CHAPTER 2

LINEAR THEORY OF ELASTICITY FOR POROUS PIEZOELECTRIC MATERIALS

2.1 Introduction


In this chapter, the linear theory of elasticity for porous piezoelectric materials is established. In Section 2, the electric enthalpy density function is defined for PPM and constitutive equations are derived for anisotropic PPM. The variational principle defined for a non-conservative system for a linear piezoelectric continuum is extended to the linear theory of elasticity for PPM in Section 3. The equations of motion and boundary conditions are also derived for PPM. In Section 4, some standard theorems of classical elasticity are generalized for the linear theory of elasticity for PPM. The uniqueness theorem and reciprocity theorem are proved for a boundary value problem defined for a porous piezoelectric body. The uniqueness of solution of initial-boundary value problem in the linear theory of electro-elastodynamics for PPM is proved in Section 5.
2.2 Constitutive Equations

Let us consider a porous piezoelectric solid (PPS). A porous solid, in general, consists of two phases, namely solid and fluid phases. Let \( f \) denotes the porosity of the PPS so that \( 1 - f \) represents proportion of the solid phase, i.e. volume of the solid phase per unit volume of the bulk material.

![Diagram of a porous material](image)

**Figure 2.1 Diagram of a porous material**

Let \( \sigma_{ij} \) and \( \varepsilon_{ij} \) \((i, j = 1, 2, 3)\) be the stress and strain tensor components for the solid phase of the porous aggregate. Similarly, \( \sigma^* \) and \( \varepsilon^* \) denote stress and strain acting on the fluid phase of the porous solid. The quantities with superscript * are associated to the fluid phase of porous bulk material. Let \( E_i (E_i^*) \) denote the electric field vector components acting on the solid (fluid) phase of the PPS. Similarly, \( D_i (D_i^*) \) denote the displacement vector components in the solid (fluid) part of the PPS.

Let \( U \) denotes the internal energy of the system at instant \( t \) and \( dU \) represents the change of the internal energy \( U \) in the time interval \( (t, t + dt) \). The first law of thermodynamics gives

\[
dU = \delta w + \delta Q, \tag{2.1}
\]

where \( \delta w \) denotes the work done on the system by the applied external forces and \( \delta Q \) represents the quantity of heat absorbed by the system per unit volume.
As, in case of an elastic solid, the work done on the system by the external forces during a deformation is stored in the form of potential energy in the system.

\[
\delta w = \delta P' = \sigma_{ij} \, d\varepsilon_{ij},
\]

where \( \delta P' \) is the change in the potential energy stored in the system and \( d\varepsilon_{ij} \) denotes strain change.

In case of a piezoelectric material, the change of electric potential energy is also taken into account, which yields

\[
\delta w = \sigma_{ij} \, d\varepsilon_{ij} + E_i \, dD_i, \quad (i, j = 1, 2, 3).
\]

In absence of the thermal effects, the equation (2.1) becomes

\[
dU = \sigma_{ij} \, d\varepsilon_{ij} + E_i \, dD_i.
\]

For a PPS material, potential energy, stored in both the solid and fluid phases of the material, is to be reckoned with. Moreover, both the mechanical and electrical forces acting on both the phases contribute towards the change in potential energy of the system. Accordingly, the equation (2.4) can be modified as

\[
dU = \sigma_{ij} \, d\varepsilon_{ij} + \sigma^* \, d\varepsilon^* + E_i \, dD_i + E_i^* \, dD_i^*.
\]

Enthalpy is a state function which measures the total energy of a thermodynamic system. In an analogous manner, the electrical enthalpy density function (\( W \)) for a PPS is defined as

\[
W = U - E_i \, D_i - E_i^* \, D_i^*.
\]

Substituting the equation (2.5) in the equation (2.7), we get

\[
dW = \sigma_{ij} \, d\varepsilon_{ij} + \sigma^* \, d\varepsilon^* - D_i \, dE_i - D_i^* \, dE_i^*.
\]

The electric enthalpy density function (\( W \)) is a function of the independent variables \( \varepsilon_{ij}, \varepsilon^*, E_i \) and \( E_i^* \).
Comparison of the equations (2.8) and (2.9) yields

\[
\frac{\partial W}{\partial \epsilon_{ij}} = \sigma_{ij}, \quad \frac{\partial W}{\partial \epsilon^*} = \sigma^*, \quad \frac{\partial W}{\partial E_i} = -D_i, \quad \frac{\partial W}{\partial E_i^*} = -D_i^*.
\]  

(2.10)

