Chapter 4

A counting problem

4.1 Dynamically Special points

The properties of dynamical systems which are preserved by topological conjugacies are called dynamical properties. The points which are unique upto some dynamical property are called dynamically special points. Said differently, a special point has a dynamical property which no other point has. The idea of special points is new to the literature.

Throughout this chapter we will be working with continuous self maps of the real line. Since $\mathbb{R}$ has order structure, we would like to consider the conjugacies preserving the order. Hence the conjugacies which we consider in this chapter are order preserving conjugacies (increasing conjugacies).

When we are working with a single system, any self conjugacy can utmost shuffle points with same dynamical behavior. Therefore a point which is unique upto its behavior must be fixed by every self conjugacy. On the other hand if a point is fixed by all self conjugacies then it must have a special property (some times it may not be known explicitly). These things motivated us to call the set of all points fixed by all
self-conjugacies as set of *special points*.

For \( x, y \in \mathbb{R} \), we write \( x \sim y \) if \( x \) and \( y \) have the same dynamical properties in the dynamical system \((\mathbb{R}, f)\). Said precisely, \( x \sim y \) if there exists an increasing homeomorphism \( h : \mathbb{R} \to \mathbb{R} \) such that \( h \circ f = f \circ h \) and \( h(x) = y \). It is easy to see that \( \sim \) is an equivalence relation. Let \([x]\) to denote the equivalence class of \( x \in \mathbb{R} \).

In a dynamical system \((X, f)\), we say that a point \( x \) is ordinary if, it is “like” points near it. That is,

**Definition 4.1.1.** An element \( x \in \mathbb{R} \) is *ordinary* in \((\mathbb{R}, f)\) if its equivalence class \([x]\) is a neighbourhood of it. i.e, the equivalence class of \( x \) contains an open interval around \( x \). A point which is not ordinary is called *non-ordinary* . Let \( N(f) \) denote the set of all non-ordinary points of \( f \).

We call a point to be *special* if \([x] = \{x\} \). Let \( S(f) \) denote the set of all special points of \( f \).

**Remark 4.1.2.** A point \( x \) in a topological space \( X \) is said to be *rigid* if it is fixed by every self homeomorphism of \( X \). For example, the point 1 is rigid in \((0, 1]\). According to the above definition all rigid points are special, even though there is no role for the map \( f \). we make this as a convention.

### 4.1.1 Examples and some characterization theorems

**Definition 4.1.3.** Let \((X, f)\) be a dynamical system. By the *full orbit* of a point \( x \in X \) we mean the set

\[
\tilde{O}(x) = \{y \in X| f^n(x) = f^m(y) \text{ for some } m, n \in \mathbb{N}\}.
\]

For any subset \( A \subset \mathbb{R} \), let

\[
\tilde{O}(A) = \bigcup_{x \in A} \tilde{O}(x) = \bigcup_{x \in A} \{y \in \mathbb{R} : f^n(y) = f^m(x) \text{ for some } m, n \in \mathbb{N}\}.
\]
Definition 4.1.4. A point \( x \) in a dynamical system \((X, f)\) is said to be a critical point if \( f \) fails to be one-one in every neighbourhood of \( x \). The set of all critical points of \( f \) is denoted by \( C(f) \).

We prove in [25], the following characterization theorem for the set \( N(f) \) (and hence for \( S(f) \)).

Theorem 4.1.5. For continuous self-maps of the real line \( \mathbb{R} \), the set of all nonordinary points is contained in the closure of the union of full orbits of critical points, periodic points and the limits at infinity (if they exist and finite).

Remark 4.1.6. In the above theorem the inclusion can be strict.

Consider the map \( f(x) = x + \sin x \) for all \( x \in \mathbb{R} \). All integral multiples of \( \pi \) are fixed points for this map but the increasing bijection \( x \mapsto x + 2\pi \) commutes with \( f \) and fixes none of them.

Theorem 4.1.7. [25] For polynomials of even degree the equality is true in the previous theorem.

Let \( D = \hat{O}(C(f) \cup P(f) \cup \{f(\infty), f(-\infty)\}) \). Where \( f(\infty) \) and \( f(-\infty) \) are the limits of \( f \) at \( \infty \) and \( -\infty \) respectively, provided they are finite, \( C(f) \) denotes the set of all critical points and \( P(f) \) denotes the set of all periodic points of \( f \).

Theorem 4.1.8. [25] For polynomial maps \( f \) of \( \mathbb{R} \), \( S(f) \) has to be either empty or a singleton or the whole \( \overline{D} \).

From the definition, it is clear that the set of special points \( S(f) \) is always closed. The following theorem is about the converse and it is proved in [25].

Theorem 4.1.9. Given any closed subset \( F \) of \( \mathbb{R} \), there exists a continuous map \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that \( S(f) = F \).
Examples

For any $f : \mathbb{R} \to \mathbb{R}$, let $G_f$ denote the set of all topological conjugacies of $f$ and let $G_{f\uparrow}$ denote the set of all increasing conjugacies of $f$.

**Proposition 4.1.10.** If $x$ is an ordinary point of $f$ and if $h$ is self topological conjugacy of $f$, then $h(x)$ is ordinary.

*Proof.* Since $x$ is ordinary there exists an open interval $V$ contained in $[x]$. We prove that the open interval (since $h$ is a homeomorphism) $h(V)$ is contained in $[h(x)]$.

Take $s \in h(V)$. Then $s = h(t)$ for some $t \in V$. Since $V \subset [x]$, there exists $\varphi \in G_{f\uparrow}$ such that $\varphi(t) = x$. Then the increasing homeomorphism $\psi = h\varphi h^{-1}$ carries $s$ to $h(x)$ and commutes with $f$.

\[\square\]

**Proposition 4.1.11.** If $x$ is a nonordinary point of $f$ and if $h$ is a self topological conjugacy of $f$, then $h(x)$ is nonordinary.

*Proof.* Note that, if $h$ is a self conjugacy of $f$ then $h^{-1}$ is also a self conjugacy of $f$. Now, the proof follows from the previous proposition.

\[\square\]

**Example 4.1.12.** (i) If $f : \mathbb{R} \to \mathbb{R}$ has a unique fixed point then it is nonordinary.

(ii) If $f : \mathbb{R} \to \mathbb{R}$ has a unique nonordinary point then it must be a fixed point.

*Proof.* (i) Since the topological conjugacies carry fixed points to fixed points, the unique fixed point must be fixed by every self conjugacy and hence special.

(ii) Suppose $x_0 \in \mathbb{R}$ is the unique nonordinary point of $f$. Then $h(x_0) = x_0$ for all $h \in G_{f\uparrow}$. Now, for any $h \in G_{f\uparrow}$ we have $h(f(x_0)) = f(h(x_0)) = f(x_0)$. That is, the point $f(x_0)$ is special. since $x_0$ is the only special point, we have $f(x_0) = x_0$. 

Example 4.1.13. If \( f : \mathbb{R} \to \mathbb{R} \) has finitely many fixed points (critical points) then all fixed (critical) points are special and hence nonordinary.

Proof. Follows from the fact that under a topological conjugacy fixed points will be mapped to fixed points and critical points will be mapped to critical points and the fact that it takes the finite set \( F \) (of fixed points) to \( F \) bijectively, preserving the order.

Example 4.1.14. If there are only finitely many periodic cycles then all periodic points are special.

