Chapter 3

The set of periods of toral automorphisms

3.1 Examples and Motivation

Let (X, f) be a dynamical system. Let $x \in X$. Recall that x is a periodic point if $f^n(x) = x$ for some $n \in \mathbb{N}$ and that least such n is called the period of x.

We denote by $Per(f)$, the set of all periods of periodic points in the dynamical system (X, f). We now ask:

Given X, which subsets of $\mathbb{N}$ arise as $Per(f)$ for some continuous self map f from X to X ?

We start with a simple example. For the class of homeomorphisms on $\mathbb{R}$ the set of periods is one among the following subsets of $\mathbb{N}$.

1. The empty set.
2. $\{1\}$.
3. $\{1, 2\}$.

This follows from the following facts:

1. For increasing maps on $\mathbb{R}$ all periodic points are fixed points.

2. For any homeomorphism f on $\mathbb{R}$, the map f^2 is increasing.

For the class of continuous maps on $\mathbb{R}$, we have:

3.1.1 Sharkovskii's Theorem

Among the uncountably many subsets of $\mathbb{N}$, there are only countably many that can arise as $Per(f)$ for some continuous self map of $[0,1]$. These sets form a chain (any two members are comparable) and can be described through a total order on $\mathbb{N}$ different from the usual. This is a celebrated theorem of Sharkovskii.

Definition 3.1.1. The following total order of $\mathbb{N}$ is called the Sharkovskii's ordering:

$$3 \succ 5 \succ 7 \succ 9 \succ ...$$

$$\succ 2 \times 3 \succ 2 \times 5 \succ 2 \times 7 \succ ...$$

$$\vdots$$

$$\succ 2^n \times 3 \succ 2^n \times 5 \succ 2^n \times 7 \succ ...$$

$$\vdots$$

$$.............2^n \succ .... \succ 2^2 \succ 1$$

Theorem 3.1.2. ***(Sharkovskii)*** *[17]*

Let $m \succ n$ in the Sharkovskii's ordering. For every continuous self-map of $\mathbb{R}$, if there is an $m-cycle$, then there is an $n-cycle$.

A converse of Sharkovskii's theorem : [18]

Let m and n be distinct positive integers. Let m not precede n in the above ordering. Then there is a continuous map f from $\mathbb{R}$ to $\mathbb{R}$, where there is an $m - cycle$ but no $n - cycles$.

A combined Statement:

$m \succeq n$ in the Sharkovskii's ordering if and only if for every continuous self-map of $\mathbb{R}$, the existence of an $m - cycle$ forces that of an $n - cycle$.

This can be restated as,

Theorem 3.1.3. *Let $A \subset \mathbb{N}$. Then $A = Per(f)$ for some continuous map $f : \mathbb{R} \to \mathbb{R}$ if and only if*

$m \in A$ and $m \succ n$ imply $n \in A$.

Remark 3.1.4. This theorem is special to some spaces like the real line. Its analogues for other spaces are not valid. For example, on the unit circle S^1, the rotation by 120 degrees admits a $3 - cycle$, but admits cycles of no other length.

It seems that the sharkovskii-ordering is forced by nature. There are spaces other than $\mathbb{R}$ where an analogous theorem is true.

3.1.2 Baker's Theorem

We reformulate a theorem of Baker as follows:

Theorem 3.1.5. *[5] Let p be a complex polynomial. Then the set of periods of p has to be one of the following five kinds of subsets of* $\mathbb{N}$:

1. *The whole set* $\mathbb{N}$
2. $\mathbb{N} \setminus \{2\}$
3. $\{1, n\}$, *where* $n \in \mathbb{N} \setminus \{1\}$

4. $\{1\}$

5. Empty set.

Moreover, the following hold:

(a) *Any polynomial* p *such that* $Per(p) = \mathbb{N} \setminus \{2\}$ *has to be conjugate to* $z^2 - z$*.*

(b) *For all polynomials* p *of degree* ≥ 2*,* $Per(p) \supset \mathbb{N} \setminus \{2\}$*.*

The following table gives the number of complex polynomials upto topological conjugacy, having the given subset of $\mathbb{N}$ as its period set.

Subset of $\mathbb{N}$	**Polynomials for which it is the period set**	**Number of conjugacy classes**
Empty set	Nontrivial translations $z + c$ where $c \neq 0$	1
$\{1\}$	Polynomials $cz + d$ satisfying $d \neq 0$; also identity z; also polynomials cz where c is not a root of unity.	Uncountably many
$\{1, n\}$, where $n \in \mathbb{N} \setminus \{1\}$.	Polynomials cz where c is a nontrivial nth root of unity.	$\phi(n)$
$\mathbb{N} \setminus \{2\}$	Polynomials of the form $az^2 + (2ab - 1)z + b^2$, where $a \neq 0$	1
$\mathbb{N}$	All other polynomials. (not in the previous rows)	Uncountably many
All other subsets not listed in the previous rows	No polynomial.	0

Incidentally, we obtain a dynamical characterization of the polynomial $z^2 - z$. Upto topological conjugacy, it is the only polynomial whose set of periods is $\mathbb{N} \setminus \{2\}$.

