Chapter 5

Periodic points Vs Critical points

5.1 Motivation

In this chapter, first we discuss some interesting results connecting critical points and periodic points. *We prove a necessary condition for a continuous self map on* $\mathbb{R}$ *to be conjugate to a polynomial.*

Recall that an element $c \in X$ is said to be a critical point of the dynamical system (X, f) if f fails to be one-one on every neighbourhood of x.

Example 5.1.1. (1) When $X = \mathbb{R}$, local maxima and minima points are critical points. These are the points where the graph of f takes a turn.

(2) For complex polynomials, the roots of the derivative are critical points.

Definition 5.1.2. Let x be a fixed point of a dynamical system (X, f). We say that x is an attracting fixed point if there is a neighbourhood V of x such that for every y in V, the trajectory of y converges to x.

Example 5.1.3. Let $X = \mathbb{R}$ or $\mathbb{C}$. Let f be a continuously differentiable function. Let x be a fixed point of (X, f). If $|f'(x)| < 1$, then x is an attracting fixed point.

Definition 5.1.4. Let x be an attracting fixed point in a dynamical system (X, f). Then the set of all points whose trajectory converges to x, is called the *basin of attraction* of x.

Example 5.1.5. For the map z^2 on $\mathbb{C}$, the number 0 is an *attracting fixed point.* It is also the unique critical point. The basin of attraction is the open unit disc.

Theorem 5.1.6. ***(Fatou)*** *[13] Every attracting periodic point of a complex polynomial contains a critical point in its basin of attraction.*

Remark 5.1.7. For a complex polynomial, there are only finitely many critical points. Some of these critical trajectories may converge. Their limits may be attracting periodic points. Our search for attracting periodic points may be confined to them. We can't find them elsewhere.

Definition 5.1.8. Let x be a periodic point of a dynamical system (X, f). Let n be its period. Then x is a fixed point of (X, f^n). If x is an attracting fixed point of (X, f^n), then x is called an *attracting periodic point* of (X, f).

Definition 5.1.9. Let (X, f) be a dynamical system. For any subset $A \subset X$, define $\overleftarrow{A} = \{x \in X |\ f^n(x) \in A \text{ for some } n \in \mathbb{N} \cup \{0\}\}$.

If $C(f)$ denotes the set of all critical points of f, then the elements of $\overleftarrow{C(f)}$ are called *precritical points* . Similarly, the elements of $\overleftarrow{P(f)}$ are called the *preperiodic points* .

Definition 5.1.10. A point x in a dynamical system (X, f) is said to be a *recurrent point* if there exists an increasing sequence of natural numbers (n_k) such that $f^{n_k}(x) \to x$. We say x is nonwandering if $f^{n_k}(x_k) \to x$ for some sequence of points $x_k \to x$ and some sequence of integers $n_k \to \infty$.

The main theorem of this chapter is similar in spirit to the following known theorems:

Theorem 5.1.11. *[6] If f is any continuous map from I to I, then the set of periodic points and the set of recurrent points have the same closure.*

Theorem 5.1.12. *[6] The set of all nonwandering points of a continuous map $f : I \to I$ is contained in the closure of the set of all preperiodic points.*

Theorem 5.1.13. *[14] Let $f : I \to I$ be continuous. If the set of sensitive points is dense in I, then the set of preperiodic points is also dense in I.*

Theorem 5.1.14. *[24] Let $f : I \to I$ be a polynomial such that f^2 is not identity. Then $\overleftarrow{C(f)}$ is dense in I if and only if $\overleftarrow{P(f)}$is dense in I.*

5.2 Main Theorem

Example 5.2.1. Consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x(1-x)$. For this map $\frac{1}{2}$ is the only critical point.

Let $\Lambda = \{x \in [0,1] \mid f^n(x) \in [0,1] \text{ for all } n \in \mathbb{N}\}$.

That is, if $J = \{x \in [0,1] \mid f(x) \notin [0,1]\}$ and if $E_n = [0,1] \setminus f^{-n}(J)$, then $\Lambda = \cap_{n=1}^{\infty} E_n$.

