**ABSTRACT**

This chapter investigates a machine repair model with standbys, working vacation and server breakdown. As soon as an operating unit fails, it is immediately replaced by a standby unit for the smooth running of the production. When there is no failed unit in the system, the server goes on vacation; in the meanwhile, the server performs some works and is said to be on working vacation. The life time and the repair time of the units are assumed to be exponentially distributed. The matrix recursive method is used to evaluate various performances measures such as the expected number of failed units in the system, the expected number of operating units in the system, machine availability, operating utilization, etc.. The cost function is established to maximize the gain. The sensitivity analysis is also carried out to examine the effect of different parameters on various system characteristics.

**4.1 Introduction**

Queueing models are of interest due to their significant role to predict queueing characteristics in many areas of day-to-day life as well as industrial scenario including computer and communication networks, manufacturing and production systems, etc.. The analysis of machining systems via queueing theory provides the quantitative prediction of these systems for handling the problems related with the blocking, delay and optimal control at various levels. For the smooth running of the production in a machining system, the provision of standbys is recommended in many industrial systems. As soon as an operating unit fails, it is immediately sent for repair and in the mean time replaced with a spare unit for avoiding the interruption in the production. The industry will like to avoid any loss of production during the period of breakdown with the provision of a proper combination of spare part support and repair facility. The queueing literature in the area of performance modeling of machine repair problems via queue theoretic approach is large enough. The important
recent past research works done in the area of machine repair problem with spares are reported in previous chapter 3.

In many machining systems, the server usually undergoes on vacation of random length as the system becomes empty. In many cases, the vacation duration of the server may be utilized for some auxiliary work and such vacation is known to be as working vacation. Concerning the real life congestion situations, the example of this kind of systems can be realized in many industrial organizations including computer systems, telecommunications, manufacturing systems, production systems, etc. Recently, queueing models with working vacation have drawn the attention of several researchers working in the field of the queueing theory (cf. Gray et al, 2000; Tian and Zhang, 2002; Zhang and Tian, 2003; Ke and Pearn, 2004). Baba (2005) considered that the server works with different rates in a GI/M/1 queue with multiple working vacations. Wu and Takagi (2006) analyzed M/G/1 queue with multiple working vacations. Analytic analysis and computation for the GI / M/1/N queue with multiple working vacations have been done by Banik et al. (2007). Li et al. (2007) presented the performance analysis of GI/M/1 queue with working vacations. Wang et al. (2009) examined optimal management of the machine repair problem with working vacation with the help of Newton’s method. A queueing model with working vacations including multiple types of server breakdowns has been discussed by Jain and Jain (2010). Zhang and Hou (2010) discussed performance analysis of M/G/1 queue with working vacations and vacation interruption. Li et al. (2010a) analyzed steady-state discrete-time batch arrival queue with working vacations. The modeling and analysis of unreliable Markovian multiserver finite-buffer queue with discouragement and synchronous working vacation policy has been done by Jain and Upadhyaya (2011). Gao and Liu (2012) treated an M/G/1 queue with single working vacation and vacation interruption under Bernoulli schedule. Whenever the system becomes empty at a service completion instant, the server goes for a single working vacation.

In many queueing models, the server is subject to breakdown. While studying the vacation policy for more realistic queueing system, the concept of server breakdown should be incorporated. There may be several reasons of the server breakdowns such as tiredness, illness, some technical error in the system or due to other environmental conditions. A remarkable work has been done by many queue
theoreticians in this area too (cf. Wartenhorst, 1995; Hsien and Andersland, 1995). Ke (2003) studied the optimal control of an M/G/1 queueing system with server vacations, startup and breakdowns. Further, Ke (2005) considered modified T vacation policy for an M/G/1 queueing system with unreliable server and startup. Li et al. (2008) gave performance analysis of GI/M/1 queue with working vacations and vacation interruption. An M/G/1 retrial queueing system with two phases of service subject to server breakdown and repair has been analyzed by Choudhury and Deka (2008). Wang and Xu (2009) examined an M/G/1 queue with second optional service and server breakdown for its well-posedness. Falin (2010) studied an M/G/1 retrial queue with an unreliable server and general repair times. A computational algorithm and parameter optimization for a multi-server system with unreliable servers and impatient customers has been performed by Wu and Ke (2010). Ke et al. (2011) explained the performance measures and randomized optimization for an unreliable server M[i]/G/1 vacation system. Choudhury and Ke (2012) examined the steady state behaviour of an M[i]/G/1 queue with general retrial time and Bernoulli vacation schedule for an unreliable server, which consists of a breakdown period and delay period.