The chart below helps us to contrast four situations:

Complex polynomials	Exactly 5 subsets of $\mathbb{N}$
Real continuous maps	A countably infinite family $\mathfrak{F}$ of subsets of $\mathbb{N}$
Complex continuous maps	All subsets of $\mathbb{N}$
Real polynomials	An infinite proper subfamily of $\mathfrak{F}$

Here $\mathfrak{F}$ is as described in theorem 3.1.3.

The third row means: Given any subset A of $\mathbb{N}$, there is a continuous map f from $\mathbb{C}$ to $\mathbb{C}$ such that $\text{Per}(f) = \mathbb{C}$.

The fourth row implies: There is a subset of $\mathbb{N}$, occurring as $\text{Per}(f)$ for a continuous self map of $\mathbb{R}$, but not as $\text{Per}(p)$ for a real polynomial. Explicitly, $\{2^k : k \in \mathbb{N}_0\}$ is one such set.

3.1.3 Period sets of Unit circle S^1 in the plane

The theorem of Sharkovskii specifies, for continuous maps of an interval, which sets of positive integers may occur as the sets of periods. Results along these lines have also been obtained for maps of the circle.

In 1980, L.S Block [8] proved the following interesting results on continuous self maps of the unit circle S^1. By $f \in C(S^1, S^1)$ we mean f is a continuous self map from S^1 to S^1.

Theorem 3.1.6. *Let $f \in C(S^1, S^1)$. Suppose $1 \in Per(f)$ and $n \in Per(f)$ for some odd integer $n > 1$, then for every integer $m > n$, $m \in Per(f)$.*

Theorem 3.1.7. *Let $f \in C(S^1, S^1)$ and suppose that $Per(f)$ is finite. Then there are integers m and n (with $m \geq 1$ and $n \geq 0$) such that*

$Per(f) = \{m, 2.m, 2^2.m, ..., 2^n.m\}$

Theorem 3.1.8. *Let $f \in C(S^1, S^1)$. If $\{1, 2, 3\} \subset Per(f)$ then $Per(f) = \mathbb{N}$. Conversely, if $S \subset \mathbb{N}$ with the property that for any $f \in C(S^1, S^1)$, $S \subset Per(f) \Rightarrow Per(f) = \mathbb{N}$ then $\{1, 2, 3\} \subset S$.*

Description of $Per(f)$ when f has fixed point:

Again in 1981, Block [7] proved the following main results about $Per(f)$ for $f \in C(S^1, S^1)$ when f has a fixed point.

Theorem 3.1.9. *Let $f \in C(S^1, S^1)$. Suppose $1 \in Per(f)$ and $n \in Per(f)$ for some integer $n > 1$. Then (atleast) one of the following holds:*

(i) For every integer m with $n < m$, $m \in Per(f)$

(ii) For every integer m with $n \succ m$, $m \in Per(f)$

(here $<$ denotes the usual order on $\mathbb{N}$).

He has also proved the converse of the above theorem. i.e,

Theorem 3.1.10. *Let $S \subset \mathbb{N}$ with $1 \in S$. Suppose that for each $n \in S$ with $n \neq 1$, atleast one of the following holds.*

(a) for every integer m with $n < m$, $m \in S$.

(b) for every integer m with $n \succ m$, $m \in S$

Then there exists a continuous map $f \in C(S^1, S^1)$ such that the set of periods of periodic points of f is exactly S.

These two theorems of Block characterize the sets of periods which can occur for a continuous map of the circle to itself having a fixed point. But not every self map of the circle has fixed point. One can see that if the $deg(f) \neq 1$ (where $f \in C(S^1, S^1)$)

then f has a fixed point. Partly for this reason, degree-one maps of the circle require special attention to study the periodic orbits. See [6] for more details.

$Per(f)$ **when degree of** $f \neq 1$**:**

In 1982, M. Misiurewicz [9] proved the result, which describes the possible sets of periods of the periodic points of a continuous degree one map (See [6]) of the circle. For any two real numbers a and b, let $M(a,b) = \{n \in \mathbb{N} : a < \frac{t}{n} < b$ for some integer $t\}$.

Definition 3.1.11. If $a \in \mathbb{R}$ and $l \in \mathbb{N} \cup \{2^\infty\}$, (Think 2^∞ as a symbol) we define a subset $S(a,l) \subset \mathbb{N}$ as follows:

If a is irrational then $S(a,l) = \emptyset$

If a is rational and if $a = \frac{t}{n}, n \in \mathbb{N}, t \in \mathbb{Z}, (t,n) = 1$ and if $l \in \mathbb{N}$ then $S(a,l)$ denotes the set of positive integers of the form ns, where $l \succ s$(in Sarkovskii ordering).

If $l = 2^\infty$ then $S(a,l)$ denotes the set of all postive integers of the form ns, where s is a power of 2.

Now we state Misiurewicz's result about continuous maps of circle.

Theorem 3.1.12. *Let f be a continuous map of the circle to itself of degree one. Then there exist $a, b \in \mathbb{R}$ with $a \leq b$ and $l, r \in \mathbb{N} \cup \{2^\infty\}$ such that $Per(f) = M(a,b) \cup S(a,l) \cup S(b,r)$.*

Conversely, for every subset A of $\mathbb{N}$ of the form $A = M(a,b) \cup S(a,l) \cup S(b,r)$ there is a continuous map of the circle to itself of degree one such that $Per(f) = A$

Remark 3.1.13. Hence we have a complete answer for describing the sets of periods for continuous self maps on circle, because

$PER(S^1) = \{Per(f) | f$ is a continuous self map on S^1 with deg $1\} \bigcup \{Per(g) | g$ is a continuous self map on S^1 with a fixed point $\}$.

