *invariant* if $f(A) \subset A$. When A is an invariant set , $(A, f|_A)$ also becomes a dynamical system.

Example 1.1.8. 1. The set of all rational numbers in $[0, 1]$ is an invariant set in the system $([0, 1], h_r)$ when r is rational. This is because $rx(1 - x)$ is rational whenever x is rational.

2. $\mathbb{Z}$ is invariant under $(\mathbb{R}, f)$, where $f(x) = x + 5$ for all $x \in \mathbb{R}$.

3. In the system $(\mathbb{C}, f)$ where $f(z) = z^2$ for all $z \in \mathbb{C}$, the singleton set $\{1\}$ and the set $\{z \in \mathbb{C} : |z| \text{ is an integer }\}$ are examples of invariant sets.

Definition 1.1.9. Let (X, f) be a dynamical system. A point $x \in X$ is said to be *periodic* if there exists a positive integer $n \in \mathbb{N}$ such that $f^n(x) = x$. Least such n is called the *period* of x and the set $\{x, f(x), f^2(x), \ldots, f^{n-1}(x)\}$ is called an *n-cycle*. The set of all periodic points of f is denoted by $P(f)$.

The points which will reach a periodic point after finitely many iterations are called *eventually periodic* .

Let $Per(f)$= $\{n \in \mathbb{N}$ such that f has a point of period n $\}$ and we call this as the *set of periods* of f or simply the *period set* of f.

Let X be a topological space and let $A \subset \mathbb{N}$. We write $A \in PER(X)$ if there exists a continuous map $f : X \to X$ such that $Per(f) = A$.

Example 1.1.10. 1. When f is the identity map or a constant map, all points are mapped to fixed points and hence $Per(f) = \{1\}$.

2. When f is the reflection map $x \mapsto -x$ on $\mathbb{R}$, the point 0 is fixed and all other points are periodic with period 2. Hence $Per(f) = \{1, 2\}$.

3. When f is the translation map $x \mapsto x + 1$ on $\mathbb{R}$, then $Per(f)$ is the empty set, since every orbit is strictly monotone.

4. On the unit circle, if we consider the rotation by the angle $\frac{2\pi}{3}$, all points are periodic with period 3. In this case $Per(f) = \{3\}$.

5. For the shift map(defined at the last section of this chapter) on $\sum_2 = \{0, 1\}^{\mathbb{N}}$ the periodic points are of the form $x = \overline{w} = www\ldots$ for some word w over $\{0, 1\}$ and $Per(f) = \mathbb{N}$.

We now in the following proposition state ten important simple results that are easy to prove.

Proposition 1.1.11. *Let (X, f) be a dynamical system, where X is a Hausdorff space. Then the following hold:*

1. The set of all fixed points is a closed subset of X.

2. In any trajectory, either all terms are distinct, or only finitely many terms are distinct.

3. Orbits of any two periodic points are either identical or disjoint.

4. If a trajectory converges, it converges to a fixed point.

5. An element is eventually periodic if and only if it has a finite orbit.

6. Every orbit is an invariant set; the orbits of periodic points are minimal invariant sets.

7. A subset of X is invariant if and only if it is a union of orbits.

8. The closure of an invariant set is also invariant.

9.The set of all periodic points is an invariant set.

10. For each subset A of X, the set $\cup_{n=0}^{\infty} f^n(A)$ is the smallest invariant set containing A.

1.2 The role of $Per(f)$ in Chaos

A question: *Can we find f continuous from $\mathbb{R}$ to $\mathbb{R}$ such that $Per(f) = A$, where A is $\{1, 2, 3\}$ or where A is $\{1, 2, 4\}$?*

The answer turns out to be 'NO' for the former and 'YES' for the latter. This is because of the following theorem:

Theorem 1.2.1. ***(Li and Yorke)***

Let f be continuous from $\mathbb{R}$ to $\mathbb{R}$. If $3 \in Per(f)$, then $Per(f) = \mathbb{N}$. [In other words , if there is a $3 - cycle$ then there is an $n - cycle$ for all n.]

This theorem is hard to prove. But here is an easy observation. If in the dynamical system $(\mathbb{R}, f)$, two points x and y move in the opposite directions, then there should be a fixed point between them.