In the present study, we consider an M/M/1 machine repair system with standbys including the working vacation and server breakdown. The presence of standby units is useful for the continuous and smooth production. It is assumed that the failed unit is immediately replaced by a standby unit if available. The server goes for working vacation when the system is empty and as soon as there are failed units in the system, he returns back from vacation. In both the states i.e. working vacation and busy state, the server may subject to breakdown. The remaining chapter is organized as follows. In section 4.2, we formulate the model by stating requisite assumptions and notations. The governing steady state equations are constructed by taking appropriate transition rates. Section 4.3 deals with the matrix recursive approach to compute various steady state probabilities. In section 4.4, various performance indices of the system are evaluated explicitly in term of probabilities. The cost analysis has been given in section 4.5. Numerical illustration and sensitivity analysis have been carried out in section 4.6. Finally, the concluding remarks are drawn in section 4.7.
4.2 Model Description

Consider an M/M/1 machine repair problem with standbys, working vacation and server breakdown. The Markov model is formulated based on the following assumptions:

- There are M identical operating units in the system along with two types of spare units, \( S_1 \) of type 1 and \( S_2 \) of type 2.
- The operating units as well as both types of spare units are subjected to breakdown according to Poisson distribution with rates \( \lambda, \alpha_1 \) and \( \alpha_2 \), respectively.
- The repair times of the failed operating as well as spare units are exponential distributed.
- The repairman goes for a working vacation when the system is empty. The duration of the working vacation of the server is independent and identically distributed random variable having exponential distribution with mean \( 1/\eta \).
- During the working vacation, the repairman is prone to breakdown; the life time and repair time of the server are exponentially distributed with mean \( \alpha_v \) and \( \beta_v \), respectively.
- During busy period, the server may breakdown in Poisson fashion with mean rate \( \alpha_b \); and is repaired exponentially with mean \( 1/\beta_b \).
- When all the spare units are exhausted, the system starts working in degraded mode; the life time of the operating units is exponentially distributed with mean \( 1/\lambda_d \).
- Let \( n \) be the number of failed units in the system. The state dependent failure rate \( \Lambda_n \) is given as follows:

\[
\Lambda_n = \begin{cases} 
M\lambda + S_2\alpha_2 + (S_1 - n)\alpha_1, & 0 \leq n < S_1 \\
M\lambda + (S_1 + S_2 - n)\alpha_2, & S_1 \leq n < S_1 + S_2 \\
(M + S_1 + S_2 - n)\lambda_d, & S_1 + S_2 \leq n < K 
\end{cases}
\]

- The state dependent repair rate when the server is on working vacation is

\[
\gamma_n^v = \begin{cases} 
\mu_v, & 0 < n \leq S_1 \\
\mu_1^v, & S_1 < n \leq S_1 + S_2 \\
\mu_2^v, & S_1 + S_2 \leq n \leq K 
\end{cases}
\]

When the server is in busy state, the repair rate is given by

\[
\gamma_n^b = \begin{cases} 
\mu, & 0 < n \leq S_1 \\
\mu_1^b, & S_1 < n \leq S_1 + S_2 \\
\mu_2^b, & S_1 + S_2 \leq n \leq K 
\end{cases}
\]
Fig. 4.1: State transition diagram
Let $P_{i,j}$ be the steady state probability that there are $i$ failed units in the system and the server is in $j^{th}$ state, where

$$j = \begin{cases} 
0, & \text{when the server is in working vacation state} \\
1, & \text{when the server is brokendown and under repair state while failed during working vacation} \\
2, & \text{when the server is in busy state} \\
3, & \text{when the server is brokendown and under repair state while failed during busy period}
\end{cases}$$

The state transition diagram is shown in fig. 4.1. The steady state equations governing the model are constructed as follows:

\begin{align*}
\Lambda_0 P_{0,0} &= \mu^1 P_{1,0} + \mu^0 P_{1,1} \\
(\Lambda_1 + \alpha_v + \eta + \mu^1) P_{1,0} &= \Lambda_0 P_{0,0} + \mu^0 P_{2,0} + \beta_v P_{1,2} \\
(\Lambda_n + \alpha_v + \eta + \gamma_n^1) P_{n,0} &= \Lambda_{n-1} P_{n-1,0} + \gamma_n^1 P_{n+1,0} + \beta_v P_{n,2}, \quad 1 < n \leq K-1 \\
(\alpha_v + \eta + \mu_2^1) P_{k,0} &= \Lambda_{k-1} P_{k-1,0} + \beta_v P_{k,2} \\
(\Lambda_1 + \alpha_b + \mu^1) P_{1,1} &= \mu^0 P_{2,1} + \beta_b P_{1,3} + \eta P_{1,0} \\
(\Lambda_n + \alpha_b + \gamma_n^1) P_{n,1} &= \Lambda_{n-1} P_{n-1,1} + \gamma_n^1 P_{n+1,1} + \beta_b P_{n,3} + \eta P_{n,0}, \quad 1 < n \leq K-1 \\
(\alpha_b + \mu_2^1) P_{k,1} &= \Lambda_{k-1} P_{k-1,1} + \beta_b P_{k,3} + \eta P_{k,0} \\
(\Lambda_1 + \beta_v) P_{1,2} &= \alpha_v P_{1,0} \\
(\Lambda_n + \beta_v) P_{n,2} &= \Lambda_{n-1} P_{n-1,2} + \alpha_v P_{n,0}, \quad 1 < n \leq K-1 \\
\beta_v P_{k,2} &= \Lambda_{k-1} P_{k-1,2} + \alpha_v P_{k,0} \\
(\Lambda_1 + \beta_b) P_{1,3} &= \alpha_b P_{1,1} \\
(\Lambda_n + \beta_b) P_{n,3} &= \Lambda_{n-1} P_{n-1,3} + \alpha_b P_{n,1}, \quad 1 < n \leq K-1 \\
\beta_b P_{k,3} &= \Lambda_{k-1} P_{k-1,3} + \alpha_b P_{k,1}
\end{align*}

### 4.3 Matrix Recursive Solution

For Markov process, the concept of matrix recursive approach can be employed to evaluate the steady state probabilities. Using equations (4.1)-(4.13), the
corresponding transition rate matrix $Q$ of the Markov model can be represented in a block-tridiagonal matrix as follows:

$$Q = \begin{bmatrix}
B_0 & C_0 & & & & \\
A_0 & B_1 & C_1 & & & \\
& A_2 & B_2 & C_2 & & \\
& & & & & \\
& & & & A_{K-1} & B_{K-1} & C_{K-1} & \\
& & & & A_K & B_K &
\end{bmatrix}$$

(4.14)

The matrix $Q$ consists of sub matrices $A_0, B_0, C_0, A_n, B_n, \text{ and } C_n$ ($1 \leq n \leq K - 1$) as follows:

$$B_0 = [-\Lambda_v], \quad C_0 = [A_0 \ 0 \ 0 \ 0]$$

$$A_0 = \begin{bmatrix}
\mu^v \\
\mu^B \\
0 \\
0
\end{bmatrix}, \quad A_n = \begin{bmatrix}
\gamma_n^v & 0 & 0 & 0 \\
0 & \gamma_n^B & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A_K = \begin{bmatrix}
\gamma_K^v & 0 & 0 & 0 \\
0 & \gamma_K^B & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

$$B_n = \begin{bmatrix}
-\left(\Lambda_n + \alpha_v + \eta + \gamma_n^v\right) & \eta & \alpha_v & 0 \\
0 & -\left(\Lambda_n + \alpha_B + \gamma_n^B\right) & 0 & \alpha_B \\
\beta_v & 0 & -\left(\Lambda_n + \beta_v\right) & 0 \\
0 & \beta_B & 0 & -\left(\Lambda_n + \beta_B\right)
\end{bmatrix}$$

$$C_n = \begin{bmatrix}
\Lambda_n & 0 & 0 & 0 \\
0 & \Lambda_n & 0 & 0 \\
0 & 0 & \Lambda_n & 0 \\
0 & 0 & 0 & \Lambda_n
\end{bmatrix}, \quad B_K = \begin{bmatrix}
-\left(\alpha_v + \eta + \gamma_K^v\right) & \eta & \alpha_v & 0 \\
0 & -\left(\alpha_B + \gamma_K^B\right) & 0 & \alpha_B \\
\beta_v & 0 & -\beta_v & 0 \\
0 & \beta_B & 0 & -\beta_B
\end{bmatrix}$$

where $A_0$ is a matrix of order $1 \times 4$, $B_0$ is a matrix of order $1 \times 1$ and $C_0$ is a matrix of order $4 \times 1$. Also $A_n$, $B_n$ and $C_n$ ($1 \leq n \leq K - 1$) are the square matrices of order 4.

Let $P$ be the vector corresponding to steady state probabilities with $Q$ as coefficient matrix. Let us partition $P$ as
\[ P = [P_\theta, P_1, P_2, \ldots] \]

where \( P_\theta = P_{0,0} \) and \( P_n = \{P_{n,0}, P_{n,1}, P_{n,2}, P_{n,3}\} \quad (1 \leq n \leq K) \).