3.1.4 Period sets of the Y-space

In 1989, L.Alseda, J. Llibre and M. Misiurewicz [1] made a generalization of Sarkovskii's theorem to characterize the possible sets of periods for continuous maps f of the space $Y = \{z \in \mathbb{C}|\ z^3 \in [0,1]\}$ into itself for which $f(0) = 0$.

Hereafter $C(Y)$ denote the set of all continuous self maps on Y.

The following theorem was proved by Mumbru in [12] for $f \in C(Y)$.

Theorem 3.1.14. *(a) If $f \in C(Y)$ and $\{2,3,4,5,7\} \subset Per(f)$ then $Per(f) = \mathbb{N}$.*

(b) If $W \subset \mathbb{N}$ is a set such that for every $f \in C(Y)$, $W \subset Per(f)$ implies $Per(f) = \mathbb{N}$, then $W \subset \{2,3,4,5,7\}$.

To describe the result of Alseda et al in [1], we need to introduce the following notations and two new orderings.

$S(k) = \{n \in \mathbb{N} : k \succ n\} \cup \{k\}$ for all $k \in \mathbb{N}$.

(here $\succ$ is the Sharkovskii order.)

$S(2^\infty) = \{2^i|\ i = 0,1,2,\ldots\}$

Definition 3.1.15. [1]

Green ordering is the ordering of $\mathbb{N} \setminus \{2\}$, denoted by $<_g$, and defined as follows:

$5 <_g 8 <_g 4 <_g 11 <_g 14 <_g 7 <_g 17 <_g 20 <_g 10 <_g 23 <_g 26 <_g 13 <_g \ldots <_g 3.3 <_g 3.5 <_g 3.7 <_g \ldots 3.2.3 <_g 3.2.5 <_g 3.2.7 <_g \ldots <_g 3.2^2.3 <_g 3.2^2.5 <_g 3.2^2.7 <_g \ldots <_g 3.2^3 <_g 3.2^2 <_g 3.2 <_g 3.1 <_g 1.$

The first part of the ordering can be understood as

6-1,6+2, 3+1, 2.6-1, 2.6+2, 2.3+1, 3.6-1, 3.6+2, 3.3+1,. . . .

Definition 3.1.16. [1]

Red ordering is the ordering of $\mathbb{N} \setminus \{2,4\}$, denoted by $<_r$ and defined as :

$7 <_r 10 <_r 5 <_r 13 <_r 16 <_r 8 <_r 19 <_r 22 <_r 11 <_r 25 <_r 28 <_r 14 <_r \ldots <_r 3.3 <_r 3.5 <_r 3.7 <_r \ldots\ldots <_r 3.2.3 <_r 3.2.5 <_r 3.2.7 <_r \ldots <_r 3.2^2.3 <_r 3.2^2.5 <_r 3.2^2.7 <_r \ldots <_r \ldots 3.2^3 <_r 3.2^2 <_r 3.2 <_r 3.1 <_r 1.$

Here the first part can be viewed as,

6-1, 6+4, 3+2, 2.6+1, 2.6+4, 2.3+2, 3.6+1, 3.6+4, 3.3+2,. . . .

Notation 3.1.17. We denote $G(n) = \{n\} \cup \{k : n <_g k\}$ for all $n \in \mathbb{N} \setminus \{2\}$

$R(n) = \{n\} \cup \{k : n <_r k\}$ for all $n \in \mathbb{N} \setminus \{2, 4\}$ and

$G(3.2^\infty) = R(3.2^\infty) = \{3.i : i \in S(2^\infty)\}$.

We also denote

$$N_s = \mathbb{N} \cup \{2^\infty\}$$

$$N_g = (\mathbb{N} \setminus \{2\}) \cup \{3.2^\infty\}$$

$$N_r = (\mathbb{N} \setminus \{2, 4\}) \cup \{3.2^\infty\}$$

Now we are ready to state the main result of Alseda, Llibre and Misiurewicz on sets of periods of Y.

Theorem 3.1.18. *(a) If $f \in C_0(Y)$ (i.e., f is a continuous self map on Y with $f(0) = 0$) then $Per(f) = S(n_s) \cup G(n_g) \cup R(n_r)$ for some $n_s \in N_s$, $n_g \in N_g$ and $n_r \in N_r$.*

If $n_s \in N_s$, $n_g \in N_g$ and $n_r \in N_r$ then there exists a continuous self map f on $C_0(Y)$ for which

$$Per(f) = S(n_s) \cup G(n_g) \cup R(n_r).$$

In [1] at the end they posed an open question to describe possible sets of periods of continuous self maps f on the space $X_k = \{z \in \mathbb{C} |\ z^k \in [0, 1]\}$ for $k > 3$, for which zero is a fixed point.

This question was taken up by S. Baldwin, and he generalized the theorem of [1]. He described the complete solution for the sets of periods of self maps on n-od with 0 as fixed point.