More precisely: If $x < f(x)$ and if $f(y) < y$, then there exists z between x and y such that $f(z) = z$. This is proved by applying intermediate value theorem to the function $f(x) - x$. This implies that if $1 \notin Per(f)$, then the motion is uni-directional and so no point can be periodic. In other words, $Per(f) \neq \emptyset \implies 1 \in Per(f)$.

This elementary result, in combination with the result of Li and Yorke, exhibit the numbers 3 and 1 in the two extremes of an order. If a $3 - cycle$ is there, all $n - cycles$ have to be there. If no $1 - cycle$ is there then no $n - cycles$ can be there.

This leads to a search of pairs of positive integers (m, n) such that if an $m - cycle$ there, $n - cycle$ has to be there. What are all such pairs?

Sharkovskii's theorem provides a complete answer to this question. We discuss this in Chapter-3 in detail.

We now, in the next section, explain the importance of the set of periodic points in the theory of chaos.

1.2.1 Chaos

The expression " chaos" became popularized through the paper of Li and Yorke [16], "Period three implies chaos".

Chaotic systems share the property of having a high degree of sensitivity to initial conditions. In other words, a very small change in initial values will multiply in such a way that the new computed system bears no resemblance to the one predicted.

Definition 1.2.2. We say that the system (X, f) (where X is a metric space with metric d) is *sensitive to the initial conditions* if there exists $\delta > 0$ such that for any $x \in X$ and for any $\epsilon > 0$ there exists a point $y \in X$ with $d(x, y) < \delta$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \delta$. This $\delta > 0$ is called a *sensitive constant* .

Another important property of dynamical systems called transitivity roughly says that, the iterates of any nonempty open set are well spread throughout the space. Said precisely,

Definition 1.2.3. A dynamical system (X, f) is said to be *transitive* if for any two nonempty open sets U and V in X there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$. We say that f is *totally transitive* if f^n is transitive for all $n \geq 1$.

It has been recognized by Sharkovsky [17], Li and Yorke[16] and many others that there is a hidden, self-organizing order in chaotic systems. A certain degree of order in chaotic systems has led to various definitions of chaos in the literature.

1.2.2 Devaney's definition of chaos

Definition 1.2.4. According to Devaney a dynamical system (X, f) is said to be *chaotic* if

1. f is transitive.
2. f has a dense set of periodic points.
3. f is sensitive to the initial conditions.

Remark 1.2.5. In [23], Banks et al showed that, when X is infinite, the conditions (1) and (2) in the above definition imply condition (3) of sensitive dependence on initial conditions. However, no other two conditions imply the third. Examples can be seen in [18]

Theorem 1.2.6. *(Banks et al)*

Let $f : X \to X$ be a continuous map on an infinite compact metric space (X, d). If f is transitive and its set of periodic points is dense, then f possesses sensitive dependence on initial conditions, i.e., f is chaotic.

Remark 1.2.7. In fact, for continuous maps on intervals in $\mathbb{R}$, transitivity implies that the set of periodic points is dense (See [19]). Hence it follows from the above theorem that in this case *transitivity implies chaos.*

The following results describe the set $Per(f)$ for the class of transitive interval maps.

Theorem 1.2.8. *[15] A subset S of $\mathbb{N}$ occurs as $Per(f)$ for some transitive map f on $[0, 1]$ if and only if it satisfies the following two conditions:*

(1) $n \in S,\ n \neq 1 \Rightarrow\ n + 2 \in S$

(2) 1 and 2 are in S.

This theorem can be deduced by combining Sarkovskii's theorem (See chapter-3) with the following two results:

Theorem 1.2.9. *[15]*

Every transitive map on $[0, 1]$ has a periodic point of period 6.

Theorem 1.2.10. *[15], [22]*

Given any odd integer $k > 4$, there exists a transitive map f on $[0, 1]$ such that $k \in Per(f)$ but $k - 2 \notin Per(f)$.

The following two theorems are about total transitivity.

Theorem 1.2.11. *[30] A subset S of $\mathbb{N}$ arises as $Per(f)$ for a totally transitive map on $[0,1]$ if and only if $S = \mathbb{N}$ or $\mathbb{N} - \{3, 5, \ldots, 2n+1\}$ for some $n \in \mathbb{N}$.*

Theorem 1.2.12. *[30] Let f be transitive on $[0,1]$. Then f is totally transitive if and only if $Per(f)$ has a finite complement in $\mathbb{N}$.*

1.2.3 Li- Yorke Chaos

In a dynamical system (X, f), let $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ be the respective orbits of two distinct points $x_0, y_0 \in X$. We ask two questions:

Q1: Do x_n and y_n come arbitrarily close to each other?