Remark 4.1.15. It is immediate from the definition that every special point is nonordinary. But every nonordinary point may not be special. For example, consider the map \( x \mapsto x + \sin x \) on \( \mathbb{R} \) which has countably many fixed points namely \( n\pi \) where \( n \in \mathbb{Z} \). Among them, the fixed points \( 2k\pi, (k \in \mathbb{Z}) \) are repelling and the fixed points \( (2k + 1)\pi, (k \in \mathbb{Z}) \) are attracting. Recall that repelling fixed points cannot be conjugate to attracting fixed points. Note that all these fixed points are nonordinary but these fixed points constitute two conjugacy classes namely, \([0] = \{2k\pi : k \in \mathbb{Z}\}\) and \([\pi]\) = \{(2k + 1)\pi : k \in \mathbb{Z}\}, (Conjugacies of the form \( x \mapsto x + 2k\pi \) serve the purpose.) hence they are not special.

The following proposition is about the converse.

Proposition 4.1.16. If \( f : \mathbb{R} \to \mathbb{R} \) has only finitely many nonordinary points then every nonordinary point is special.

Proof. Let \( N(f) \) to denote the set of all nonordinary points of \( f \). Since \( N(f) \) is finite, it follows from the previous proposition that \( h(N(f)) = N(f) \) for all \( h \in G_{f^1} \). Then
we must have \( h(x) = x \) for all \( x \in N(f) \), because of the order preserving nature of \( h \).
Hence all points of \( N(f) \) are special.

Thus, for the class of maps with finitely many nonordinary points the idea of special points and the idea of nonordinary point, coincide.

**Proposition 4.1.17.** For maps with finitely many nonordinary points, \( f(x) \) is nonordinary whenever \( x \) is nonordinary.

**Proof.** Since \( x \) is nonordinary and since there are only finitely many nonordinary points, we have \( h(x) = x \) for all \( h \in G_{f_{1}} \).

Now for any \( h \in G_{f_{1}} \), we have \( h(f(x)) = f(h(x)) = f(x) \). Hence \( f(x) \) is nonordinary.

**Definition 4.1.18.** For any subset \( A \) of \( \mathbb{R} \), we write \( \partial A = \overline{A} \cap (X - A) \) and call the boundary of \( A \). Here \( \overline{A} \) denotes the closure of \( A \) in \( \mathbb{R} \).

Recall that the properties which are preserved under topological conjugacies are called dynamical properties. Hence, if two points \( x, y \) in the dynamical system \( (X, f) \), differ by a dynamical property, then no conjugacy can map one to the other, from which it follows that,

**Proposition 4.1.19.** For any dynamical property \( P \), the points of \( \partial S_{P} \) are nonordinary. Here \( S_{P} \) denotes the set of all points in \( (X, f) \) having the dynamical property \( P \).

**Corollary 4.1.20.** Let \( f : \mathbb{R} \to \mathbb{R} \) be constant in a neighbourhood of a point \( x_{0} \). Then the end points of the maximal interval around \( x \) on which \( f \) is constant, are non-ordinary.
Lemma 4.1.21. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. Suppose \( a < b \) and \( (a, b) \cap N(f) = \emptyset \). Then \( x \sim y \) for all \( x, y \in (a, b) \).

Proof. Assume WLOG that \( x < y \). Suppose \( x \not\sim y \), so \( z = \sup(\{x \cap (\mathbb{R} \setminus [x] \}) \) exists. Clearly \( z \in [x] \). If \( z = y \) then \( z \in [y] \subset \mathbb{R} \setminus [x] \). Otherwise \( z < y \) and \( [z, y) \cap (\mathbb{R} \setminus [x]) \neq \emptyset \) for every \( y - x > \epsilon > 0 \) which again shows \( z \in \mathbb{R} \setminus [x] \). Hence \( z \in \partial([x]) \), so \( z \in N(f) \) by proposition 4.1.19. But \( a < x \leq z \leq y < b \) so \( z \in (a, b) \cap N(f) \) contradicting our hypothesis. \( \square \)

Theorem 4.1.22. Let \( f : \mathbb{R} \to \mathbb{R} \) be continuous. If \( |N(f)| = n \) then \(|\{[x] : x \in \mathbb{R}\}| = 2n + 1\).

Proof. Let \( N(f) = \{x_1, x_2, \ldots, x_n\} \) where \( x_1 < x_2 < \cdots < x_n \). By proposition 4.1.16, each \( \{x_i\} \) is an equivalence class. By remark 4.2.1, each of these intervals \((-\infty, x_1), (x_1, x_2), \cdots, (x_{n-1}, x_n), (x_n, \infty)\) is invariant under every element of \( G_f \), so all remaining equivalence classes are contained in one of these intervals. Lemma 4.1.21 above now shows that each of these interval is an equivalence class, giving \(|\{[x] : x \in \mathbb{R}\}| = 2n + 1\). \( \square \)

Remark 4.1.23. If the definition of \( \sim \) is weakened to allow decreasing conjugacies (i.e, \( x \sim y \) if there exists \( h \in G_f \) such that \( h(x) = y \)), similar reasoning yields the inequality

\[
n + 1 \leq |\{[x] : x \in \mathbb{R}\}| \leq 2n + 1
\]

and both bounds can be obtained. This is illustrated by the maps \( f(x) = 2x \) for which \( |N(f)| = 1 \) and \(|\{[x] : x \in \mathbb{R}\}| = 2\) and \( f(x) = |x| \) for which \( |N(f)| = 1 \) and \(|\{[x] : x \in \mathbb{R}\}| = 3\).
Remark 4.1.24. Note that, being a point in a particular equivalence class \([x]\) is a dynamical property.

Remark 4.1.25. There are maps \(f : \mathbb{R} \to \mathbb{R}\) having finitely many equivalence classes, but infinitely many nonordinary points. For example, consider the map \(f(x) = x + \sin x\) on \(\mathbb{R}\). As noted above, (See remark 4.1.15) there are two classes of fixed points. Since increasing orbits must map to increasing orbits under increasing conjugacies, points like \(\frac{\pi}{2}\) (increasing orbit) and \(\frac{3\pi}{2}\) (decreasing orbit) cannot be equivalent. Hence there must be at least four equivalence classes. To see that there are exactly four equivalence classes, let \(I_k = (2k\pi, (2k + 1)\pi)\), \(D_k = ((2k + 1)\pi, 2(k + 1)\pi)\) and observe that \(I_k \cap N(f) = \emptyset = D_k \cap N(f)\) for each \(k \in \mathbb{Z}\) by proposition 4.1.34. Hence by lemma 4.1.21, each \(I_k\) and \(D_k\) is contained in a single equivalence class. Conjugacies of the form \(x \mapsto x + 2k\pi\) complete the argument.

Recall that, if \(f : \mathbb{R} \to \mathbb{R}\) has a unique non-ordinary point then it is a fixed point.

We say that a function \(f : \mathbb{R} \to \mathbb{R}\) is locally constant at a point \(x_0 \in \mathbb{R}\) if \(f\) is constant in some open interval around \(x_0\).

Proposition 4.1.26. Let \(f : \mathbb{R} \to \mathbb{R}\) be a continuous function. Then

(i) If \(x \in \mathbb{R}\) is both critical and ordinary then \(f\) is locally constant at \(x\).

(ii) If \(x\) is ordinary and \(f\) is not locally constant at \(x\) then \(f(x)\) is ordinary.

Proof. (i) Let \(x_0 \in \mathbb{R}\) be both critical and ordinary.

Claim: \(f\) is constant in some neighbourhood of \(x_0\).

Since \(x_0\) is ordinary, there exist \(\eta > 0\) such that all points in \((x_0 - \eta, x_0 + \eta)\) will look alike. So it is enough to prove that \(f\) is somewhere constant in \((x_0 - \eta, x_0 + \eta)\).