3.1.5 Sets of periods of n-od

The n-od, denoted by X_n is the subspace of the complex plane, which is described as $X_n = \{z \in \mathbb{C} |\ z^n \in [0,1]\}$ [Note that 1-od and 2-od are homeomorphic]. This can be viewed as the set obtained by attaching n copies of unit interval to the central point(i.e, at 0.)

Some definitions and notations:

In order to study the structure of sets of periods of continuous maps $f : X_n \to X_n$ we need to define partial ordering $\leq_t$ for all positive integers t.

The ordering $\leq_1$ is defined by

$$2^i \leq_1 2^{i+1} \leq_1 2^{j+1}(2m+3) \leq_1 2^{j+1}(2m+1) \leq_1 2^j(2k+3) \leq_1 2^j(2k+1)$$

for all integers $i, j \geq 0$ and $k, m \geq 0$.

In other words $\leq_1$ is the usual Sharkovskii ordering.

If $n > 1$ then the ordering $\leq_n$ is defined as follows:

Let m, k be positive integers.

Case: 1

$k = 1$ then $m \leq_n k$ if and only if $m = 1$.

Case: 2

k is divisible by n then $m \leq_n k$ if and only if either $m = 1$ or m is divisible by n and $(\frac{m}{n} \leq_n \frac{k}{n})$.

Case: 3

$k > 1$, k is not divisible by n. Then $m \leq_n k$ if and only if either $m = 1, m = k$ or $m = ik + jn$ for some integers $i \geq 0,\ j \geq 1$.

In [3], some diagrams illuminating these partial orderings are given. From the definition we can see that $\leq_1$ and $\leq_2$ coincide with the Sharkovskii ordering. If $n > 2$, then $\leq_n$ is not a linear ordering. Define a set $S \subset \mathbb{N}$ to be an initial segment of $\leq_n$ if whenever k is an element of S and $m \leq_n k$ then m is also an element of S, i.e, S is closed under $\leq_n$ predecessors. Now we state the theorem of Baldwin in [3].

Theorem 3.1.19. *Let X be an n-od.*

1. Let $f : X_n \to X_n$ be a continuous map. Then $Per(f)$ is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq n\}$.

2. Conversely, if S is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq n\}$ then there is a continuous map $f : X_n \to X_n$ such that $f(0) = 0$ and $Per(f) = S$.

Remark 3.1.20. The n-od result is the same, regardless of whether the branching point is required to be fixed or not.

3.1.6 Period sets for Tree maps

By a tree in the plane we mean a connected graph without any cycles. A tree with its relative topology (treating as a subset of $\mathbb{R}^2$) in $\mathbb{R}^2$ is called a tree space.

The main result of [4] is the extension of n-od theorem to every continuous self map on a tree T having all branching points fixed. It is of interest to ask what $Per(f)$ can be if all branching points of T are fixed. The result on n-od has been extended to all trees(without assumptions on the branching points) by Baldwin, but these results do not specify which sets are possible if the branching points are remaining fixed. For similar results on graphs which characterize sets of periods without specifying which sets of periods correspond to which graphs, See [10]. Now we state the main result of [4].

Given a tree T, let $e(T)$ and $b(T)$ be the number of end points and branching points respectively.

Theorem 3.1.21. *Let T be a tree.*

(a) Let $f : T \to T$ be a continuous map with all branching points fixed. Then $Per(f)$ is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq e(T)\}$.

(b) Conversely, if S is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq e(T)\}$ then there is a continuous map $f : T \to T$ with all branching points fixed such that $Per(f) = S$.

This theorem solves a problem which was originally posed by Alseda et al in [1].

3.1.7 Saradhi's result

Theorem 3.1.22. *[27] Let X be a noncompact convex subset of $\mathbb{R}^2$ with nonempty interior. Then $PER(X) = \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$.*

Theorem 3.1.23. *[27]*

Let S be a closed disc. Then $PER(S) = \{$ all subsets of $\mathbb{N}$ containing 1 $\}$.

We conclude our survey here and we now prove our main theorem of this chapter.

3.2 Period sets of hyperbolic toral automorphisms

Proposition 3.2.1. *[21]*

For $2-$dimensional hyperbolic toral automorphisms, the eigen values are real and irrational.

Proof. Let A be the matrix of a toral automorphism and $(x, y)(\neq (0, 0))$ be an eigenvector of A. If λ is the corresponding eigenvalue, we have

$$ax + by = \lambda x \quad .....(1)$$

$$cx + dy = \lambda y \quad .....(2)$$

First, $x \neq 0$; for if $x = 0$, then $b = 0$ from (1). $Det(A) = \pm 1$ and $b = 0$ implies $ad = \pm 1$. Therefore $d = \pm 1$ and hence from (2) we obtain $\lambda = \pm 1$ (since $y \neq 0$), contradiction to the hyperbolicity.

Next, letting $t = \frac{y}{x}$ and using it to eliminate x and y from (1) and (2), we get $a + bt = \lambda$ and $c + dt = \lambda$.