Q2: Do x_n and y_n keep a minimum positive distance from each other for infinitely many n?

Note that, by the nature of the questions, the two answers simultaneously cannot be 'No'. So the respective answers can be (Yes, No), (No, Yes) or (Yes, Yes).

Example 1.2.13. If f is an isometry like rotation of the unit circle or if f is a contraction map like $x \mapsto \frac{x}{2}$ on $[0,1]$, then for any two points x_0, y_0 the answer pair is (No, Yes).

Example 1.2.14. If x_0 and y_0 are two distinct periodic points in a dynamical system, then the answer pair is (No, Yes).

Example 1.2.15. For the function $x \mapsto \sin(\frac{\pi x}{2})$ on $[-1,1]$, -1,0,1 are fixed points, $\sin(\frac{\pi x}{2}) > x$ for $0 < x < 1$ and $\sin(\frac{\pi x}{2}) < x$ for $-1 < x < 0$. So the orbit of x converges to 1 for $0 < x < 1$ and the orbit of x converges to -1 for $-1 < x < 0$. Therefore for two distinct points $x_0, y_0 \in [-1,1]$, the answer pair is (Yes, No) if $x_0 y_0 > 0$, and the answer pair is (No, Yes) if $x_0 y_0 \le 0$.

Example 1.2.16. Let x_0 be a fixed point and y_0 be a point with dense orbit in an infinite dynamical system (X, f). Then the answer pair is (Yes, Yes).

Remark 1.2.17. For any two points in a dynamically simple system, the answer pair will be either (Yes, No) or (No, Yes). But an (Yes, Yes) answer pair indicates some complexity in the relative behavior of the two orbits. In the study of *scrambled sets* we are interested in pairs of points for which both the two questions have affirmative answers.

Definition 1.2.18. Let (X, f) be a dynamical system and let $S \subset X$ be a set with at least two points. Then, S is a *scrambled set* for f if for any two distinct points $x, y \in S$

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

and

$$\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$$

From the famous paper of Li and Yorke [16] there ensued a definition of chaos:

Definition 1.2.19. A continuous self map of the interval is said to be Li-Yorke Chaotic if it has an uncountable scrambled set.

Li and Yorke proved that the occurrence of a period-3 point forces chaos.

Theorem 1.2.20. *If an interval map has a point of period three, then it is Li-Yorke chaotic.*

Here is a theorem connecting the set $Per(f)$ and Li-Yorke chaos.

Theorem 1.2.21. *[30] Let f be a self map of an interval I in $\mathbb{R}$. Then*

(i) If $Per(f)$ properly contains the set $\{1, 2, 2^2, 2^3, \ldots\}$, then f is Li-Yorke chaotic.

(ii) If $Per(f)$ is properly contained in $\{1, 2, 2^2, 2^3, \ldots\}$, then f is not Li-Yorke chaotic.

Corollary 1.2.22. *For any interval map f, Devaney's chaos implies Li-Yorke chaos. (we use theorem 1.2.9)*

Remark 1.2.23. We cannot say anything if $Per(f) = \{1, 2, 2^2, 2^3, \ldots\}$. In [20] it is shown that there are continuous interval maps f, g such that $Per(f) = Per(g) = \{1, 2, 2^2, 2^3, \ldots\}$, f is Li-Yorke chaotic but g is not Li-Yorke chaotic.

1.3 Topological Conjugacies

In order to classify dynamical systems we need a notion of equivalence. The notion of topological conjugacy in dynamical systems is analogous to the notion of isomorphism among groups and to "homeomorphisms" among topological spaces. i.e, We say that two dynamical systems are " having the same dynamical properties" or " dynamically same" if they are topologically conjugate.

Roughly speaking, by saying (X, f) and (Y, g) are topologically conjugate, we mean:

(1) X and Y have the same kind of topology.

(2) f and g have the same kind of dynamics.

Definition 1.3.1. Two dynamical systems (X, f) and (Y, g) are said to be *topologically conjugate* (or simply *conjugate*) if there exists a homeomorphism $h : X \to Y$(called topological conjugacy) such that $f \circ h = h \circ g$. we say simply, f is conjugate to g, and we write it as $f \sim g$. The case when h happens to be an increasing homeomorphism (For example, when $X = \mathbb{R}$ or an interval) we say that f and g are *increasingly conjugate* or *order conjugate* .