Case: 1
Now, suppose some point of $(x_0 - \eta, x_0 + \eta)$ is point of local maximum. Then we can prove easily that every point of $(x_0 - \eta, x_0 + \eta)$ is a point of local maximum. That is there exist $\epsilon > 0$ such that $f(x_0) \geq f(t) \forall t \in (x_0 - \epsilon, x_0 + \epsilon)$. Next, choose $\delta < \epsilon, \eta$. Then there exist $y \in [x_0 - \delta, x_0 + \delta]$ such that $f(y) \leq f(t) \forall t \in [x_0 - \delta, x_0 + \delta]$ .......(1).

But $y$ is a point of local maximum (since $\delta < \eta$). That is there exist $\alpha > 0$ such that $f(y) \geq f(s) \forall s \in (y - \alpha, y + \alpha)$ ............(2).

From equations (1) and (2), it follows that $f$ is constant in some neighbourhood $y$ and hence constant in some neighbourhood of $x_0$.

**Case: 2**

No point is a point of local maximum. That is, in every subinterval $f$ attains its maximum at one of the end points.

If $f$ assumes supremum always on the right end or always on the left end then $f$ is strictly monotone.

Note that, it is enough if we prove monotone somewhere. Take a neighbourhood $(\alpha, \beta)$ of $x_0$ such that $(\alpha, \beta) \subset (x_0 - \eta, x_0 + \eta)$ and let sup $f$ on $(\alpha, \beta)$ is attained at the right end point $\beta$. Suppose sup $f$ is attained at the right end point in every subinterval of $(\alpha, \beta)$ containing $x_0$. Then $f$ is increasing in $(x_0, \beta)$. We are done.

Suppose there is a subinterval say $(x_0 - \epsilon_1, x_0 + \epsilon_2)$ of $(\alpha, \beta)$ on which $f$ attains its supremum at the left end point. Then $f$ attains its infimum on $(x_0 - \epsilon_1, \beta)$ at some interior point. We now argue as in Case.1, with infimum instead of supremum.

**Proof of (ii):** We make use of (i).

Assume that $f$ is not constant on any neighbourhood of $x$. Because $x$ is ordinary, there exist an open interval $J$ around $x$ in which all points are equivalent such that $f$ is not constant on $J$. It follows that $f$ is not constant on any non-trivial subinterval of $J$, because the end points of intervals of constancy are non-ordinary. From (i), it follows
that $J$ has no critical point. Therefore $f(J)$ is an open interval. We claim that any two
elements of $f(J)$ are equivalent. Let $f(y)$ be a general element of $f(J)$ where $y \in J$,
$y \neq x$. By choice of $J$, there exists a self conjugacy $h$ of $f$ such that $h(y) = x$. Which
implies $hf(y) = fh(y) = f(x)$. Therefore $f(y)$ is equivalent to $f(x)$. This proves $f(x)$
is ordinary.

\[\Box\]

**Example 4.1.27.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $\sup f(\mathbb{R})$, $\inf f(\mathbb{R})$, $\lim_{x \to \infty} f(x)$
and $\lim_{x \to -\infty} f(x)$ are nonordinary (in fact, special points) provided they are finite.
Note that, For maps with finitely many nonordinary points both $\lim_{x \to \infty} f(x)$ and
$\lim_{x \to -\infty} f(x)$ always exists in $\mathbb{R} \cup \{\infty, -\infty\}$.

*Proof.* For any $h \in G_{f^\uparrow}$, $h(f(\mathbb{R})) = f(h(\mathbb{R})) = f(\mathbb{R})$. That is $h$ takes the range of $f$
to itself. Since $h$ is increasing, $h(\sup f) = \sup f$ and $h(\inf f) = \inf f$.

**To prove** $\lim_{x \to \infty} f(x)$ **is special:**

First we prove that for maps with finitely many nonordinary points, $\lim_{x \to \infty} f(x)$
always exists in $\mathbb{R} \cup \{\infty, -\infty\}$.

For, let $t_0$ be the largest nonordinary point and let $A$ be the set of all critical points
$> t_0$.

Suppose $A$ is empty. Then $f$ is monotone on $[t_0, \infty)$ and hence $\lim_{x \to \infty} f(x)$ exists.

Suppose $A$ is nonempty. Then $\partial A$ is nonempty. But every element of $\partial A$ is nonor-
dinary. Hence $\partial A = \{t_0\}$. Therefore $A = (t_0, \infty)$. Therefore $f$ is constant on $A$ (We
argue as in the proof of Case-2 of (i), in the previous proposition). Hence $\lim_{x \to \infty} f(x)$
exists.

Now to prove $\lim_{x \to \infty} f(x)$ is special:

Denote $\lim_{x \to \infty} f(x)$ by $l$. Let $h \in G_{f^\uparrow}$. Note that for any sequence $(x_n) \to \infty$ the
sequence $f(x_n) \to l$ and the sequence $h(x_n) \to \infty$. 
Let \((x_n) \to \infty\). Then \(f(x_n) \to l\). Hence \(h(f(x_n)) = f(h(x_n)) \to h(l)\). But the sequence \(h(x_n) \to \infty\). Hence by the definition of \(l\), \(f(h(x_n)) = h(f(x_n)) \to l\). Hence \(h(l) = l\). This completes the proof.

\[\]

**Proposition 4.1.28.** The maps \(x + 1\) and \(x - 1\) on \(\mathbb{R}\) are topologically conjugate; but not order conjugate.

**Proof.** The maps \(x + 1\) and \(x - 1\) are conjugate to each other through \(-x + \frac{1}{2}\).

If possible, let \(h\) be an order conjugacy from \(f(x) = x + 1\) to \(g(x) = x - 1\). Then \(h(x + 1) = h(f(x)) = g(h(x)) = h(x) - 1\). i.e, \(h(x + 1) - h(x) = -1 < 0\). Which is a contradiction to the assumption that \(h\) is increasing.

\[\]

**Remark 4.1.29.** Note that for the map \(x + 1\) on \(\mathbb{R}\), all points are ordinary. For, if \(a, b \in \mathbb{R}\), then the map \(x + b - a\) is the order conjugacy of \(x + 1\) which maps \(a\) to \(b\).

The following proposition is proved in [26].

**Proposition 4.1.30.** Let \(f : \mathbb{R} \to \mathbb{R}\) be a homeomorphism without fixed points. Then

(i) If \(f(0) > 0\) then \(f\) is order conjugate \(x + 1\).

(ii) If \(f(0) < 0\) then \(f\) is order conjugate \(x - 1\).

**Proof.** Define \(h : \mathbb{R} \to \mathbb{R}\) as follows. Assume \(f(0) > 0\). Define \(h(t) = \frac{t}{f(0)}\), \(0 \leq t < f(0)\). We know that \((f^n(0))\) increases and diverges to \(\infty\) and \((f^{-n}(0))\) decreases and diverges to \(-\infty\) for all \(n \in \mathbb{N}\). Moreover for \(t \in \mathbb{R}\) there exist unique \(n \in \mathbb{Z}\) such that, \(f^n(0) \leq t < f^{n+1}(0)\). Define \(h(t) = h(f^{-n}(t)) + n\). Then \(h \circ f(t) = h(t) + 1 \ \forall \ t \in \mathbb{R}\).

This \(h\) gives a conjugacy from \(f\) to \(x + 1\).

If \(f(0) < 0\) then we can give a similar proof.
Corollary 4.1.31. Let \( f, g : (a, b) \to (a, b) \) be homeomorphisms without fixed points. Then \( f \) is order conjugate to \( g \) if and only if both \( \text{graph}(f) \) and \( \text{graph}(g) \) are on the same side of the diagonal.

In particular,

(i) If \( f(x) > x \) for all \( x \in (a, b) \) then \( f \) is order conjugate to \( x + 1 \).