When the function \( W \) is expanded with respect to \( \epsilon_{ij}, \epsilon^*, E_i \) and \( E_i^* \), within the scope of linear interactions, we can write

\[
W = \frac{1}{2} \left( \epsilon_{ij} \frac{\partial}{\partial \epsilon_{ij}} + \epsilon^* \frac{\partial}{\partial \epsilon^*} + E_i \frac{\partial}{\partial E_i} + E_i^* \frac{\partial}{\partial E_i^*} \right) \left( \epsilon_{kl} \frac{\partial}{\partial \epsilon_{kl}} + \epsilon^* \frac{\partial}{\partial \epsilon^*} + E_k \frac{\partial}{\partial E_k} + E_k^* \frac{\partial}{\partial E_k^*} \right) W.
\]  

(2.11)

\[
\Rightarrow 2W = \epsilon_{ij} \epsilon_{kl} \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}} + \epsilon_{ij} \epsilon^* \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon^*} + \epsilon_{ij} E_k \frac{\partial^2 W}{\partial \epsilon_{ij} \partial E_k} + \epsilon_{ij} E_k^* \frac{\partial^2 W}{\partial \epsilon_{ij} \partial E_k^*} +
\]

\[
\epsilon^* \epsilon_{kl} \frac{\partial^2 W}{\partial \epsilon^* \partial \epsilon_{kl}} + \epsilon^* \epsilon^* \frac{\partial^2 W}{\partial \epsilon^* \partial \epsilon^*} + \epsilon^* E_k \frac{\partial^2 W}{\partial \epsilon^* \partial E_k} + \epsilon^* E_k^* \frac{\partial^2 W}{\partial \epsilon^* \partial E_k^*} +
\]

\[
E_i \epsilon_{kl} \frac{\partial^2 W}{\partial E_i \partial \epsilon_{kl}} + E_i \epsilon^* \frac{\partial^2 W}{\partial E_i \partial \epsilon^*} + E_i E_k \frac{\partial^2 W}{\partial E_i \partial E_k} + E_i E_k^* \frac{\partial^2 W}{\partial E_i \partial E_k^*} +
\]

\[
E_i^* \epsilon_{kl} \frac{\partial^2 W}{\partial E_i^* \partial \epsilon_{kl}} + E_i^* \epsilon^* \frac{\partial^2 W}{\partial E_i^* \partial \epsilon^*} + E_i^* E_k \frac{\partial^2 W}{\partial E_i^* \partial E_k} + E_i^* E_k^* \frac{\partial^2 W}{\partial E_i^* \partial E_k^*}.
\]  

(2.12)

We define the followings:

\[
c_{ijkl} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}},
\]

(2.13a)

\[
R = \frac{\partial^2 W}{\partial \epsilon^* \partial \epsilon^*},
\]

(2.13b)

\[
m_{ij} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon^*} = \frac{\partial \sigma}{\partial \epsilon_{ij}},
\]

(2.13c)

\[
-\varepsilon_{ijk} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial E_k} = -\frac{\partial D_k}{\partial \epsilon_{ij}},
\]

(2.14a)

\[
-\varepsilon^*_{ik} = \frac{\partial^2 W}{\partial \epsilon^* \partial E_k} = -\frac{\partial D_k^*}{\partial \epsilon^*},
\]

(2.14b)

\[
-\varepsilon_{ijk} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial E_k^*} = -\frac{\partial D_k^*}{\partial \epsilon_{ij}}.
\]

(2.14c)
\[-\tilde{\Gamma}_k = \frac{\partial^2 W}{\partial \varepsilon^* \partial E_k} = -\frac{\partial D_k}{\partial \varepsilon^*} = \frac{\partial \sigma^*}{\partial E_k}; \quad (2.14d)\]

and
\[
\begin{align*}
\xi_{ij} &= -\frac{\partial^2 W}{\partial E_i \partial E_j} = \frac{\partial D_i}{\partial E_j} = \frac{\partial D_j}{\partial E_i}, \quad \tag{2.15a} \\
\xi_{ij}^* &= -\frac{\partial^2 W}{\partial E_i^* \partial E_j^*} = \frac{\partial D_i^*}{\partial E_j^*} = \frac{\partial D_j^*}{\partial E_i^*}, \quad \tag{2.15b} \\
A_{ij} &= -\frac{\partial^2 W}{\partial E_i \partial E_j^*} = \frac{\partial D_i}{\partial E_j^*} = \frac{\partial D_j^*}{\partial E_i}. \quad \tag{2.15c}
\end{align*}
\]

These above defined coefficients characterize elastic, piezoelectric and dielectric properties of a piezoelectric material. In case of a homogeneous body, these coefficients are considered as constants. The coefficients defined in the equations (2.13) characterize elastic behaviour of the material. \(c_{ijkl}\) are the elastic stiffness coefficients for the solid phase of the porous aggregate. The elastic constant \(R\) is a measure of the pressure required to force a volume of the fluid into the aggregate while the total volume remains unchanged. \(m_{ij}\) are the elastic coupling coefficients which take into account the elastic interaction between the solid and fluid phases of the porous aggregate. \(m_{ij}\) represent the change in the stress in the solid (fluid) phase with respect to the strain in the fluid (solid) phase.