Remark 1.3.2. When $Y = X$ and $g = f$, we say that h is a self-conjugacy of f. Being conjugate(and as well as increasingly conjugate) is an equivalence relation among dynamical systems.

Let (X, f) and (Y, g) be two dynamical systems. Then a topological conjugacy from f to g carries orbits of f to "similar " g-orbits. Said precisely,

Theorem 1.3.3. *[13]*

Let (X, f) and (Y, g) be two dynamical systems and let $h : X \to Y$ be a topological conjugacy. Then

1. $h^{-1} : Y \to X$ is a topological conjugacy.

2. $h \circ f^n = g^n \circ h$ for all $n \in \mathbb{N}$.

3. $x \in X$ is a periodic point of f if and only if $h(x)$ is a periodic point of g.

4. If x is a periodic point of f with stable set $W^s(x)$, then the stable set of $h(x)$ is $h(W^s(x))$.

5. The periodic points of f are dense in X if and only if the periodic points of g are dense in Y.

6. f is topologically transitive on X if and only if g is topologically transitive on Y.

7. f is chaotic on X if and only if g is chaotic on Y.

1.3.1 Dynamical properties

Definition 1.3.4. Properties preserved by topological conjugacy are called *dynamical properties* . Many kinds of examples are provided below.

Example 1.3.5. (Dynamical properties of a point)

(a) Fixed point

(b) periodic point

(c) periodic point of period n_0

(d) Eventually fixed point

(e) Eventually periodic point

(f) Point whose orbit has exactly n elements.

Example 1.3.6. (Dynamical properties of subsets)

(a) Invariant subset

(b) The property that $f(A) = A$

(c) The property that $f^{-1}(A) = A$

(d) Dense set

(e) Having a unique limit point

(f) Finite set.

Example 1.3.7. (Dynamical properties dynamical systems)

(a) Having no fixed point

(b) Having no invariant sets, except the whole set and the empty set

(c) Having exactly n periodic points

(d) Having every point periodic

(e) Surjectivity

(f) Injectivity

(g) Having dense range.

(h) Transitivity.

1.3.2 The shift map - An example

Let $\sum_2 = \{(s_0, s_1, s_2, ...)|\ s_i = 0 \text{ or } 1 \text{ for all } i\}$ be the set of all sequences of 0's and 1's. If $s = s_0, s_1, s_2, ...$ and $t = t_0, t_1, t_2, ...$, then define $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$. Note that $(\sum_2, d)$ is a metric space and $d(s,t) \leq 2$ for all $s, t \in \sum_2$.

The following proposition which can be proved easily , asserts that, if we start with any sequence from $\sum_2$, by keep on changing the initial terms we can go arbitrarily close to any point in the space $\sum_2$.

Proposition 1.3.8. *Let $s, t \in \sum_2$. Then*

1. *If the first $n+1$ digits in s and t are identical, then $d(s,t) \leq \frac{1}{2^n}$.*
2. *If $d(s,t) \leq \frac{1}{2^n}$, then the first n digits in s and t are identical.*

Definition 1.3.9. The shift map $\sigma : \sum_2 \rightarrow \sum_2$ is defined by,

$$\sigma(s_0 s_1 s_2 ....) = s_1 s_2 s_3 \ldots .$$

In other words, the shift map forgets the first term.

It follows from the above proposition that the shift map is continuous.

Theorem 1.3.10. *[13]*

The shift map has the following properties.

1. *The set of periodic points of the shift map is dense in $\sum_2$.*
2. *The shift map has 2^n periodic points whose period divides n.*
3. *The set of eventually periodic points of the shift map that are not periodic is dense in $\sum_2$.*
4. *There is an element of $\sum_2$ whose orbit is dense in $\sum_2$.*
5. *The set of points that are neither periodic nor eventually periodic is dense in $\sum_2$.*

Proof. 1. Suppose $s = s_0 s_1 s_2 ....$ is a periodic point of σ with period k. Then $\sigma^k(s) = s$. That is $s_k s_{k+1} s_{k+3} ... = s_0 s_1 s_2 ....$ This implies that $s_{n+k} = s_n$ for all n. That is s is a periodic point with period k if and only if s is a sequence formed by repeating the k-digits $s_0 s_1 s_2 ... s_{k-1}$ infinitely often.