Note that, for any $n \in \mathbb{N}$, $f^{-n}\{\frac{1}{2}\}$ is a set of 2^n points in $[0,1]$. Also, note that $f^{-n}(J)$ is a disjoint union of 2^n open intervals such that each component interval contains exactly one element of $f^{-n}\{\frac{1}{2}\}$.

Since $f^{-n}(J) \cap f^{-m}(J) = \emptyset$ whenever $m \neq n$, we conclude that the set $\cup_{n=0}^{\infty} f^{-n}\{\frac{1}{2}\} = \overleftarrow{C(f)}$ is a discrete set.

Now, We can prove that $\overline{\overleftarrow{C(f)}} = \overleftarrow{C(f)} \cup \Lambda$.(We use the fact that Λ is perfect and totally disconnected.)

Since the map $f|_\Lambda : \Lambda \to \Lambda$, is topologically conjugate to the shift map (See [13]), we have $\overline{\overleftarrow{P(f)}} = \Lambda$.

Thus, $\overline{\overleftarrow{C(f)}} \Delta \overline{\overleftarrow{P(f)}}$ is discrete.

Proposition 5.2.2. *For monotone maps, all the periodic points are fixed points or period-2 points.*

Proof. Proof follows from the fact that for an increasing map the orbits are uni-directional. When the given map f is decreasing consider f^2 which is increasing.

□

Proposition 5.2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and piecewise monotone. Let A be any backward invariant subset of $\mathbb{R}$ and let J be any component interval of $\mathbb{R} \setminus \overline{A}$. Then $f(\overline{J}) \subset \overline{K}$ for some component interval K of $\mathbb{R} \setminus \overline{A}$.*

Proof. Since A is backward invariant $int f(J)$ cannot meet $\overline{A}$ and therefore $int f(J) \subset K$ for some component interval K of $\mathbb{R} \setminus \overline{A}$. Now $f(\overline{J}) \subset \overline{f(J)} = \overline{int f(J)} \subset \overline{K}$. □

As noted earlier, the main theorem gives a necessary condition for a continuous self map on $\mathbb{R}$ to be conjugate to a polynomial.

Theorem 5.2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial such that $f^2 \neq id$(the identity map on $\mathbb{R}$). Then $\overline{\overleftarrow{C(f)}} \Delta \overline{\overleftarrow{P(f)}}$ is a discrete set in the relative topology.*

Proof. ***Part: 1***

First we prove that $\overline{\overleftarrow{P(f)}} \setminus \overline{\overleftarrow{C(f)}}$ is a discrete set. For this , consider $\mathbb{R} \setminus \overline{\overleftarrow{C(f)}}$ which is open and hence a countable union of disjoint intervals(including open rays) say, $\mathbb{R} \setminus \overline{\overleftarrow{C(f)}} = \bigcup_{\alpha \in S} I_\alpha$. Note that for any I_α, the *open* intervals $I_\alpha, f(I_\alpha), f^2(I_\alpha), \cdots$ are disjoint from $\overline{\overleftarrow{C(f)}}$, as $\overline{\overleftarrow{C(f)}}$ is backward f invariant(that is $f^{-1}(A) \subset A$) and hence induces a map $f^* : S \to S$.

Step: 1

We claim that each I_α can contain atmost finitely many periodic points.

1. Suppose I_α has a periodic point for some $\alpha \in S$. Let $k \in \mathbb{N}$ be the least such that $f^k(I_\alpha) \subset I_\alpha$. From the choice of k it is clear that a point $x \in I_\alpha$ is f-periodic if and only if it is f^k-periodic. We know that $f^k : I_\alpha \to I_\alpha$ is strictly monotone. Hence by proposition 5.2.2, the periodic points of f^k are fixed points and period-2 points. Since f is a polynomial such that $f^2 \neq Id$, they are finite in number. This completes the proof of step-1.