(ii) If \( f(x) < x \) for all \( x \in (a, b) \) then \( f \) is order conjugate to \( x - 1 \).

Remark 4.1.32. In fact, in the previous corollary, the interval \((a, b)\) can be replaced by any open ray in \( \mathbb{R} \).

Remark 4.1.33. For an increasing bijection \( f : \mathbb{R} \to \mathbb{R} \) with finitely many nonordinary points, all nonordinary points are fixed points.

Proof. We know that, for maps with finitely many nonordinary points, all nonordinary points are fixed by every order conjugacy. Here \( f \) itself is a self conjugacy.

\[ \square \]

For a continuous map \( f : \mathbb{R} \to \mathbb{R} \), let \( \text{Fix}(f) \) denote the set of all fixed points of \( f \).

It follows from the continuity of \( f \) that \( \text{Fix}(f) \) is closed.

Recall that for any subset \( A \) of a topological space \( X \),

\[ \partial A^c = \text{int}(A) \cup \text{int}(A^c) \tag{4.1} \]

The following theorem gives a characterization for the nonordinary points of increasing homeomorphisms.

Proposition 4.1.34. Let \( f : \mathbb{R} \to \mathbb{R} \) be an increasing bijection and let \( x \in \mathbb{R} \). Then \( x \) is non-ordinary if and only if \( x \) is in the boundary of \( \text{Fix}(f) \).
Proof. Necessary part:

Let \( x \in \partial \text{Fix}(f) \). Then \( x \) is non-ordinary, since every open interval around \( x \) contains fixed and non-fixed points.

Sufficient part:

Suppose \( x \notin \partial \text{Fix}(f) \). We shall prove that \( x \) is ordinary.

Now, \( x \notin \partial \text{Fix}(f) \) implies \( x \in (\partial \text{Fix}(f))^c = \text{int}(\text{Fix}(f)) \cup \text{int}((\text{Fix}(f))^c) \) by equation (4.1). Hence \( x \in \text{int}(\text{Fix}(f)) \) or \( x \in \text{int}(\text{Fix}(f))^c \).

Case: 1

Suppose \( x \in \text{int}(\text{Fix}(f)) \). Then Choose \( a, b \in \mathbb{R} \) such and \( x \in (a,b) \subset \text{Fix}(f) \). Let \( y \in (a,b) \) be such that \( y \neq x \).

Then define \( \phi_y : \mathbb{R} \to \mathbb{R} \) by,

\[
\phi_y(t) = \begin{cases} 
  t & \text{if } t \notin (a,b) \\
  y & \text{if } t = x \\
  \text{extend linearly at other places}
\end{cases}
\]

This \( \phi_y \) is an increasing continuous bijection on \( \mathbb{R} \) which maps \( y \) to \( x \). Both \([a,b]\) and its complement are invariant under both \( \phi_y \) and \( f \). This \( \phi_y \) commutes with \( f \) since on \([a,b]\), \( f \) is identity and on the complement of \([a,b]\), \( \phi_y \) is identity. This proves \( x \) is an ordinary point.

Case: 2

Suppose \( x \in \text{int}(\text{Fix}(f))^c \). Let \((a,b)\) be the component interval(open) of \((\text{Fix}(f))^c\) containing \( x \). Then \( f(a) = a \) and \( f(b) = b \) and since \( f \) is increasing, the map \( f|_{(a,b)} \) is a fixed point free self map of \((a,b)\). Hence by corollary 4.1.31, the map \( f|_{(a,b)} \) is order conjugate to either \( x + 1 \) or \( x - 1 \), for which all points are ordinary. This completes the proof.
4.2 Counting homeomorphisms

Note that, under a topological conjugacy a point can be mapped to a point with similar dynamics. By definition the points of \([x]\) are dynamically same. i.e, all have the dynamics similar to that of \(x\).

We now consider the systems for which there are only finitely many equivalence classes. This means there are only finitely many kinds of orbits upto conjugacy. For this reason we call such systems as simple systems.

In this chapter, we try to understand some simple systems on \(\mathbb{R}\).

Recall that, if \(S_P\) denote the set of all points having the dynamical property \(P\) then the points of \(\partial S_P\) (the boundary of \(S_P\)) are nonordinary. In particular, being a point in a particular equivalence class is a dynamical property of the point. Hence by the very nature of of the order conjugacies, it follows that when there are finitely many nonordinary points(therefore special points) there are only finitely many equivalence classes. These are the simple systems we study in this chapter.

We describe completely, the homeomorphisms on \(\mathbb{R}\), having finitely many nonordinary points and give a general formula for counting.

By remark 4.1.24, for systems with finitely many nonordinary points there are only finitely many equivalence classes. We now study, in the next section, the class of simple systems induced by homeomorphisms having finitely many nonordinary points.
4.2.1 Counting increasing homeomorphisms

Remark 4.2.1. Note that the complement of $\text{Fix}(f)$ is a countable union of open intervals (including rays) whose end points are fixed points. Since $f$ is increasing and the end points are fixed, no point in a component interval can be mapped to a point in any other component interval by $f$.

Hence, it is observed that, for an increasing bijection $f$ on $\mathbb{R}$, if $\text{Fix}(f)^c = \sqcup I_n$ then $f|_{I_n}$ is a self map of $I_n$.

**Proposition 4.2.2.** Let $f, g$ be two increasing bijections such that $\text{Fix}(f) = \text{Fix}(g)$ and let $\text{Fix}(f)^c = \sqcup I_n$. Suppose $f|_{I_n} \sim g|_{I_n}$ for every $n$ then $f \sim g$.

**Proof.** For each $n \in \mathbb{N}$ let $h_n : I_n \to I_n$ be an order conjugacy from $f|_{I_n}$ to $g|_{I_n}$.

Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} h_n(x) & \text{if } x \in I_n \\ x & \text{otherwise.} \end{cases}$$

Then $h$ is an increasing bijection such that $hf = gh$. Hence the proposition. \qed

The above proposition can be generalized as,

**Proposition 4.2.3.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous with $\text{Fix}(f) = \text{Fix}(g)$, let $\text{Fix}(f)^c = \sqcup I_n$ and suppose each $I_n$ is invariant under both $f$ and $g$. Then $f \sim g$ if and only if $f|_{I_n} \sim g|_{I_n}$ for every $n$.

An alphabet is a finite set of letters with at least two elements. A finite sequence of letters from an alphabet is often referred to as a word. For example, if $\Sigma = \{a, b\}$ be an alphabet then $abab, aaabbbab$ are words over $\Sigma$. Number of letters (may not be
distinct) in a word is called its length. Any word of consecutive characters in a word \( w \) is said to be a subword of \( w \).

Throughout this section we will be working with the alphabet \( \{A, B, O\} \). Let \( \tilde{A} = B \), \( \tilde{B} = A \) and \( \tilde{O} = O \). If \( w = w_1w_2...w_n \), then the dual of \( w \) is defined as

\[
\tilde{w} = \tilde{w}_n\tilde{w}_{n-1}...\tilde{w}_1.
\]

If \( \tilde{w} = w \) then \( w \) is said to be self conjugate. Here \( A \) stands for “above the diagonal” and \( B \) stands for “below the diagonal” and \( O \) stands for “on the diagonal”.