To prove that the periodic points of σ are dense in $\sum_2$, we must show that for all points $t \in \sum_2$ and all $\epsilon > 0$, there is a periodic point s of σ such that $d(t, s) < \epsilon$. For this , if $t = t_0t_1t_2...$ then we choose n such that $\frac{1}{2^n} < \epsilon$, then we can let $s = t_0t_1t_2...t_nt_0t_1t_2...t_nt_0t_1t_2...t_n....$. As t and s agree on the first $n + 1$ digits, by previous proposition 1.3.8, we get $d(s, t) \leq \frac{1}{2^n} < \epsilon$

Proof of 2 and 3 are similar to that of 1.

4. The sequence which begins with 0 1 00 01 10 11 and then includes all possible blocks of 0 and 1 with three digits, followed by all possible blocks of 0 and 1 with four digits, and so forth - called the *Morse sequnce* - has dense orbit.

5. Since the set of nonperiodic points includes as a subset the orbit of the Morse sequence, proof follows from (4).

□

Theorem 1.3.11. *Let s be any point in $\sum_2$ and $\epsilon > 0$. Then there is $t \in \sum_2$ and n_0 such that $d(s, t) < \epsilon$ and $d(\sigma^n(s), \sigma^n(t)) = 2$ for all $n \geq n_0$.*

Proof. Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0}} < \epsilon$. We then choose t such that,

$$t_i = \begin{cases} s_i & \text{if } i \leq n_0 \\ (s_i + 1)\ mod\ 1 & \text{if } i > n_0 \end{cases}$$

That is $s_i = t_i$ if and only if $i \leq n_0$. Then for any $n \geq n_0$, $\sigma^n(s)$ and $\sigma^n(t)$ differ on every digit. Hence $d(\sigma^n(s), \sigma^n(t)) = 2$.

□

Theorem 1.3.12. *The shift map σ is chaotic on $\sum_2$.*

Remark 1.3.13. To study a particular dynamical system, we often look for a conjugacy with a better-understood model. We illustrate this with the following example.

Example 1.3.14. Let K denote the Cantor middle third set (with 0 and 1 are identified). Let $f : K \to K$ be defined by $f(x) = 3x(mod 1)$. Then f is topologically conjugate to the shift map.

For, every number $a \in K$ can be written uniquely in the form $a = \frac{a_0}{3} + \frac{a_1}{3^2} + \frac{a_2}{3^3} + \ldots + \frac{a_n}{3^{n+1}} + \ldots$ where each $a_n \in \{0, 2\}$. [This is called the ternary expansion of a.] To this element a, we associate the sequence $s_0 s_1 s_2 \ldots \in \sum_2$ where $s_n = \frac{a_n}{2}$ for all n. This defines a map T from K to $\sum_2$. One can prove that this map T is, in fact a topological conjugacy from f to the shift map σ. Hence in the view of theorem 1.3.3 all the properties of the shift map specified in theorem 1.3.10 are true for the map f and in particular, f is chaotic.

1.4 Organization of the Thesis / Synopsis

The main results of chapter 2 and 3 are about the continuous automorphisms of the torus. The torus $\mathbb{T}^2 = \frac{\mathbb{R}^2}{\mathbb{Z}^2}$ is here viewed as the topological group $[0, 1) \times [0, 1)$ with coordinate wise addition modulo 1.

For the well known systems like, tent map $f : [0, 1] \to [0, 1]$ defined by $f(x) = 1 - |1 - 2x|$, and for shift map on $\{0, 1\}^{\mathbb{N}}$ the set of periodic points is well studied (See [21]). It is already known [21] that for a hyperbolic(having no eigen values with absolute value 1) continuous toral automorphism, the periodic points are precisely the rational points.

In **chapter-2**, we calculate the set of periodic points for other continuous toral automorphisms; this happens to be the subgroup generated by $\mathbb{Q} \times \mathbb{Q} \cup$ (a line with rational slope).

In fact, for all non-hyperbolic continuous toral automorphism, there are uncountably many periodic points.

Notation 1.4.1. Let $\mathbb{Q}_1$ be the set of all rational points in $[0,1)$. Given $r \in \mathbb{Q}$, we write $S_r = \{(x,y) \in \mathbb{T}^2$ such that $rx + y$ is rational $\}$ and let $S_\infty = \mathbb{Q}_1 \times [0,1)$. Note that $S_0 = [0,1) \times \mathbb{Q}_1$.