Now eliminating λ, we have a quadratic in t, namely $bt^2 + (a - d)t - c = 0$. The discriminant of this quadratic equation is $(a + d)^2 \pm 4$, never a perfect square unless $a+d = 0$. This is because the Diophantine equation $X^2 - Y^2 = 4$ has no integer solution except when $Y = 0$. But $a + d = 0$ implies $Ch(A) = x^2 \pm 1$. This not possible since A is hyperbolic. Hence from equation (2) it follows that eigen values are irrational.

□

Let p_n denote the number of solutions(in $\mathbb{T}^2$) of the equation $A^n X = X$ and let $q_n = |Trace(A^n)|$. Then it follows from the proposition 2.1.6 that $p_n = |Det(A^n - I)|$. It is observed in the following lemma that this sequence (q_n) follows a nice pattern for hyperbolic automorphisms.

Proposition 3.2.2. *Let T_A be a hyperbolic toral automorphism induced by the matrix $A \in GL(2, \mathbb{Z})$. Let t denote the trace of A and let $q_n = |Tr(A^n)|$. Then*

$q_{n+1} + Det(A)q_{n-1} = |t|q_n$ for all $n \geq 2$.

Proof. By Cayley Hamilton Theorem we have $A^2 - tA + (Det(A))I = 0$.

Multiplying by A^{n-1} we get

$A^{n+1} + (Det(A))A^{n-1} = tA^n$

The result follows immediately by taking the trace both sides.

Here we use the following facts :

(i) If the $Trace(A) \geq 0$ then $Trace(A^n) \geq 0$ for all $n \in \mathbb{N}$.

(i) If the $Trace(A) < 0$ then $Trace(A^n) < 0$ for all odd numbers $n \in \mathbb{N}$ and $Trace(A^n) > 0$ for all even numbers $n \in \mathbb{N}$.

□

Lemma 3.2.3. For $n \geq 3$, $n^2 - 2 > 2n$.

Proof. Proof follows by induction on n. □

Theorem 3.2.4. *For any hyperbolic toral automorphism $T_A : \mathbb{T}^2 \to \mathbb{T}^2$, $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.*

Proof. Let $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ be hyperbolic. Note that, to prove $n \in Per(T_A)$ it is enough to prove that $p_{n+1} > p_1 + p_2 + ... + p_n$, where p_k = the number of solutions of the equation $A^kX = X$ which is equal to $|Det(A^k - I)|$, by proposition 2.1.6.

In fact, it is enough to prove: $p_{n+1} > p_1 + p_2 + ... + p_{n-1}$, because if x is a point of period n for T_A and $T_A^m(x) = x$ then n must divide m.

Let α be an eigen value of A with $|\alpha| > 1$. By proposition 3.2.1, $\alpha \in \mathbb{R}$ and hence either $\alpha < -1$ or $\alpha > 1$.

Case(1) : $\alpha > 1$

Subcase(1) : $Det(A) = 1$

Then the eigen values are $\alpha, \frac{1}{\alpha}$. Then $t = \alpha + \frac{1}{\alpha} \geq 3$ as t is an integer and $(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}})^2 > 0$. By proposition 2.1.6, $p_n = q_n - 2 \; \forall \; n \in \mathbb{N}$. From proposition 3.2.2, we have $q_{n+1} + q_{n-1} = tq_n \; \forall \; n \geq 2$.

Adding these equations for $n = 2, 3, \ldots, k$ we get,

$$q_1 + q_2 + 2(q_3 + q_4 + \ldots + q_{k-1}) + q_k + q_{k+1} = t(q_2 + q_3 + \ldots + q_k)$$

i.e.,

$$q_{k+1} = -q_1 + (t-1)q_2 + (t-2)[q_3 + q_4 + \ldots + q_{k-1}] + (t-1)q_k$$

$$\geq -q_1 + 2q_2 + q_3 + \ldots + q_{k-1} + 2q_k, \text{ since } t \geq 3.$$

$$= (-q_1 + 2q_2) + q_3 + \ldots + q_{k-1} + 2q_k$$

$> q_1 + q_2 + \ldots + q_{k-1} + q_k$ (since $t \geq 3$, by lemma-3.2.3 we have $t^2 - 2 > 2t$. Hence $q_2 = t^2 - 2 > 2t = 2q_1$ and therefore $q_2 - q_1 > q_1$.)

Therefore, $q_{k+1} - 2 > (q_1 - 2) + (q_2 - 2) + \ldots + (q_{k-1} - 2) + (q_k - 2)$

i.e, $p_{k+1} > p_1 + p_2 + ... + p_k$ for all $k \geq 2$.

Hence $k \in Per(T_A)$ for all $k \geq 3$ and hence $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

Subcase 2 : $Det(A) = -1$

Then eigen values are $\alpha, \frac{1}{\alpha}$. Then $t = \alpha - \frac{1}{\alpha} > 0$ and hence $t \geq 1$, as t is an integer. Using proposition 2.1.6,

$$p_n = \begin{cases} q_n - 2 & \text{if } n \textit{ is even} \\ q_n & \text{if } n \textit{ is odd} \end{cases}$$

From proposition 3.2.2, we have $q_{n+1} - q_{n-1} = tq_n$ for all $n \geq 2$.