The piezoelectric behaviour of a PPS body is characterized by the coefficients defined in the equations (2.14). \(e_{ijk}\) and \(e_{k}^*\) are the piezoelectric constants for solid and fluid phases, respectively. The piezoelectric constants \(e_{ijk}\) represent rate of change of the electric displacement with strain in the solid phase or change in negative mechanical stress with electric field in the solid phase. Similar meanings hold for piezoelectric constants \(e_{k}^*\) in the fluid phase of the PPS. \(\zeta_{ijk}\) and \(\tilde{\zeta}_k\) are the constants which take into account the piezoelectric coupling between the two phases of the PPS. \(\zeta_{ijk}\) represent the rate of change of electric displacement in the fluid phase with strain in the solid phase or rate of change of negative stress in the solid phase.
with electric field in the fluid phase. $\tilde{\zeta}_k$ represent the rate of change of electric displacement in the solid phase with strain in the fluid phase or rate of change of pressure in the fluid phase with electric field in the solid phase. The negative sign with these piezoelectric constants signifies the fact that a piezoelectric material tends to expand when electric field is applied on it and then a pressure (negative stress) is required to keep the dilatation checked.

The dielectric behaviour of a PPS is characterized by the coefficients defined in the equations (2.15). $\xi_{ij}$ and $\xi^*_{ij}$ are dielectric constants for the solid and fluid phases, respectively. $\xi_{ij}$ represent the change in the electric displacement in the solid phase when electric field is applied on it. A similar meaning holds for dielectric constants $\xi^*_{ij}$ in the fluid phase of the PPS. $A_{ij}$ are the constants which take into account the dielectric coupling between the two phases of the porous aggregate. $A_{ij}$ represent the change in the electric displacement in the solid (fluid) phase when the electric field is applied on the fluid (solid) phase, respectively. The different kinds of interactions which can take place in a PPS are shown in the figure 2.2.

Using the equations (2.13)-(2.15), the equation (2.12) can be expressed as

$$W = \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2} R \varepsilon_i \varepsilon_i^* + m_{ij} \varepsilon_{ij} \varepsilon^* - \varepsilon_{kij} E_k^* + \varepsilon_{ij} - \frac{1}{2} \xi_{ij} E_i^* E_j - \frac{1}{2} \xi^*_{ij} E_i^* E_j^* - A_{ij} E_i^* E_j.$$

From the above equation, we get

$$\frac{\partial W}{\partial \varepsilon_{ij}} = c_{ijkl} \varepsilon_{kl} + m_{ij} \varepsilon^* - \varepsilon_{kij} E_k^*,$$  

$$\frac{\partial W}{\partial \varepsilon_i^*} = m_{ij} \varepsilon_{ij} + R \varepsilon_i^* - \varepsilon_i^* E_i^*,$$  

$$\frac{\partial W}{\partial E_k} = -(\varepsilon_{kij} \varepsilon_{ij} + \varepsilon_{kij} E_j^* + \xi_{ij} E_j^* + A_{ij} E_j^*),$$  

$$\frac{\partial W}{\partial E_k^*} = -(\varepsilon_{kij} \varepsilon_{ij} + \varepsilon_k^* E_j^* + A_{ij} E_j^* + \xi^*_{ij} E_j^*).$$
Figure 2.2 Schematic diagram showing different kinds of interactions in a porous piezoelectric solid
Using the equations (2.10), the equations (2.17)-(2.20) can be written as

\[
\sigma_{ij} = c_{ijkl} \varepsilon_{kl} + m_{ij} \varepsilon_{i}^e - e_{kij} E_k - \zeta_{kij} E_k^e , \tag{2.21}
\]
\[
\sigma_i^e = m_j \varepsilon_{ij} + R \varepsilon_i^e - \zeta_{ij} E_j - e_i^e E_j^e , \tag{2.22}
\]
\[
D_t = e_{ijk} \varepsilon_{jk} + \varepsilon_{ij} \varepsilon_i^e + \varepsilon_{ij} E_j^e + A_{ijl} E_{j}^e , \tag{2.23}
\]
\[
D_{ij}^e = \zeta_{ijk} \varepsilon_{jk} + e_{ij} \varepsilon_i^e + A_{ijl} E_j^e + \zeta_{ij} E_j^e , \quad (i,j,k,l = 1, 2, 3) . \tag{2.24}
\]

These are constitutive equations for anisotropic PPS, saturated with a fluid.