Definition 1.4.2. For $m, n \in \mathbb{Z}$, we define

$$A_{m,n} = \begin{cases} \begin{pmatrix} m & n \\ \frac{-(m-1)^2}{n} & 2-m \end{pmatrix} & \text{if } n \neq 0 \\ \begin{pmatrix} 1 & 0 \\ m-1 & 1 \end{pmatrix} & \text{if } n = 0 \end{cases}$$

Note that $Det(A_{m,n}) = 1$ and $Trace(A_{m,n}) = 2$ for all $m, n \in \mathbb{Z}$.

Proposition 1.4.3. *If $A \in GL(2,\mathbb{Z})$ is such that $Det(A) = 1$ and $Trace(A) = 2$ then $A = A_{m,n}$ for some $m, n \in \mathbb{Z}$ such that n divides $(m-1)^2$.*

Theorem 1.4.4. *For $m, n \in \mathbb{Z}$, the set of all periodic points of $T_{A_{m,n}}$ is either the set S_r for some $r \in \mathbb{Q} \cup \{\infty\}$ or $\mathbb{T}^2$, as given in the following table:*

$A_{m,n}$	$P(T_{A_{m,n}})$
$m \neq 1$ and $n \neq 0$	$S_{\frac{m-1}{n}}$
$m = 1$ and $n = 0$	$\mathbb{T}^2$
$m \neq 1$ and $n = 0$	S_∞
$m = 1$ and $n \neq 0$	S_0

Remark 1.4.5. The set S_r can be thought of as the points on the line through the origin with slope $-r$ and its rational translates in $[0,1) \times [0,1)$. From the above proposition, $r = \frac{(m-1)}{n}$when $m \neq 1$. On the other hand given any rational $r = \frac{p}{q} \in \mathbb{Q}$, we can find $m, n \in \mathbb{Z}$ with $n|(m-1)^2$ such that $\frac{p}{q} = \frac{m-1}{n}$. Choose $m = pq + 1$, $n = q^2$. Hence every S_r arises as $P(A_{m,n})$ for some $m, n \in \mathbb{Z}$.

From the above remark it follows that,

Proposition 1.4.6. *The following are equivalent for a subset of the torus.*

(1) It is $P(T_{A_{m,n}})$ *for some* $A_{m,n} \in GL(2, \mathbb{Z})$.

(2) It is S_r *for some* $r \in \mathbb{Q} \cup \{\infty\}$.

Theorem 1.4.7. *For any continuous toral automorphism* T_A*, the set* $P(T_A)$ *of periodic points of* T_A *is one of the following:*

1. $\mathbb{Q}_1 \times \mathbb{Q}_1$.
2. S_r *for some* $r \in \mathbb{Q} \cup \{\infty\}$*; where* $S_r = \{(x, y) \in \mathbb{T}^2 \mid rx + y \text{ is rational } \}$.
3. $\mathbb{T}^2$

In **Chapter-3**, we first discuss some well known results about $Per(f)$ (the set of periods of f), due to Sharkovski [17] and Baker [5] and many others, which are similar to our main result of this chapter. We prove that there are exactly 8 subsets of $\mathbb{N}$ which can occur as $Per(T)$ for some continuous toral automorphism T. We solve the problem separately for hyperbolic and nonhyperbolic automorphisms.

Theorem 1.4.8. *For any hyperbolic toral automorphism* $T_A : \mathbb{T}^2 \to \mathbb{T}^2$*, the set of periods* $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

Theorem 1.4.9. *Let* T_A *be a toral automorphism. Then* $Per(T_A)$ *is one of the following 8 subsets of* $\mathbb{N}$.

(1) $\{1\}$

(2) $\{1, 2\}$

(3) $\{1, 3\}$

(4) $\{1, 2, 4\}$

(5) $\{1, 2, 3, 6\}$

(6) $2\mathbb{N} \cup \{1\}$

(7) $\mathbb{N} \setminus \{2\}$

(8) $\mathbb{N}$.

In **chapter-4**, we study certain simple systems on $\mathbb{R}$. Since $\mathbb{R}$ has order structure, we would like to consider the conjugacies preserving the order. Let $f : \mathbb{R} \to \mathbb{R}$ be a dynamical system and let $x, y \in \mathbb{R}$. Then, we write $x \sim y$ if there exists an order conjugacy $h : \mathbb{R} \to \mathbb{R}$ of f such that $h(x) = y$. Note that $\sim$ is an equivalence relation. Let $[x]$ to denote the equivalence class of x.