The matrix equations are constructed as follows:

\[ P_\theta B_0 + P_1 A_0 = 0 \]

\[ P_\theta C_0 + P_1 B_1 + P_2 A_2 = 0 \]

\[ P_{n-1} C_{n-1} + P_n B_n + P_{n-1} A_{n-1} = 0, \text{ for } 2 \leq n \leq K-1 \]

\[ P_{K-1} C_{K-1} + P_K B_K = 0 \]

After backward substitution, we obtain

\[ P_K = -P_{K-1} C_{K-1} B_K^{-1} = P_{K-1} X_K \quad (4.15) \]

\[ P_n = P_{n-1} X_n \quad \text{for } 2 \leq n \leq K-1 \quad (4.16) \]

\[ P_1 = -P_1 C_0 (B_1 + X_2 A_2) \quad (4.17) \]

\[ P_\theta (B_0 - C_0 (B_1 + X_2 A_2) A_0) = 0 \quad (4.18) \]

where, \( X_n = -\lambda^{-1}_{n-1} (B_n + X_{n+1} A_{n+1}) \), \( 2 \leq n \leq K-1 \) is a \( 4 \times 4 \) matrix. Also \( X_K = -\lambda^{-1}_{K-1} B_K^{-1} \).

The matrix \( B_K \) must be nonsingular. By using equation (4.18), we obtain \( P_\theta \) up to a multiplicative constant while the other equations (4.15)-(4.17) give us \( P_K, P_{K-1}, \ldots, P_2, P_1 \) up to same constant, which can be uniquely determined by the following normalizing equation

\[ P_{0,0} + \sum_{n=1}^{K} P_n e = 1 \quad (4.19) \]

where \( e \) is a column vector of appropriate dimension with all the elements equal to 1.

For the determination of \( P_{0,0}, P_{i,j} \) for \( j = 0,1,2,3 \) and \( 1 \leq i \leq K \), we do coding of computer program in the software MATLAB.
4.4 Performance Measures

The main goal of our investigation is to predict various performance metrics in terms of the steady state probabilities. Some indices to characterize the system performance are as follows:

- The expected number of failed units in the system when the server is on working vacation, is

  \[ E(V) = \sum_{n=0}^{K} nP_{n,0} \quad (4.20) \]

- The expected number of failed units in the system when the server is in the busy state, is

  \[ E(B) = \sum_{n=1}^{K} nP_{n,1} \quad (4.21) \]

- The expected number of failed units in the system when the server is broken down and under repair state while failed during the working vacation, is

  \[ E(D_V) = \sum_{n=1}^{K} nP_{n,2} \quad (4.22) \]

- The expected number of failed units in the system when the repairman is broken down and under repair state while failed during the busy state, is

  \[ E(D_B) = \sum_{n=1}^{K} nP_{n,3} \quad (4.23) \]

- The expected total number of failed units in the system, is

  \[ E(N) = E(V) + E(B) + E(D_V) + E(D_B) \quad (4.24) \]

- The throughput of the system is given by

  \[ TP = \sum_{n=1}^{K} \mu_n \left( P_{n,o} + P_{n,1} \right) \quad (4.25) \]

- The machine availability of the system is

  \[ MA = 1 - \frac{E(N)}{K} \quad (4.26) \]
4.5 Cost Analysis

Now we construct a cost function per unit time in terms of cost elements related to different states of the machining system. Let us denote

\[ C_1 = \text{Cost per unit time per failed unit when the server is on working vacation} \]

\[ C_2 = \text{Cost per unit time per failed unit when the server is in busy state} \]

\[ C_3 = \text{Cost per unit time per failed unit when the server is broken down working vacation} \]

\[ C_4 = \text{Cost per unit time per failed unit when the server is broken down from busy state} \]

\[ C_5 = \text{Cost incurred per unit time for the repair of the server broken down during working vacation} \]

\[ C_6 = \text{Cost incurred per unit time for the repair of the server broken down during the busy state} \]

Using above cost elements, the expected total cost per unit time is given by:

\[ C(\beta_v, \beta_b) = C_1 E(V) + C_2 E(B) + C_3 E(D_v) + C_4 E(D_b) + C_5 \beta_v + C_6 \beta_b \]