### 2.3 Equations of Motion

Consider a porous piezoelectric body occupying a region \( V \) bounded by a surface \( S \). Let \( \mathbf{n}(\mathbf{n}) \) be the unit outward normal vector to a surface element \( dS \). Let \( u_i \) and \( u_i^e \) be the mechanical displacement components in solid and fluid phases of the porous aggregate, respectively. \( \Phi \) and \( \Phi^e \) denote the electric potentials corresponding to solid and fluid phases, respectively. \( \tilde{t}_j \) and \( \tilde{t}_j^e \) denote the prescribed tractions acting on the solid and fluid phases of \( S \). \( \tilde{c} \) and \( \tilde{c}^e \) are the surface charge per unit area on the solid and fluid phases of a porous aggregate, respectively. Let \( K' \) denotes the kinetic energy.

The Lagrangian \( (L) \), depending on the independent variables and their derivatives, is defined as

\[
L = K' - P' . \tag{2.25}
\]

The Hamilton principle states that, for a conservative system, out of all possible paths by which a system point move from the position at time \( t_0 \) to the position at the time \( t_1 \), the system will actually move along the path for which the value of the integral \( \int_{t_0}^{t_1} L dt \) is stationary.

The necessary and sufficient condition for the integral \( \int_{t_0}^{t_1} L dt \) to be stationary is

\[
\delta \int_{t_0}^{t_1} L dt = 0 , \tag{2.26}
\]
If the force system is not conservative, then the Hamilton principle can be generalized as

\[ \delta \int_{t_0}^{t_1} L dt + \int_{t_0}^{t_1} \delta w' dt = 0, \]  

(2.27)

where \( \delta w' \) is the virtual work done by the non-conservative forces in a virtual displacement consistent with the constraints.

The virtual work per unit area done by the prescribed surface tractions \( \tilde{t}_j \) and \( \tilde{t}_j^* \) in a small virtual displacements \( \delta u_j \) and \( \delta u_j^* \) corresponding to solid and fluid phases is \( \tilde{t}_j \delta u_j + \tilde{t}_j^* \delta u_j^* \). Similarly, the virtual work per unit area done by the prescribed surface charges \( \tilde{c} \) and \( \tilde{c}^* \) in a small variation \( \delta \Phi \) and \( \delta \Phi^* \) of electric potentials in solid and fluid phases is \( \tilde{c} \delta \Phi + \tilde{c}^* \delta \Phi^* \). The total virtual work \( \delta w' \) done by all non conservative forces in a virtual displacement is given as

\[ \delta w' = \int_S \left( \tilde{t}_j \delta u_j + \tilde{t}_j^* \delta u_j^* + \tilde{c} \delta \Phi + \tilde{c}^* \delta \Phi^* \right) dS. \]  

(2.28)

In an electro-mechanical medium, the potential energy is replaced by electric enthalpy \( (W') \) in the Lagrangian function.

\[ \therefore \text{The equation (2.25) becomes} \]

\[ L = K' - W'. \]  

(2.29)

The kinetic energy \( (K') \) and electric enthalpy \( (W') \) can be written as

\[ K' = \int K dV, \quad W' = \int W dV, \]  

(2.30)

where \( K \) is the kinetic energy density and for a porous piezoelectric body (Biot, 1956a), this can be expressed as

\[ K = \frac{1}{2} \left\{ \rho_{11} \ddot{u}_j \dot{u}_j + 2 \rho_{12} \dot{u}_j \dot{u}_j^* + \rho_{22} \dot{u}_j^* \dot{u}_j^* \right\}. \]  

(2.31)

The dynamical mass coefficients \( \rho_{11}, \rho_{12} \) and \( \rho_{22} \) depend upon the porosity \( (f) \), density of porous aggregate \( (\rho) \), pore fluid density \( (\rho_{fp}) \) and solid grain density \( (\rho_{sp}) \). \( \rho_{11} \) represents the total effective mass of the solid moving in the fluid. This total mass must be equal to the mass of the solid per unit volume of the aggregate plus an additional mass \( (\rho_a) \) due to the fluid, i.e.

\[ \rho_{11} = (1 - f) \rho_{sp} + \rho_a. \]  