Let \( f : \mathbb{R} \to \mathbb{R} \) be an increasing homeomorphism with finitely many nonordinary (hence special points) say, \( x_1 < x_2 < ... < x_n \) for some \( n \in \mathbb{N} \). This finite set of points gives rise to an ordered partition \( \{(-\infty, x_1), (x_1, x_2), ...(x_n, \infty)\} \) of \( \mathbb{R} \setminus \{x_1, x_2, ..., x_n\} \). Note that, On each component interval exactly one of the following holds, by proposition 4.1.34 (Since the only subsets of \( \mathbb{R} \) with empty boundary are the empty set and \( \mathbb{R} \)).

\[
\begin{align*}
(i) \quad f(t) &> t \quad \forall \ t \\
(ii) \quad f(t) &< t \quad \forall \ t \\
(iii) \quad f(t) &= t \quad \forall \ t.
\end{align*}
\]

This gives rise to a word \( w(f) \) over \( \{A, B, O\} \) of length \( n + 1 \) by associating \( A \) to (i), \( B \) to (ii) and \( O \) to (iii).

Next, note that the sub word \( OO \) is forbidden. For, suppose \( O \) is occurring at \( i^{th} \) and \( (i+1)^{th} \) place then in both \((x_i, x_{i+1})\) and \((x_{i+1}, x_{i+2})\) all points are fixed. Then \( x_{i+1} \) becomes ordinary. A contradiction to the assumption that \( x_{i+1} \) is a nonordinary point.

Conversely,

Suppose a word \( w \) of length \( (n + 1) \) (in which \( OO \) is forbidden) is given.

Then we can construct an increasing bijection on \( \mathbb{R} \) such that its associated word is \( w \), as follows:

Take the points \( 0, 1, 2, ..., n - 1 \) and consider the partition \( \{(-\infty, 0), (0, 1), (1, 2), ..., (n-1, \infty)\} \) of \( \mathbb{R} \). If \( w = w_1w_2...w_{n+1} \) then associate \( w_1 \) to \((-\infty, 0)\), \( w_2 \) to \((0, 1)\), \( ..., \) and
$w_{n+1}$ to $(n-1, \infty)$. Now it is easy to construct an increasing bijection $f : \mathbb{R} \to \mathbb{R}$ such that $w(f) = w$. To be precise if $i-1 < t < i$ then

$$f(t) = \begin{cases} 
    i - 1 + (t - i + 1)^2 & \text{if } w_i = B \\
    i - 1 + \sqrt{t - i + 1} & \text{if } w_i = A \\
    t & \text{if } w_i = O 
\end{cases}$$

The increasing bijection so constructed is unique upto order conjugacy. This follows from the following proposition.

**Proposition 4.2.4.** Let $f, g$ be two increasing bijection on $\mathbb{R}$ with finitely many (same number of) nonordinary points. Then $f$ and $g$ are order conjugate if and only if $w(f) = w(g)$.

**Proof.** Suppose $w(f) = w(g) = w_1w_2...w_n$. Let $x_1 < x_2 < ... < x_n$ and $y_1 < y_2 < ... < y_n$ be the non-ordinary points $f$ and $g$ respectively.

The former gives the ordered partition $\{(-\infty, x_1), (x_1, x_2), ..., (x_n, \infty)\}$ of $\mathbb{R} \setminus \{x_1, x_2, ..., x_n\}$ and the later gives the ordered partition $\{(-\infty, y_1), (y_1, y_2), ..., (y_n, \infty)\}$ of $\mathbb{R} \setminus \{y_1, y_2, ..., y_n\}$.

Now, from proposition 4.1.34, it follows that, for each $i$, both $f|_{(x_i, x_{i+1})}$ and $g|_{(y_i, y_{i+1})}$ are fixed point free self maps (homeomorphisms) and hence by corollary 4.1.31, both are order conjugate to $x + 1$ if $w_{i+1} = A$, and order conjugate to $x - 1$ if $w_{i+1} = B$. Hence, by the proposition 4.2.2 $f$ is order conjugate to $g$.

Converse follows from corollary 4.1.31. \qed

Thus we have proved:

**Proposition 4.2.5.** There is a one to one correspondence between the set of all increasing continuous bijections (upto order conjugacy) on $\mathbb{R}$ with exactly $n$ non-ordinary points and the set of all words of length $n+1$ on three symbols $A, B, O$ such that $OO$ is forbidden.
Proposition 4.2.6. Let \( a_n \) be the number of words of length \( n+1 \) over \( \{A,B,O\} \) where OO is forbidden. Then

\[
a_n = C_1 (1 + \sqrt{3})^n + C_2 (1 - \sqrt{3})^n \quad \text{where} \quad C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}} \quad \text{and} \quad C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}}.
\]

Proof. Let \( A_n \) be the set all words of length \( n+1 \) over \( \{A,B,O\} \) in which OO is forbidden. A general element in \( A_{n+2} \) is of the form

(i) \( Aw \) or \( Bw \) for some \( w \in A_{n+1} \)

or

(ii) \( OA_v \) or \( OB_v \) for some \( v \in A_n \).

Therefore \( a_{n+2} = a_{n+1} + a_{n+1} + a_n + a_n \), since \( A_{n+2} \) is the disjoint union of four types of the elements described above. Hence \( a_{n+2} = 2(a_n + a_{n+1}) \). This is a linear homogeneous recurrence relation with constant coefficients. The corresponding characteristic equation is \( \alpha^2 - 2\alpha - 2 = 0 \) which has the two distinct roots \( \alpha_1 = 1 + \sqrt{3} \) and \( \alpha_2 = 1 - \sqrt{3} \). It follows that \( a_n = C_1 (1 + \sqrt{3})^n + C_2 (1 - \sqrt{3})^n \) where the constants \( C_1 \) and \( C_2 \) can be determined by using the boundary conditions \( a_0 = 3 \) and \( a_1 = 8 \).

Here \( C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}} \) and \( C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}} \).

The following is one of the principal theorems of this chapter.

Theorem 4.2.7. The number of all increasing continuous bijections (upto order conjugacy) on \( \mathbb{R} \) with exactly \( n \) non-ordinary points is \( a_n = C_1 (1 + \sqrt{3})^n + C_2 (1 - \sqrt{3})^n \).

Where \( C_1 = \frac{(5+3\sqrt{3})}{2\sqrt{3}} \) and \( C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}} \).

Proof. Follows from propositions 4.2.5 and 4.2.6.

The following proposition is an analogue of proposition 4.2.4.

Proposition 4.2.8. Let \( f, g \) are two increasing bijection on \( \mathbb{R} \) with finitely many (same number of) nonordinary points. Then \( f \) and \( g \) are decreasingly conjugate if and only if \( w(g) = \overline{w(f)} \).
Let \( t_n \) to denote the number of increasing homeomorphisms upto “topological conjugacy”. Then by proposition 4.2.4 and proposition 4.2.8, we get
\[
\begin{align*}
    t_n &= \frac{a_n - \text{Number of self conjugate words of length } n + 1}{2} + \\
         & \quad \text{Number of self conjugate words of length } n + 1.
\end{align*}
\]

**Case: 1**  
When \( n \) is even, say \( n = 2m \). A self conjugate word \( w \) of length \( 2m + 1 \) (OO is forbidden) is of the form
\[
w_1w_2...w_mw_{m+1}w_{m+2}...w_{2m+1}
\]
such that \( w_{m+1} = O \) and \( w_m, w_{m+2} \in \{A, B\} \) such that \( w_m \neq w_{m+2} \). Therefore the number of self conjugate words is \( 2a_{m-2} \).

Hence \( t_{2m} = \frac{a_{2m} + 2a_{m-2}}{2} \) for all \( m \geq 2 \).

**Case: 2**  
When \( n \) is odd, say \( n = 2m + 1 \).

In this case any self conjugate word of length \( 2m + 2 \) (OO is forbidden) is of the form
\[
w_1w_2...w_mw_{m+1}w_{m+2}...w_{2m+2}
\]
such that \( w_{m+1}, w_{m+2} \in \{A, B\} \) and such that \( w_{m+1} \neq w_{m+2} \). Hence the number of self conjugate words of length \( 2m + 2 \) is \( 2a_{m-1} \).