Adding these relations for $n = 2, 3, ..., k$, we get,

$$q_{k+1} = q_1 + (t+1)q_2 + t(q_3 + q_4 + ... + q_{k-1}) + (t-1)q_k$$

$$> q_1 + q_2 + ... + q_{k-1} \text{ (since } t \geq 1.)$$

Hence $p_{k+1} > p_1 + p_2 + ... + p_{k-1}$. for all $k \geq 2$

Which implies that $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

<u>**Case (2)**</u> : $\alpha < -1$

Subcase 1 : $Det(A) = 1$

Therefore the eigen values are $\alpha, \frac{1}{\alpha}$. Then $-t = -\alpha - \frac{1}{\alpha} \geq 3$ as $(\sqrt{-\alpha} - \frac{1}{\sqrt{-\alpha}})^2 > 0$ and t is an integer. Using proposition 2.1.6, we get,

$$p_n = \begin{cases} q_n - 2 & \text{if } n \textit{ is even} \\ q_n + 2 & \text{if } n \textit{ is odd} \end{cases}$$

From proposition 3.2.2, we have $q_{n+1} + q_{n-1} = -tq_n \ \forall n \geq 2$. Adding these relation for $n = 2, 3, ..., k$, we get,

$$q_1 + q_2 + 2(q_3 + q_4 + ... + q_{k-1}) + q_k + q_{k+1} = -t(q_2 + q_3 + ... + q_k).$$

Hence

$$\begin{aligned} q_{k+1} &= -q_1 - (t+1)q_2 - (t+2)(q_3 + q_4 + ... + q_{k-1}) - (t+1)q_k. \\ &\geq -q_1 + 2q_2 + q_3 + q_4 + ... + q_{k-1} + 2q_k, \text{ as } -t \geq 3. \\ &> q_1 + q_2 + ... + q_k. \end{aligned}$$

Therefore $p_{k+1} > p_1 + p_2 + ... + p_{k-1} \ \forall k \geq 2$. Hence $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

Subcase 2 : $Det(A) = -1$

Here the eigen values are $\alpha, \frac{-1}{\alpha}$. Then $t = \lambda - \frac{1}{\lambda} < 0$ and hence $-t \geq 1$, since t is an integer. Using proposition 2.1.6,

$$p_n = \begin{cases} q_n - 2 & \text{if } n \textit{ is even} \\ q_n & \text{if } n \textit{ is odd} \end{cases}$$

From proposition 3.2.2, we have $q_{n-1} - q_{n+1} = tq_n$, for all $n \geq 2$.

Adding these relations for $n = 2, 3, ...k$, we get,

$$\begin{aligned} q_{k+1} &= q_1 + (1-t)q_2 - t(q_3 + q_4 + ... + q_{k-1}) - (t+1)q_k \\ &> q_1 + q_2 + \ldots + q_{k-1} \text{ (Since } -t \geq 1, \text{ we have } 1 - t > 1.) \end{aligned}$$

Which implies that $p_{k+1} > p_1 + p_2 + ... + p_{k-1}$ for all $k \geq 2$.

Hence $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

□

Remark 3.2.5. From the above proposition , it is clear that for a hyperbolic automorphism T_A, the period set $Per(T_A)$ is either $\mathbb{N}$ or $\mathbb{N} \setminus \{2\}$. In proposition 3.3.5 and proposition 3.3.6 we prove that for certain class of hyperbolic automorphisms $Per(T_A)$ is $\mathbb{N}$ and for some other class of automorphisms, it is $\mathbb{N} \setminus \{2\}$.

3.3 The nonhyperbolic case

Note that $1 \in Per(T_A), \forall A \in GL(2, \mathbb{Z})$. Suppose that $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ is a non-hyperbolic toral automorphism. Then $t = |\alpha + \beta| \leq |\alpha| + |\beta| \leq 2$. That is $t \in \{-2, -1, 0, 1, 2\}$ and $Det(A) = \pm 1$. Thus, any $A \in GL(2, \mathbb{Z})$ which is nonhyperbolic will fall under one of these 10 cases . For $A \in GL(2, \mathbb{Z})$, let $ch(A)$ denote the characteristic polynomial of A .

Proposition 3.3.1. *Let $A \in GL(2, \mathbb{Z})$ be such that $Det(A) = -1$, $Trace(A) = 0$. Then $Per(T_A) = \{1, 2\}$.*

Proof. The $ch(A) = x^2 - 1$. Then by Cayley-Hamilton theorem $A^2 = I$ and $A \neq I$. Hence $Per(T_A) = \{1, 2\}$. □

Proposition 3.3.2. *Let $A \in GL(2, \mathbb{Z})$ be such that $Det(A) = 1$, $Trace(A) = 0$. Then $Per(T_A) = \{1, 2, 4\}$.*

Proof. The $ch(A) = x^2 + 1$. Then $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2 + bc = -1$.