(2.32)
Similarly, $\rho_{22}$ is the sum of the mass of the fluid per unit volume of the aggregate and an additional mass ($\rho_a$), i.e.

$$\rho_{22} = \int \rho_f \, dV + \rho_a. \quad (2.33)$$

$\rho_{12}$ represents a mass coupling parameter between fluid and solid phases of the porous aggregate, i.e.

$$\rho_{12} = -\rho_a. \quad (2.34)$$

Using the equations (2.28)-(2.30) into the equation (2.27), we obtain

$$\delta \int_{t_0}^{t_1} dt \int (K - W) \, dV + \int_{t_0}^{t_1} dt \int (\dddot{i}_j \delta u_j + \dddot{r}_j \delta u_j + \dddot{e} \delta \Phi + \dddot{e}^* \delta \Phi^*) \, dS = 0. \quad (2.35)$$

Making use of the equation (2.31), the first term in the first integral of the above equation can be written as

$$\delta \int_{t_0}^{t_1} dt \int K dV = -\rho_{11} \int_{t_0}^{t_1} dt \int \dddot{u}_j \delta u_j \, dV + \rho_{22} \int_{t_0}^{t_1} dt \int \dddot{u}_j \delta u_j \, dV + \rho_{12} \int_{t_0}^{t_1} dt \int (\dddot{u}_j \delta u_j + \dddot{u}_j \delta u_j) \, dV. \quad (2.36)$$

From the definition of $W$, we can write

$$\delta \int_{t_0}^{t_1} dt \int W \, dV = \int_{t_0}^{t_1} dt \int \left( \frac{\partial W}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} + \frac{\partial W}{\partial \varepsilon^*} \delta \varepsilon^* + \frac{\partial W}{\partial E_j} \delta E_j + \frac{\partial W}{\partial E^*} \delta E^* \right) \, dV. \quad (2.37)$$

The strain tensor $\varepsilon_{ij} (\varepsilon^*)$ is related to mechanical displacements $u_i (u^*_i)$, as

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad \text{and} \quad \varepsilon^* = u^*_{i,j}. \quad (2.38)$$

In case of quasi-static approximation, electric field appears static in comparison to electromagnetic waves or electric field is irrotational. Thus, electric field can be expressed as negative of gradient of scalar potential function. The electric field components $E_i (E^*_i)$ can be related to the electric potentials $\Phi (\Phi^*)$ as

$$E_i = -\Phi_j, \quad E^*_i = -\Phi^*_j. \quad (2.39)$$

Applying Gauss divergence theorem and using of the equations (2.10), (2.38)-(2.39) in the equation (2.37), we get
\[ \delta \int_{t_0}^{t_f} dt \int_{V} W \, dV = \int_{t_0}^{t_f} dt \int_{S} (\sigma_{ij, i} \delta u_j + \sigma_{j}^{*} \delta u_j^{*} + D_{j}^{*} \delta \Phi + D_{j}^{*} \delta \Phi^{*}) \, n_j \, dS - \]
\[ \int_{t_0}^{t_f} dt \int_{V} (\sigma_{ij, i} \delta u_j + \sigma_{j}^{*} \delta u_j^{*} + D_{j}^{*} \delta \Phi + D_{j}^{*} \delta \Phi^{*}) \, dV. \]  
(2.40)

Making use of the equations (2.36) and (2.40), the equation (2.35) can be written as

\[ \int_{t_0}^{t_f} dt \int_{V} [(\sigma_{ij, i} - \rho_{11} \ddot{u}_j - \rho_{12} \ddot{u}_j^{*}) \delta u_j + (\sigma_{j}^{*} - \rho_{22} \ddot{u}_j^{*} - \rho_{12} \ddot{u}_j) \delta u_j^{*} + D_{j,j} \delta \Phi + D_{j,j}^{*} \delta \Phi^{*}] \, dV + \]
\[ \int_{t_0}^{t_f} dt \int_{S} [(\ddot{u}_j - \sigma_{ij, i} \bar{n}_i) \delta u_j + (\ddot{u}_j^{*} - \sigma_{j}^{*} \bar{n}_j) \delta u_j^{*} + (\ddot{c} - D_{j,j} \bar{n}_j) \delta \Phi + (\ddot{c}^{*} - D_{j,j}^{*} \bar{n}_j) \delta \Phi^{*}] \, dS = 0. \]
(2.41)

The above equation implies that inside \( V \),

\[ \sigma_{ij, i} - \rho_{11} \ddot{u}_j - \rho_{12} \ddot{u}_j^{*} = 0, \]  
(2.42)

\[ \sigma_{j}^{*} - \rho_{22} \ddot{u}_j^{*} - \rho_{12} \ddot{u}_j = 0, \]  
(2.43)

\[ D_{j,j} = 0, \]  
(2.44)

\[ D_{j,j}^{*} = 0. \]  
(2.45)

The equations (2.42)-(2.43) are the equations of motion for a PPS body in the absence of body forces. The equations (2.44)-(2.45) are equivalent to the Gauss equations in the theory of piezoelectricity.