We introduce the notions of *special points* and *nonordinary points*, which are new to the literature. The properties which are preserved under topological conjugacies are called *dynamical properties*. Having every point periodic, having a point with dense orbit, having exactly n periodic points are examples of dynamical properties.

Under a topological conjugacy a point can be mapped to a point with similar dynamics. Therefore, if a point is unique upto some dynamical property, then it must be fixed by all order conjugacies. This motivates us to define,

Definition 1.4.10. A point $x \in (\mathbb{R}, f)$ is said to be "dynamically special" if $[x] = \{x\}$.

We call a point to be *ordinary* if it is like points near it. That is, $[x]$ is a neighbourhood of x. The points which are not ordinary are called *nonordinary*.

It is observed that for systems with finitely many nonordinary points, the idea of nonordinary points and the idea of special points coincide.

Definition 1.4.11. Let (X, f) be a dynamical system. By the *full orbit* of a point $x \in X$ we mean the set

$\widetilde{O}(x) = \{y \in X | f^n(x) = f^m(y) \text{ for some } m, n \in \mathbb{N}\}$.

Definition 1.4.12. A point x in a dynamical system (X, f) is said to be a *critical point* if f fails to be one-one in every neighbourhood of x. The set of all critical points of f is denoted by $C(f)$.

We proved in [25], the following characterization theorem for the set of special points $S(f)$.

Theorem 1.4.13. *For continuous self-maps of the real line $\mathbb{R}$, the set of all nonordinary points (and hence the set of all special points) is contained in the closure of the union of full orbits of critical points , periodic points and the limits at infinity (if they exist and are finite).*

We prove that: If S_P denote the set of all points having the dynamical property P then the points of ∂S_P (the boundary of S_P) are nonordinary. In particular, being a point in a particular equivalence class is a dynamical property of the point. Hence by the very nature of the order conjugacies, it follows that when there are finitely many nonordinary points(therefore special points) there are only finitely many equivalence classes. These are the simple systems we study in this chapter.

We describe completely, the homeomorphisms on $\mathbb{R}$, having finitely many nonordinary points and give a general formula for counting.

Notation 1.4.14.

a_n = The number of increasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto order conjugacy.

t_n = The number of increasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto topological conjugacy.

s_n = The number of decreasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto order conjugacy.

k_n = The number of decreasing bijections, with exactly n nonordinary points, on $\mathbb{R}$, upto topological conjugacy.

Theorem 1.4.15. *For every positive integer n, we have*

$a_n = c_1(1+\sqrt{3})^n + c_2(1-\sqrt{3})^n$ *where* $c_1 = \frac{5+3\sqrt{3}}{2\sqrt{3}}$ *and* $c_2 = \frac{3\sqrt{3}-5}{2\sqrt{3}}$.

$$s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ a_{\frac{n-1}{2}} & n \text{ is odd} \end{cases}$$

$$t_n = \begin{cases} \frac{a_n + 2a_{\frac{n-4}{2}}}{2} & \text{if } n \text{ is even} \\ \frac{a_n + 2a_{\frac{n-3}{2}}}{2} & n \text{ is odd} \end{cases}$$

$$k_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ t_{\frac{n-1}{2}} & n \text{ is odd} \end{cases}$$

Where $t_0 = 2$, $t_1 = 5$ *and* $t_2 = 12$ *by direct computation.*

Also we prove that,

Theorem 1.4.16. *Upto order conjugacy, there are exactly 26 self maps on $\mathbb{R}$ with a unique nonordinary point.*

The main result of **chapter-5**, is obtained while making an attempt to answer the question: *Exactly which maps on $\mathbb{R}$ are topologically conjugate to a polynomial?.* The main theorem of this chapter, gives a necessary condition for a continuous self map of $\mathbb{R}$ to be conjugate to a polynomial.

Points which will reach a critical point (res. periodic point) after finitely many instants of time are called precritical points (res. preperiodic points).

Theorem 1.4.17. *Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial such that $f^2 \neq id$(the identity map on $\mathbb{R}$). Then $\overleftarrow{C(f)} \Delta \overleftarrow{P(f)}$ is a discrete set in the relative topology.*

Where $\overleftarrow{C(f)}$ denotes the set of all precritical points of f and $\overleftarrow{P(f)}$ denotes the set of all preperiodic points of f.