4.6 Numerical Illustration

In this section numerical results are facilitated to explore the effects of various parameters on the system performance. For this purpose, the program for matrix recursive method is coded in software MATLAB. In order to compute the performance indices, we set default parameters as \( M = 5, S_1 = 3, S_2 = 2, \lambda_d = 0.7, \alpha_2 = 0.5, \alpha_v = \alpha_b = 0.6, \beta_v = \beta_b = 3, \mu^v = 2, \mu_1^v = 3, \mu_2^v = 3, \mu^b = 3, \mu_1^b = 4, \mu_2^b = 4. \)

The different sets of cost elements for cost analysis are given in table 4.1. Figs 4.2(a)-4.2(b) reveal the behavior of the expected total number of failed units \( E(N) \) by varying the values of \( \lambda \) and \( \mu^v \) respectively, for different values of \( \eta \). It is seen that \( E(N) \) increases sharply for the increasing values of \( \lambda \) but decreases gradually as \( \mu^v \) increases. Figs 4.3(a)-4.3(b) show the effect of \( \lambda \) and \( \mu^v \) on the machine availability (MA) corresponding to the different values of \( \eta \). As \( \lambda \) increases, the
<table>
<thead>
<tr>
<th>Cost factors</th>
<th>Set 1</th>
<th>Set 2</th>
<th>Set 3</th>
<th>Set 4</th>
<th>Set 5</th>
<th>Set 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>$C_2$</td>
<td>150</td>
<td>200</td>
<td>200</td>
<td>150</td>
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<td>$C_4$</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>50</td>
<td>100</td>
</tr>
<tr>
<td>$C_5$</td>
<td>15</td>
<td>15</td>
<td>30</td>
<td>30</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>$C_6$</td>
<td>15</td>
<td>30</td>
<td>15</td>
<td>15</td>
<td>30</td>
<td>30</td>
</tr>
</tbody>
</table>

Table 4.1: Cost sets

Machine availability initially decreases sharply then after gradually. MA shows increasing pattern as $\mu^V$ increases. Figs 4.4(a)-4.4(b) demonstrate the effect of increasing $\lambda$ and $\mu^V$ on the throughput (TP). It is found that for different values of $\eta$, TP slows down sharply as $\lambda$ increases but increases linearly as $\mu^V$ increases.

Figs 4.5 (a-f) show the combined effect of $\beta_v$ and $\beta_b$ on the expected total cost per unit time $C(\beta_v, \beta_b)$ for different sets of cost elements. It can be easily seen that $C(\beta_v, \beta_b)$ first goes down and then increases remarkably on increasing the values $\beta_v$ for all sets of cost elements. Based on numerical simulation, we conclude that

- the expected total number of failed units $E(N)$ increases sharply for the large values of $\lambda$ and decreases as $\mu^V$ increases.
- as expected, the machine availability MA and the throughput TP of the system decrease sharply as $\lambda$ increases while MA and TA show increasing pattern with $\mu^V$ for higher values of $\eta$.
- the repair rate $\mu^V$ plays a significant role to reduce the expected total number of failed units $E(N)$ and to enhance the machine availability MA and the throughput TP of the system.
4.7 Conclusion

In this chapter, the concepts of working vacation and sever breakdown have been incorporated while analyzing the M/M/1 machine repair problem with standby. The provision of standby units is recommended for the smooth and continuous functioning of multi-component machining system. The noble idea of inclusion of working vacation may be applicable in those organizations which operate in machining environment wherein the server utilizes his idle time for secondary works. The incorporation of many factors together, such as provision of standbys, working vacation, server breakdown makes our model more realistic, versatile and economic under certain reliability constraint. The matrix recursive method used to compute the steady state probabilities and other system performance indices viz. expected number of failed units, throughput, machine availability etc., are computationally tractable as validated by taking numerical example. The investigation done may be helpful for the system designers and industrial engineers for improving the grade of service of the system in many organizations operating in congestion scenario under certain techno-economic and reliability constraints.
Fig. 4.2(a): Expected number of failed units vs $\lambda$ for different values of $\eta$

Fig. 4.2(b): Expected number of failed units vs $\mu^V$ for different values of $\eta$

Fig. 4.3 (a): Machine availability vs $\lambda$ for different values of $\eta$

Fig. 4.3(b): Machine availability vs $\mu^V$ for different values of $\eta$

Fig. 4.4(a): Throughput vs $\lambda$ for different values of $\eta$

Fig. 4.4(b): Throughput vs $\mu^V$ for different values of $\eta$
Fig. 4.5 Effect of $\beta_V$ and $\beta_B$ on the total cost for (a) Set 1 (b) Set 2 (c) Set 3 (d) Set 4 (e) Set 5 (f) Set 6.