In addition, the following boundary conditions should be specified on the surface \( S \) enclosing the region \( V \):

Either \( \delta u_j \) is arbitrary and \( \sigma_{ij} \bar{n}_i = \ddot{u}_j \), or \( u_j \) is prescribed so that \( \delta u_j = 0 \),
(2.46)

Either \( \delta u_j^{*} \) is arbitrary and \( \sigma_{j}^{*} \bar{n}_j = \ddot{u}_j^{*} \), or \( u_j^{*} \) is prescribed so that \( \delta u_j^{*} = 0 \),
(2.47)

Either \( \delta \Phi \) is arbitrary and \( D_{j} \bar{n}_j = \ddot{c} \), or \( \Phi \) is prescribed so that \( \delta \Phi = 0 \),
(2.48)

Either \( \delta \Phi^{*} \) is arbitrary and \( D_{j}^{*} \bar{n}_j = \ddot{c}^{*} \), or \( \Phi^{*} \) is prescribed so that \( \delta \Phi^{*} = 0 \).
(2.49)

The equations (2.42)-(2.49) taken together define a boundary value problem (BVP) in the theory of linear electro-elasticity of porous materials.
2.4 Uniqueness of Solution of a Boundary Value Problem

In this section, the uniqueness of a solution of a Boundary Value Problem (BVP) in the linear theory of electro-elasticity for PPM is established.

Let
\{ u_i^*, u_i^\prime, \sigma_{ij}^*, \sigma_{ij}'', \Phi^*, \Phi'''', E_i^*, E_i'''', D_j, D_j''', \} \text{ and } \{ u_i^*, u_i''', \sigma_{ij}^*, \sigma_{ij}'''', \Phi^*, \Phi'''''', E_i^*, E_i'''''', D_j, D_j'''''', \}

be two sets of solution of the following BVP

\[
\begin{align*}
\sigma_{ij,i} &= \rho_{11} \dddot{u}_j + \rho_{12} \dddot{u}_j', \\
\sigma_{ij,j}^* &= \rho_{12} \dddot{u}_j + \rho_{22} \dddot{u}_j', \\
D_j, j &= 0, \\
D_j^*, j &= 0; \\
\sigma_{ij} \dddot{n}_i &= \dddot{t}_j, \\
\sigma_{ij}^* \dddot{n}_i &= \dddot{t}_j^*, \\
D_j \dddot{n}_j &= \dddot{c}, \\
D_j^* \dddot{n}_j &= \dddot{c}^*.
\end{align*}
\]

We define the following:

\[
\begin{align*}
u_i &= u_i' - u_i''', \quad u_i^* = u_i''' - u_i''', \\
\Phi &= \Phi' - \Phi^*, \quad \Phi^* = \Phi''' - \Phi''', \\
E_i &= E_i' - E_i''', \\
E_i^* &= E_i'' - E_i'''', \\
\sigma_{ij} &= \sigma_{ij}' - \sigma_{ij}''', \\
\sigma^* &= \sigma^''' - \sigma'''', \\
D_j &= D_j' - D_j''', \\
D_j^* &= D_j'' - D_j''''.
\end{align*}
\]

(2.52)

As all the equations involved in the BVP are linear, therefore by the principle of superposition, the set
\[ \{ u_i, u_i^*, \sigma_{ij}, \sigma^*, \Phi, \Phi^*, E_i, E_i^*, D_j, D_j^* \} \]

is also a solution of the problem (2.50) satisfying the boundary conditions

\[
\begin{align*}
\sigma_{ij} \dddot{n}_i &= 0, \quad \sigma^* \dddot{n}_j = 0, \quad D_j \dddot{n}_j = 0, \quad D_j^* \dddot{n}_j = 0, \quad \text{on the surface } S. 
\end{align*}
\]

(2.53)

The equations (2.50a)-(2.50d) imply that

\[
\begin{align*}
(\sigma_{ij,i} - \rho_{11} \dddot{u}_j - \rho_{12} \dddot{u}_j') \dddot{u}_j &= 0, \\
(\sigma_{ij,j}^* - \rho_{12} \dddot{u}_j - \rho_{22} \dddot{u}_j') \dddot{u}_j &= 0, \\
\dot{D}_j, j \Phi &= 0, \\
\dot{D}_j^*, j \Phi^* &= 0.
\end{align*}
\]