Therefore, \( t_{2m+1} = \frac{a_{2m+1} + 2a_{m-1}}{2} \) for all \( m \geq 1 \).

Thus we have proved:

**Theorem 4.2.9.** If \( t_n \) denotes the number of increasing homeomorphisms upto “topological conjugacy”. Then, for \( n \geq 3 \) we have:
\[
t_n = \begin{cases} 
    \frac{a_n + 2a_{\frac{n}{2}} - 4}{2} & \text{if } n \text{ is even} \\
    \frac{a_n + 2a_{\frac{n-3}{2}}}{2} & \text{if } n \text{ is odd}
\end{cases}
\]

Where \( t_0 = 2, t_1 = 5 \) and \( t_2 = 12 \) by direct computation.
4.2.2 Counting decreasing homeomorphisms

We now ask:

Given a whole number \( n \), how many decreasing bijections are there on \( \mathbb{R} \) upto order conjugacy having exactly \( n \) non-ordinary points?

**Proposition 4.2.10.** Two decreasing bijections \( f \) and \( g \) are order conjugate (res. topologically conjugate) if and only if \( f^2|_{[a,\infty)} \) and \( g^2|_{[b,\infty)} \) are order conjugate (res. topologically conjugate).

Here \( a \) and \( b \) are the fixed points of \( f \) and \( g \) respectively. [Note that every decreasing homeomorphism has a unique fixed point.]

**Proof.** Suppose \( f \) and \( g \) are order conjugate (res. topologically conjugate) then the same conjugacy between \( f \) and \( g \) when restricted, form an order conjugacy (res. topological conjugacy) between \( f^2|_{[a,\infty)} \) and \( g^2|_{[b,\infty)} \).

Conversely, suppose \( f^2|_{[a,\infty)} \) and \( g^2|_{[b,\infty)} \) are increasingly conjugate through the increasing homeomorphism \( h_1 \). Then \( h_1([a,\infty)) = [b,\infty) \) and \( h(a) = b \). Also note that \( f((-\infty,a]) = [a,\infty) \) and \( g((-\infty,b]) = [b,\infty) \). That is \( f^{-1}([a,\infty)) = (-\infty,a] \) and \( g^{-1}([b,\infty)) = (-\infty,b] \).

Define \( h : \mathbb{R} \to \mathbb{R} \) by,

\[
h(x) = \begin{cases} 
  h_1(x) & \text{if } x \in [a,\infty) \\
  g^{-1}h_1f(x) & \text{if } x < a
\end{cases}
\]

Then \( h \) is the required increasing conjugacy from \( f \) to \( g \). For, if \( t < a \), by definition \( h \circ f(t) = g \circ h(t) \).

If \( t > a \) then \( f(t) < a \). Therefore \( h(f(t)) = g^{-1}(h_1(f(f(t)))) = g^{-1}(h_1(f^2(t))) = g^{-1}(g^2(h_1(t))) = g(h_1(t)) = g(h(t)) \). Hence the proof.

\( \square \)
Similarly we can prove:

**Proposition 4.2.11.** Two decreasing bijections $f$ and $g$ are increasingly conjugate (res. topologically conjugate) if and only if $f^2|_{(-\infty,a]}$ and $g^2|_{(-\infty,b]}$ are increasingly conjugate (res. topologically conjugate).

Here $a$ and $b$ are the fixed points of $f$ and $g$ respectively.

**Definition 4.2.12.** A map $f: \mathbb{R} \to \mathbb{R}$ is said to be *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

**Proposition 4.2.13.** Let $f: \mathbb{R} \to \mathbb{R}$ be an increasing bijection which is odd. Then there exists a decreasing homeomorphism $f_r$ such that $f_r^2 = f$. (Such a decreasing homeomorphism we call as a decreasing square root of $f$).

**Proof.** Note that $f(0) = 0$. Define

$$f_r(x) = \begin{cases} -f(x) & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Clearly $f_r$ is a decreasing bijection. Then for $x \geq 0$, we have $f_r(x) \leq 0$, Therefore $f_r(f_r(x)) = -f_r(x) = f(x)$. For $x < 0$, we have $f_r(f_r(x)) = f_r(-x) = -f(-x) = f(x)$.

**Remark 4.2.14.** The hypothesis of the above proposition is not true in general.

Let

$$h(x) = \begin{cases} \frac{x}{2} & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$$

Clearly, $h$ is an increasing bijection from $\mathbb{R}$ to $\mathbb{R}$. There is no decreasing bijection $f: \mathbb{R} \to \mathbb{R}$ such that $f \circ f = h$. Let if possible $f$ be one such function. Then for all $x < 0$ we have $f(f(x) = h(x) = x$. Choose $y > 0$ such that $f(y) < 0$. Therefore
Since $f$ is one-one $f^2(y) = y$. Therefore $h(y) = y$. Which is contradiction since $h(y) = \frac{y}{2}$.

**Proposition 4.2.15.** Let $f : (0, \infty) \to (0, \infty)$ be an increasing bijection. Then there exists a unique decreasing bijection $f_r : \mathbb{R} \to \mathbb{R}$ upto order conjugacy such that $f_r \circ f_r | (0, \infty) = f$.

**Proof.** Let $f : (0, \infty) \to (0, \infty)$ be an increasing bijection. This forces that $f(0) = 0$. Any map $f : (0, \infty) \to (0, \infty)$ can be extended uniquely to an odd function $\tilde{f} : \mathbb{R} \to \mathbb{R}$. Then by proposition 4.2.13, there exist $f_r : \mathbb{R} \to \mathbb{R}$ such that $f_r \circ f_r | (0, \infty) = f$. By proposition 4.2.11, $f$ is unique upto order conjugacy. □

**Proposition 4.2.16.** Let $f : \mathbb{R} \to \mathbb{R}$ be a decreasing bijection. Then all non-ordinary points of $f \circ f$ are non-ordinary points of $f$ and conversely.

**Proof.** Suppose $x$ is an ordinary point for $f$. Then the results follows from the fact that if $h$ commutes with $f$ then it commute with $f \circ f$ also.

Conversely, suppose $x$ is an ordinary point for $f \circ f$. Let the unique fixed point of $f$ to be zero. I.e, $f(x) = 0$ iff $x = 0$ and let $x > 0$. Then there exist a neighbourhood $(x - \delta, x + \delta)$ such that for all $y$ in $(x - \delta, x + \delta)$, there exists $h \in G_{f \circ f}$ such that $h(x) = y$. Then $h|_{(0, \infty)}$ is a topological conjugacy between $f \circ f |_{(0, \infty)}$ and $f \circ f |_{(0, \infty)}$. Then by proposition 4.2.11, $h$ induces $\tilde{h}$, a conjugacy between $f$ and $f$. By the way $\tilde{h}$ is defined, we have $\tilde{h}(x) = h(x) = y$. Therefore, $x$ is an ordinary point of $f$. □

**Proposition 4.2.17.** If $f$ is a decreasing bijection from $\mathbb{R}$ to $\mathbb{R}$ with fixed point $a$. Then $f$ has $2n + 1$ non-ordinary points if and only if $(f \circ f)|_{(a, \infty)} : (a, \infty) \to (a, \infty)$ has $n$ non-ordinary points.
**Chapter 4. A Counting Problem**

**Proof.** Suppose that $f$ has $2n + 1$ non-ordinary points. Let them be $x_1 < x_2 < \ldots < x_n < x_{n+1} < x_{n+2} < \ldots < x_{2n+1}$. Let $N = \{x_1, x_2, \ldots, x_{2n+1}\}$. Then $f(N) \subset N$ by proposition 4.1.17. Since $f$ is a decreasing bijection we have $f(N) = N$ and $a = x_n$. Hence $(f \circ f)\mid_{(a, \infty)} : (a, \infty) \to (a, \infty)$ has $n$ non-ordinary points.