Then by Cayley-Hamilton theorem, $A^2 = -I$ and hence $A^4 = I$. Note that $p_1 = |Det(A - I)| = 2$ and $p_2 = |Det(A^2 - I)| = |Det(-2I)| = 4$. Hence $2 \in Per(T_A)$. Hence $Per(T_A) = \{1, 2, 4\}$. □

Proposition 3.3.3. *Let $A \in GL(2, \mathbb{Z})$ be such that $Det(A) = 1$, $Trace = -1$. Then $Per(T_A) = \{1, 3\}$.*

Proof. The $ch(A)$ is x^2+x+1. Then$A = \begin{pmatrix} a & b \\ c & -1-a \end{pmatrix}$ with $a^2+a+bc = -1$. Now $p_1 = |Det(A-I)| = 3$ and $p_2 = |Det(A_2 - I)| = 3$. Hence $2 \notin Per(T_A)$. Note that $A^3 = I$. This implies $3 \in Per(T_A)$ and hence $Per(T_A) = \{1, 3\}$. □

Proposition 3.3.4. *Let $A \in GL(2,\mathbb{Z})$ be such that $Det(A) = 1$ and $Trace(A) = 1$. Then $Per(T_A) = \{1, 2, 3, 6\}$.*

Proof. The $ch(A)$ is $x^2 - x + 1$. Then $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ with $a - a^2 - bc = 1$. By Cayley-Hamilton theorem, $A^2 - A + I = 0$.

Now $p_1 = |Det(A-I)| = 1$. Hence A has unique fixed point namely 0, $p_2 = |Det(A^2-I)| = 3 > p_1 = 1$ showing that $2 \in Per(T_A)$, $p_3 = |Det(A^3-I)| = 4 > p_1 = 1$ showing that $3 \in Per(T_A)$. Again, $p_4 = |Det(A^4-I)| = 3 \not> p_2 = 3$ showing that $4 \notin Per(T_A)$. $p_5 = |Det(A^5-I)| = 1 \not> p_1 = 1$ showing that $5 \notin Per(T_A)$. Now, $A^6 = I$. This implies that $6 \in Per(T_A)$ and hence $Per(T_A) = \{1, 2, 3, 6\}$. □

Proposition 3.3.5. *Let $A \in GL(2,\mathbb{Z})$ be such that $Det(A) = -1$ and $Trace(A) = \pm 1$. Then $Per(T_A) = \mathbb{N} \setminus \{2\}$.*

Proof. **Case: 1** Suppose $Det(A) = -1$ and $Trace(A) = 1$. Then $ch(A)$ is $x^2 - x - 1$ and hence the eigen values are $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$, showing that A is hyperbolic. Therefore by theorem 3.2.4 we have $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

We now prove that $2 \notin Per(T_A)$.

From the hypothesis, it is clear that $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ for some integers a, b, c with $a^2 - a + bc = 1$.

Therefore,
$$\begin{aligned} p_2 &= |Det(A^2 - I)| \\ &= |Det(A-I)||Det(A+I)| \end{aligned}$$

$= p_1|Det(A + I)|$

$= p_1$ (Since $|Det(A + I)| = |a^2 + a - bc| = 1$.)

Hence $2 \notin Per(T_A)$.

Case: 2

Suppose $Det(A) = -1$ and $Trace(A) = -1$.

Proof of this is similar to that of Case: 1

□

Proposition 3.3.6. *Let $A \in GL(2, \mathbb{Z})$ be such that $Det(A) = -1$ and $Trace(A) = \pm 2$. Then $Per(T_A) = \mathbb{N}$.*

Proof. **Case: 1** Suppose $Det(A) = -1$ and $Trace(A) = 2$. Then $ch(A)$ is $x^2 - 2x - 1$ and hence the eigen values are $1 \pm \sqrt{2}$, showing that A is hyperbolic. Therefore by theorem 3.2.4 we have $Per(T_A) \supset \mathbb{N} \setminus \{2\}$.

We now prove that $2 \in Per(T_A)$.

From the hypothesis, it is clear that $A = \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix}$ for some integers a, b, c with $a^2 - 2a + bc = 1$.

Therefore, $p_2 = |Det(A^2 - I)|$

$= |Det(A - I)||Det(A + I)|$

$= p_1|Det(A + I)|$

$= p_1.2$ (Since $|Det(A + I)| = 2$.)

Hence $2 \in Per(T_A)$.

□

Proposition 3.3.7. *Let $Det(A) = 1$ and $Trace(A) = 2$. Then $Per(T_A) = \mathbb{N}$, provided $A \neq I$, in which case $Per(T_A) = \{1\}$.*

Proof. $Ch(A)$ is $x^2 - 2x + 1$. Then $A = \begin{pmatrix} a & b \\ c & 2-a \end{pmatrix}$ with $-a^2 + 2a - bc = 1$.

Now $Det(A - I) = 0$ and therefore A has infinitely many fixed points.

By induction on n, we have $A^n - I = n(A - I)$ for all $n \in \mathbb{N}$ and hence the system $A^n X = X$ has infinitely many solutions in $\mathbb{T}^2$.

To prove $Per(T_A) = \mathbb{N}$, it is enough to prove that for every $k(> 1) \in \mathbb{N}$ there exist a $X = (x, y) \in \mathbb{T}^2$ such that $A^k X = X$ and $A^i X \neq X \ \forall\ 1 \leq i < k$. This is evident from the fact that the equation $A^n X = X$ is equivalent to the following conditions:

$$\left.\begin{aligned} n[(a-1)x + by] \in \mathbb{Z} \\ n[cx + (1-a)y] \in \mathbb{Z} \end{aligned}\right\} \tag{3.1}$$

For, Suppose $k \in \mathbb{N}$ and $k > 1$

Case: 1

Suppose $b \neq 0$ and $1 - a \neq 0$.