Step: 2

Claim: Each I_α can have only finitely many preperiodic points. For, suppose I_α has a preperiodic point. Let k^1 be the least such that $f^{k^1}(I_\alpha)$ has a periodic point. Then $f^{k^1}(I_\alpha) \subset I_\beta$ for some $\beta \in S$. By step:1 this I_β can have atmost finitely many periodic points. Let this finite set be F. Then $f^{-k^1}(F)$ is also a finite set , as f is a polynomial. To complete the argument it is enough to prove that any preperiodic point in I_α will land in I_β as a periodic point after k^1 instants of time. For this, let $x \in I_\alpha$ be a preperiodic point. We claim that $f^{k^1}(x)$ is a periodic point. Now $f^{k^1}(x) \in I_\beta$ and I_β has a periodic point. Let p be the f^*-period of β. ($f^p : I_\beta \to I_\beta$ is strictly monotone and hence all preperiodic points are periodic.) Therefore $f^{k^1}(x)$ is f^p-periodic and hence f-periodic.

Now, Suppose $x \in I_\alpha \bigcap \overleftarrow{P(f)}$ for some α. Then x must be a preperiodic point because I_α contains only finitely many preperiodic points. Hence $I_\alpha \bigcap \overleftarrow{P(f)}$ is finite.

Part: 2

We now prove that $\overleftarrow{C(f)} \setminus \overleftarrow{P(f)}$ is a discrete set.

Write $\mathbb{R} \setminus \overleftarrow{P(f)} = \bigcup_{\alpha \in T} J_\alpha$, a disjoint union of open intervals . Then by proposition 5.2.3, for every $\alpha \in T$ there exists $\beta \in T$ such that $f(\overline{J_\alpha}) \subset \overline{J_\beta}$. This induces a map

$f^* : T \to T$. Consider a component interval J. Fix $c \in C(f)$. It is enough to prove that $|\overleftarrow{c} \cap K| < \infty$ for every compact subinterval $K \subset J$.

Suppose $c \in f^n(J)$ for some $n \in \mathbb{N} \cup \{0\}$. Then $f^n(J) \subset \overline{J_\beta}$ for some $\beta \in T$. If this β is not f^*-periodic, then $\overleftarrow{c} \cap K = f^{-n}(c) \cap K$ which is a finite set.

If β is f^*-periodic, let p be the least such that f^p maps J_β to J_β. Then we can find $\delta > 0$ such that $| x - f^p(x) | > \delta$ on $f^n(K)$, as $f^n(K)$ cannot contain any periodic point. Thus the motion under f^p on $f^n(K)$ is unidirectional. Hence if $c_1 \in f^n(K)$ and if $f^{tp}(c_1) = c$ for some t, then this t can be chosen to be $\leq \frac{length\ of\ f^n(K)}{\delta}$. This forces that, there can be only finitely many precritical points in K which reach the critical point c. This completes the proof. □

Remark 5.2.5. It is clear that the result is true for all continuous functions which are topologically conjugate to a polynomial. Still, among interval maps this can fail for nonpolynomials. This is illustrated in the following proposition.

Proposition 5.2.6. *Let F be a nonempty closed subset of I. Then there exists a strictly increasing continuous function $f : I \to I$ such that $Fix(f) = F$*

Proof. In each component interval of F^c, we keep a copy of the map $x^2 : [0, 1] \to [0, 1]$.

By a copy of $x^2 : [0, 1] \to [0, 1]$ on $[a, b]$ we mean the map $a + \frac{(x-b)^2}{b-a} : [a, b] \to [a, b]$. □

Remark 5.2.7. Note that the theorem is not true in case of functions on $\mathbb{C}$. For, if $f(z) = z^2$ then $C(f) = \{0\}$, $\overleftarrow{C(f)} = \{0\}$, $\overleftarrow{P(f)} = S^1 \cup \{0\}$ and hence $\overleftarrow{C(f)} \Delta \overleftarrow{P(f)} = S^1$.