Conversely, Suppose $(f \circ f)\mid_{(a, \infty)} : (a, \infty) \to (a, \infty)$ has $n$ nonordinary points. Then observe that $N(f) = N(f^2\mid_{(a, \infty)}) \cup f(N(f^2\mid_{(a, \infty)})) \cup \{a\}$ (Here we use the previous proposition). Thus, $f$ has $2n + 1$ non-ordinary points.

**Remark 4.2.18.** From the above proposition it follows that there does not exist a decreasing homeomorphism with even number of nonordinary points.

**Theorem 4.2.19.** If $s_n$ denotes the number of decreasing homeomorphisms upto order conjugacy, then

$$s_n = \begin{cases} 
0 & \text{if } n \text{ is even} \\
\frac{a_{n-1}}{2} & \text{if } n \text{ is odd}
\end{cases}$$

for all $n$.

**Proof.** $s_{2n} = 0 \ \forall \ n \in \mathbb{N}$

Follows from proposition 4.2.17.

$s_{2n+1} = a_n \ \forall \ n \in \mathbb{N}$

Let $f : \mathbb{R} \to \mathbb{R}$ be a decreasing bijection with $2n + 1$ nonordinary points. Without loss of generality, we can assume that 0 is the unique fixed point. Then $g = f^2\mid_{(0, \infty)} : (0, \infty) \to (0, \infty)$ is an increasing bijection with $n$ nonordinary points. Since $(0, \infty)$ is homeomorphic to $\mathbb{R}$, we get a increasing homeomorphism $g' : \mathbb{R} \to \mathbb{R}$ (unique upto order conjugacy) with $n$ nonordinary points, which is order conjugate to $g$.

On the other hand, suppose $h : \mathbb{R} \to \mathbb{R}$ is an increasing bijection with $n$ nonordinary points. Since $(0, \infty)$ is homeomorphic to $\mathbb{R}$, corresponding to each $h$ we have a
unique (upto order conjugacy) increasing bijection $h': (0, \infty) \to (0, \infty)$ with $n$ nonordinary points. Then by Proposition 4.2.13 there exist unique (upto order conjugacy) decreasing square root $f: \mathbb{R} \to \mathbb{R}$ for $h'$ upto order conjugacy, such that $f \circ f| (0, \infty) = h'$.

By proposition 4.2.17, $f$ has $2n + 1$ non-ordinary points.

Thus, there is a one-one correspondence between the set of all increasing bijection with $n$ non-ordinary points (upto order conjugacy) and the set of all decreasing bijection with $2n + 1$ non-ordinary points (upto order conjugacy). Hence $s_{2n+1} = a_n$.

**Theorem 4.2.20.** If $k_n$ denotes the number of decreasing homeomorphisms having $n$ nonordinary points upto “topological conjugacy” then

$$k_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ t_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

for all $n$.

**Proof.** Case: 1  When $n$ is even.

We have $k_n = 0$ since $s_n = 0$.

Case: 2  When $n$ is odd.

We will argue as in theorem 4.2.19 to prove that:

There is a one-one correspondence between the set of all increasing bijections (upto topological conjugacy) on $\mathbb{R}$ having $n$ nonordinary points and the set of all decreasing bijections (upto topological conjugacy) on $\mathbb{R}$ with $2n + 1$ nonordinary points.

Thus $k_{2n+1} = t_n$.

We conclude this section with the following table:
4.3 Counting continuous maps

In this section we prove that there are exactly 26 continuous maps on \( \mathbb{R} \) with a unique nonordinary point up to increasing conjugacy.

It follows from proposition 4.1.26(i), that a continuous map on \( \mathbb{R} \) with a single nonordinary point \( a \) must be either constant or injective on each of the intervals \( (-\infty, a) \) and \( (a, \infty) \).
4.3.1 Some basic conjugacy results

Proposition 4.3.1. Let \( f : (-\infty, 0] \rightarrow (-\infty, 0] \) be an increasing bijection (It follows that \( f(0) = 0 \)). Then

1. If \( f(x) > x \) for all \( x \in (-\infty, 0) \) then \( f \) is increasingly conjugate to \( \frac{x}{2} \).
2. If \( f(x) < x \) for all \( x \in (-\infty, 0) \) then \( f \) is increasingly conjugate to \( 2x \).

Proof. Proof of (1):

Let \( f : (-\infty, 0] \rightarrow (-\infty, 0] \) be an increasing bijection satisfying \( f(x) > x \forall x < 0 \). (It follows that \( f(0) = 0 \)). Note that for any such map \( \bigsqcup_{n \in \mathbb{Z}} [f^n(x), f^{n+1}(x)] = (-\infty, 0) \) for all point \( x \in (-\infty, 0] \).

Then \( f \) is topologically conjugate to the map \( x/2 \). We construct a topological conjugacy \( h : (-\infty, 0] \rightarrow (-\infty, 0] \) as follows: Take any point other than 0, say \(-1\) in the domain. We take an arbitrary increasing homeomorphism \( h \) from \([-1, f(-1)) \) to \([-1, -1/2)\). Then as noted above, \( \bigsqcup_{n \in \mathbb{Z}} [f^n(-1), f^{n+1}(-1)] = (-\infty, 0) \). That is, for every \( x \in (-\infty, 0) \), there exists a unique \( n_0 \in \mathbb{Z} \) such that \( f^{n_0}(x) \in [-1, f(-1)) \). We define \( h(x) = 2^{n_0}h(f^{n_0}(x)) \). This is well defined. It is an increasing homeomorphism from \((-\infty, 0)\) to \((-\infty, 0)\). This \( h \) commutes with \( f \). This \( h \) is a conjugacy from \( f \) to the map \( x/2 \).

Similarly, we can prove (2).

Proposition 4.3.2. Let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be such that

1. \( f(0) = g(0) = 0 \).
2. \( f|_{(0, \infty)}, g|_{(0, \infty)} : (0, \infty) \rightarrow (0, \infty) \) are increasing bijections.
3. \( f|_{(-\infty, 0)}, g|_{(-\infty, 0)} : (-\infty, 0) \rightarrow (0, \infty) \) are decreasing bijections.

Then \( f \) is order conjugate to \( g \) if and only if \( f|_{(0, \infty)} \) is order conjugate to \( g|_{(0, \infty)} \).
CHAPTER 4. A COUNTING PROBLEM

Proof. Suppose $h : (0, \infty) \to (0, \infty)$ is an order conjugacy from $f|_{(0,\infty)}$ to $g|_{(0,\infty)}$.

For $x < 0$, define $h(x) = (g|_{(-\infty,0)})^{-1}hf(x)$.

Remark 4.3.3. The above proposition still holds (with identical proof) if the hypothesis (2) is generalized to “$[0, \infty)$ is invariant under both $f$ and $g$.”

Proposition 4.3.4. Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that

1. $f(0) = g(0) = 0$.
2. $f|_{(-\infty,0)}, g|_{(-\infty,0)} : (-\infty, 0) \to (-\infty, 0)$ are increasing bijections.
3. $f|_{(0,\infty)}, g|_{(0,\infty)} : (0, \infty) \to (-\infty, 0)$ are decreasing bijections.

Then $f$ is order conjugate to $g$ if and only if $f|_{(-\infty,0)}$ is order conjugate to $g|_{(\infty,0)}$.

Proof. Suppose $h : (-\infty, 0) \to (-\infty, 0)$ is an order conjugacy from $f|_{(-\infty,0)}$ to $g|_{(-\infty,0)}$.