Then $X = (0, \frac{1}{k.gcd(b,1-a)}) \in \mathbb{T}^2$ satisfies our requirement and hence it is a point of period k.

Case: 2

Suppose $b = 0$ and $1 - a \neq 0$.

In this case, $X = (0, \frac{1}{k(1-a)}) \in \mathbb{T}^2$ is point of period k.

Case: 3

Suppose $b \neq 0$ and $1 - a = 0$.

Let

$$X = \begin{cases} (\frac{1}{kc}, 0) & \text{if } c \neq 0 \\ (0, \frac{1}{kb}) & \text{if } c = 0 \end{cases}$$

Then X is point of period k.

Case: 4

Suppose $b = 0$ and $1 - a = 0$

If $c \neq 0$ then take $X = (\frac{1}{kc}, 0)$. If $c = 0$ then $A = I$, in which case $Per(T_A) = \{1\}$.

□

Proposition 3.3.8. *Let $Det(A) = 1$ and $Tr(A) = -2$. Then $Per(T_A) = 2\mathbb{N} \cup \{1\}$, provided $A \neq -I$, in which case $Per(T_A) = \{1, 2\}$.*

Proof. The $ch(A) = x^2 + 2x + 1$. Then$A = \begin{pmatrix} a & b \\ c & -2-a \end{pmatrix}$ with $a^2 + 2a + bc = -1$.

It can be shown by induction, with the use of Cayley-Hamilton's theorem that

$$A^n = \begin{cases} \begin{pmatrix} -na-n+1 & -nb \\ -nc & na+n+1 \end{pmatrix} & \text{if } n \textit{ is even} \\ \begin{pmatrix} na+n-1 & nb \\ nc & -na-n-2 \end{pmatrix} & \text{if } n \textit{ is odd} \end{cases}$$

Now ,

$$|Det(A^n - I)| = \begin{cases} 4 & \text{if } n \textit{ is odd} \\ 0 & \text{if } n \textit{ is even} \end{cases}$$

It is immediate that, if $n \neq 1$ is odd then $n \notin Per(T_A)$. Also, when n is even, the equation $A^n X = X$ has infinitely many solutions. Now, the equation $A_n X = X$ is equivalent to

$$\left.\begin{aligned} n[(a+1)x + by] \in \mathbb{Z} \\ n[cx - (a+1)y] \in \mathbb{Z} \end{aligned}\right\} \tag{3.2}$$

Now, we argue as in the previous proposition to show that $2\mathbb{N} \subset Per(T_A)$, except the case when $a + 1 = b = c = 0$, in which case $A = -I$, for which $Per(T_A) = \{1, 2\}$. Thus $Per(T_A) = 2\mathbb{N} \cup \{1\}$. □

Thus we have proved,

Theorem 3.3.9. *Let T_A be a nonhyperbolic toral automorphism. Then $Per(T_A)$ is one of the following 7 subsets of $\mathbb{N}$.*

(1) $\{1\}$

(2) $\{1, 2\}$

(3) $\{1, 3\}$

(4) $\{1, 2, 4\}$

(5) $\{1, 2, 3, 6\}$

(6) $2\mathbb{N} \cup \{1\}$

(7) $\mathbb{N}$

In the following table the set $Per(T_A)$ is listed in terms of the minimal polynomial of the induced matrix A for nonhyperbolic automorphisms.

Minimal polynomial of A	$Per(T_A)$
$x^2 - 1, x + 1$	{1,2}
$x^2 + 1$	{1,2,4}
$x^2 + x + 1$	{1,3}
$x^2 - x + 1$	{1,2,3,6}
$x^2 - 2x + 1$	$\mathbb{N}$
$x^2 + 2x + 1$	$2\mathbb{N} \cup \{1\}$
$x - 1$	{1}

Main Theorem

Theorem 3.3.10. *Let T_A be a toral automorphism. Then $Per(T_A)$ is one of the following 8 subset of $\mathbb{N}$.*

(1) $\{1\}$

(2) $\{1, 2\}$

(3) $\{1, 3\}$

(4) $\{1, 2, 4\}$

(5) $\{1, 2, 3, 6\}$

(6) $2\mathbb{N} \cup \{1\}$

(7) $\mathbb{N} \setminus \{2\}$

(8) $\mathbb{N}$

Proof. Proof follows from theorems 3.2.4 and 3.3.9. □

The following table summarizes the known results, similar to our main result of this chapter.

A class of dynamical systems	**Family of period sets**
Homeomorphisms on $\mathbb{R}$	Empty set, $\{1\}$, $\{1, 2\}$
Continuous maps on $\mathbb{R}$	As in theorem 3.1.3
Continuous self maps of the upper half plane in $\mathbb{R}^2$	All subsets of $\mathbb{N}$.
Continuous maps on the closed unit disc.	All subsets of $\mathbb{N}$ containing 1.
Circle maps with degree 1	As in theorem 3.1.12
Complex polynomials.	Five subsets of $\mathbb{N}$ as in theorem 3.1.5 .
Automorphism on abelian, torsion free groups	Subsets of $\mathbb{N}$, containing 1 and closed under l.c.m.
Continuous toral automorphisms.	*8 subsets of* $\mathbb{N}$ as in theorem 3.3.10.