(2.54)
Integration of these equations over the volume \( V \) gives
\[
\int_{V} \left[ (\sigma_{ij} - \rho_{11} \ddot{u}_j - \rho_{12} \ddot{u}_{ij}^* - \rho_{22} \ddot{u}_{ij}^* + \sigma_{ij}^* - \rho_{12} \ddot{u}_{ij} - \rho_{22} \ddot{u}_{ij}^* ) \right] dV = 0. \tag{2.55}
\]

On applying Gauss divergence theorem, we obtain
\[
\int_{S} (\sigma_{ij} \ddot{n}_j + \sigma_{ij}^* \ddot{n}_j - \dot{D}_j \Phi \ddot{n}_j - \dot{D}_j^* \Phi^* ) dS -
\int_{V} \left[ (\sigma_{ij} \ddot{u}_{ij} + \sigma_{ij}^* \ddot{u}_{ij}^* - \dot{D}_j \Phi \ddot{u}_{ij} - \dot{D}_j^* \Phi^* ) \right] dV -
\frac{1}{2} \int_{V} \frac{d}{dt} \left( \rho_{11} \ddot{u}_j + 2 \rho_{12} \ddot{u}_j \ddot{u}_{ij} + \rho_{22} \ddot{u}_j \ddot{u}_{ij} \right) dV = 0. \tag{2.56}
\]

Using the equations (2.31) and (2.53), the equation (2.56) can be written as
\[
\int_{V} (\sigma_{ij} \ddot{u}_{ij} + \sigma_{ij}^* \ddot{u}_{ij}^* - \dot{D}_j \Phi \ddot{u}_{ij} - \dot{D}_j^* \Phi^* ) dV + \int_{V} \frac{dK}{dt} dV = 0 .
\]
\[
\Rightarrow \int_{V} (\sigma_{ij} \ddot{e}_{ij} + \sigma_{ij}^* \ddot{e}_{ij}^* + \dot{D}_j E_j + \dot{D}_j^* E_j^* ) dV + \int_{V} \frac{dK}{dt} dV = 0 .
\]
\[
\Rightarrow \int_{V} \left[ (\sigma_{ij} \ddot{e}_{ij} + \sigma_{ij}^* \ddot{e}_{ij}^* - D_j \dot{E}_j - D_j^* \dot{E}_j^* ) + \frac{d}{dt} (D_j E_j + D_j^* E_j^* ) \right] dV + \int_{V} \frac{dK}{dt} dV = 0 .
\]
\[
\Rightarrow \int_{V} \left[ \frac{\partial W}{\partial e_{ij}} \ddot{e}_{ij} + \frac{\partial W}{\partial e_{ij}^*} \ddot{e}_{ij}^* + \frac{\partial W}{\partial E_j} \dot{E}_j + \frac{\partial W}{\partial E_j^*} \dot{E}_j^* + \frac{d}{dt} (D_j E_j + D_j^* E_j^* ) \right] dV + \int_{V} \frac{dK}{dt} dV = 0 .
\]
\[
\Rightarrow \int_{V} \frac{d}{dt} (W + D_j E_j + D_j^* E_j^* ) dV + \int_{V} \frac{dK}{dt} dV = 0 .
\]
\[
\Rightarrow \int_{V} (U + K) dV = \text{constant} . \tag{2.57}
\]

Using the equations (2.16) and (2.23)-(2.24), we can write
\[
U = \frac{1}{2} c_{ijkl} e_{ij} e_{kl} + \frac{1}{2} R \varepsilon^* \varepsilon^* + m_{ij} e_{ij} e_{ij} + \frac{1}{2} \varepsilon_{ij} E_i E_j + \frac{1}{2} \varepsilon_{ij}^* E_i^* E_j^* + A_{ij} E_i E_j . \tag{2.58}
\]

The internal energy function \( U \) of a system must be a positive definite quadratic form. \( K \) given by the equation (2.31) is also positive definite because \( \rho_{11}, \rho_{12} \) and \( \rho_{22} \) are non-negative quantities.
The quantities \( u_i, u_i^*, \dot{u}_i, \dot{u}_i^*, E_i \) and \( E_i^* \) are zero, initially, which implies that the constant in the equation (2.57) is zero. This implies that

\[
\int (U + K) \, dV = 0 .
\]  
\[ (2.59) \]

\[
\therefore K = U = 0 \text{ inside } V .
\]  
\[ (2.60) \]

\[
\Rightarrow \dot{u}_i = \dot{u}_i^* = 0 ,
\]  
\[ (2.61) \]

\[
\varepsilon_{ij} = \varepsilon_{**} = 0 ,
\]  
\[ (2.62a) \]

\[
E_i = E_i^* = 0 .
\]  
\[ (2.62b) \]

The equation (2.61) states that we are dealing with a static case and the equation (2.62a) means that mechanical deformation of the body is not present. Thus the displacements \( u_i, u_i^* \) represent a rigid body motion. Since the displacements are zero initially, therefore the mechanical displacements are unique.