For $x > 0$, define $h(x) = (g|_{(0,\infty)})^{-1}hf(x)$.

Remark 4.3.5. The above proposition still holds (with identical proof) if the hypothesis (2) is generalized to “$(-\infty, 0]$ is invariant under both $f$ and $g$.”

4.3.2 Somewhere constant maps

A map $f : \mathbb{R} \to \mathbb{R}$ is said to be somewhere constant if it is constant in an open interval, and we say $f$ is locally constant at $a \in \mathbb{R}$ if it is constant in some open interval containing $a$.

Theorem 4.3.6. There are exactly 9 somewhere constant maps with a unique non-ordinary point, up to order conjugacy.

Proof. From corollary 4.1.20 it follows that $f$ must be constant on a ray.

Case:1
Suppose \( f \) is constant on a right ray. Without loss of generality we can assume that \( f \) is constant on \([0, \infty)\). [For, Suppose \( f \) is constant on \([a, \infty)\), then consider \( g(t) = f(a + t) - a \) for \( t \geq 0 \). Then \( g(t) = 0 \) iff \( t \geq 0 \) and \( f \sim g \) increasingly.]

From proposition 4.1.17, it follows that \( f(0) = 0 \).

Then by proposition 4.3.1 and by proposition 4.3.2 (means that the same formula used in the proof will work) it is clear that there are exactly five maps (including the zero map) up to increasing conjugacy whose interval of constancy is \([0, \infty)\).

They are,

\[
\begin{align*}
  f_1(x) &= \begin{cases} 
  0 & \text{if } x \geq 0 \\
  -x & \text{if } x < 0
  \end{cases} \quad f_2 \equiv 0 \\
  f_3(x) &= \begin{cases} 
  0 & \text{if } x \geq 0 \\
  \frac{x}{2} & \text{if } x < 0
  \end{cases} \\
  f_4(x) &= \begin{cases} 
  0 & \text{if } x \geq 0 \\
  x & \text{if } x < 0
  \end{cases} \\
  f_5(x) &= \begin{cases} 
  0 & \text{if } x \geq 0 \\
  2x & \text{if } x < 0
  \end{cases}
\end{align*}
\]

Case:2

Suppose \( f \) is constant on a left ray. This is dual to case:1. Hence there are four such maps.

\[
\begin{align*}
  f_6(x) &= \begin{cases} 
  0 & \text{if } x \leq 0 \\
  -x & \text{if } x > 0
  \end{cases} \quad f_7(x) = \begin{cases} 
  0 & \text{if } x \leq 0 \\
  \frac{x}{2} & \text{if } x > 0
  \end{cases} \\
  f_8(x) &= \begin{cases} 
  0 & \text{if } x \leq 0 \\
  x & \text{if } x > 0
  \end{cases} \\
  f_9(x) &= \begin{cases} 
  0 & \text{if } x \leq 0 \\
  2x & \text{if } x > 0
  \end{cases}
\end{align*}
\]

4.3.3 Nowhere constant maps

**Proposition 4.3.7.** Let \( f : \mathbb{R} \to \mathbb{R} \) be strictly monotone with a unique nonordinary point. Then \( f \) must be onto.

**Proof.** Let \( f \) be increasing. Then \( f \) is an increasing homeomorphism.
CHAPTER 4. A COUNTING PROBLEM

Suppose $f(\mathbb{R})$ is a bounded interval then both end points are nonordinary, which is a contradiction.

Suppose $f(\mathbb{R})$ is a ray say, $(a, \infty)$. Then $a$ is nonordinary and since it is the only nonordinary point we have $f(a) = a$. Since $f$ is increasing we have $f([a, \infty) = [a, \infty))$. Hence if $x < a$ then $f(x) \in [a, \infty)$. Thus $f$ fails to be one-one.

Proof is similar, if $f$ is decreasing.

\[\square\]

**Theorem 4.3.8.** There are exactly 17 nowhere constant maps with a unique non-ordinary point upto order conjugacy.

**Proof.** Suppose $f$ is nowhere constant with a unique nonordinary point. Then from proposition 4.1.26 it follows that $f$ can have at most one critical point.

**Case: 1** Suppose $f$ is a map without a critical point. Then $f$ is strictly monotone.

**Case: 1(a)**

Suppose $f$ is strictly increasing. Then $f$ must be a homeomorphism. Hence there are exactly $a_1 = 8$ such maps, by the word argument.

\[
f_{10}(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases} 
\]

\[
f_{11}(x) = \begin{cases} \frac{x}{2} & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases} 
\]

\[
f_{12}(x) = \begin{cases} x & \text{if } x \leq 0 \\ \frac{x}{2} & \text{if } x > 0 \end{cases} 
\]

\[
f_{13}(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ \frac{x}{2} & \text{if } x > 0 \end{cases} 
\]

\[
f_{14}(x) = \frac{x}{2} \text{ for all } x 
\]

\[
f_{15}(x) = \begin{cases} x & \text{if } x \leq 0 \\ 2x & \text{if } x > 0 \end{cases} 
\]

\[
f_{16}(x) = 2x \text{ for all } x 
\]

\[
f_{17}(x) = \begin{cases} \frac{x}{2} & \text{if } x \leq 0 \\ 2x & \text{if } x > 0 \end{cases} 
\]

**Case: 1(b)**

Suppose $f$ is decreasing. Since $f$ has unique nonordinary point, $f$ must be onto.
and hence it is a homeomorphism. Hence there are $s_1 = a_0 = 3$ such maps upto order conjugacy.

\[ f_{18}(x) = -x \text{ for all } x \]
\[ f_{19}(x) = \begin{cases} 
-2x & \text{if } x \leq 0 \\
-x & \text{if } x > 0
\end{cases} \quad f_{20}(x) = \begin{cases} 
-x & \text{if } x \leq 0 \\
-x & \text{if } x > 0
\end{cases} \]

**Case: 2**

Suppose the unique nonordinary point is the critical point. Let the unique nonordinary point be zero. Now the unique nonordinary(critical) point is either a point of local maximum or a point of local minimum.

**Case: 2(a)**

Suppose 0 is a point of local minimum. Then $f$ must be decreasing from $(-\infty, 0)$ on to $(0, \infty)$ and increasing from $(0, \infty)$ on to $(0, \infty)$ (since sup $f$ and inf $f$ are nonordinary points).

Then it follows from proposition 4.3.2 that there are exactly three such maps upto order conjugacy.

\[ f_{21}(x) = |x| \text{ for all } x \]
\[ f_{22}(x) = |2x| \text{ for all } x \]
\[ f_{23}(x) = \frac{|x|}{2} \text{ for all } x \]

**Case: 2(b)**

Suppose 0 is a point of local maximum(infact, global maximum). Then there are exactly three such maps by a proposition 4.3.4.

\[ f_{24}(x) = \begin{cases} 
-x & \text{if } x \leq 0 \\
-x & \text{if } x > 0
\end{cases} \quad f_{25}(x) = \begin{cases} 
-x & \text{if } x \leq 0 \\
-x & \text{if } x > 0
\end{cases} \]

\[ f_{26}(x) = \begin{cases} 
2x & \text{if } x \leq 0 \\
-x & \text{if } x > 0
\end{cases} \]

Thus there are 17 such maps.
CHAPTER 4. A COUNTING PROBLEM

4.4 Main theorem

Theorem 4.4.1. There are exactly 26 maps on $\mathbb{R}$ with a unique non-ordinary point, upto order conjugacy.

Proof. Proof follows from Theorems 4.3.6 and 4.3.8.
Figure 4.1: Maps with one nonordinary point
Figure 4.2: Maps with one nonordinary point
Figure 4.3: Maps with one nonordinary point