The equation (2.62b) implies that

\[
\therefore \Phi = \Phi^* = \text{constant} .
\]

Hence the two solutions of the BVP defined by the equations (2.50)-(2.51) are identical with in a static rigid body displacement and a constant potential.

### 2.5 General Theorems

In this section, some general theorems for initial-boundary value problem in the linear theory of electro-elastodynamics for PPM are proved.

Consider a porous piezoelectric body occupying a regular region \( V \) in space, enclosed by a boundary \( S \). Let \( \overline{V} \) denotes the closure of \( V \).

Let

\[
u_i, u_i^*, \Phi, \Phi^* \in C^2(V \times T) \cap C^1(\overline{V} \times T) ,
\]

and

\[
\sigma_{ij}, \sigma^*, D_i, D_i^* \in C(\overline{V} \times T) , \text{ where } T = [0, \infty) .
\]  
\[ (2.63) \]

An initial-boundary value problem in the linear theory of electro-elastodynamics for a PPM can be defined as a set of following equations:
Equations of motion;
\[
\begin{align*}
\sigma_{ij, i} &= \rho_1 \ddot{u}_j + \rho_2 \dddot{u}_j, \\
\sigma_{ij}^* &= \rho_1 \ddot{u}_j + \rho_2 \dddot{u}_j, \\
D_{j, i} &= 0, \\
D_{j, i}^* &= 0;
\end{align*}
\]  \quad \text{in } V \times T.  \quad (2.64)

Constitutive equations;
\[
\begin{align*}
\sigma_{ij} &= c_{ijkl} \varepsilon_{kl} + m_{ij} \varepsilon^* - \varepsilon_{kij} E_k - \varepsilon_{kij}^* E_k^*, \\
\sigma^* &= m_{ij} \varepsilon_{ij} + R \varepsilon^* - \varepsilon_{ij} E_i - \varepsilon_{ij}^* E_i^*, \\
D_i &= e_{ijk} \varepsilon_{jk} + \varepsilon_{ij} \varepsilon^* + \varepsilon_{ij} E_j + A_{ij} E_j^*, \\
D_i^* &= \varepsilon_{ijk} \varepsilon_{jk} + \varepsilon_{ij} \varepsilon^* + A_{ij} E_j + \varepsilon_{ij}^* E_j^*;
\end{align*}
\]  \quad \text{in } \bar{V} \times T  \quad (2.65)

Boundary Conditions;
On the surface \( S \);
Either of following three sets of boundary conditions can be prescribed:

(a) First boundary value problem;
In this case, the components of surface tractions and normal components of electrical displacements are prescribed.
\[
\begin{align*}
\sigma_{ij} \vec{n}_j &= \vec{t}_j, \\
\sigma^* \vec{n}_j &= \vec{t}_j^*, \\
D_{j, i} \vec{n}_j &= \vec{c}, \\
D_{j, i}^* \vec{n}_j &= \vec{c}^*.
\end{align*}
\]  \quad \text{on the surface } S.  \quad (2.66)

(b) Second boundary value problem;
In this case, the mechanical displacements and electric potentials are prescribed on the boundary \( S \).
\[
\begin{align*}
\vec{u}_j &= \vec{\ddot{u}}_j, \\
\vec{u}_j^* &= \vec{\ddot{u}}_j^*, \\
\Phi &= \vec{\Phi}, \\
\Phi^* &= \vec{\Phi}^*;
\end{align*}
\]  \quad \text{on the surface } S.  \quad (2.67)
(c) Mixed boundary value problem;

In this case, the boundary conditions are

\[
\sigma_{ij} \tilde{n}_i = \tilde{t}_j, \quad \text{on} \quad S_\sigma, \quad (2.68a)
\]
\[
u_j = \tilde{u}_j, \quad \text{on} \quad S_u, \quad (2.68b)
\]
\[
\sigma^* \hat{n}_j = \hat{t}_j^*, \quad \text{on} \quad S_{\sigma^*}, \quad (2.68c)
\]
\[
u_j^* = \hat{u}_j^*, \quad \text{on} \quad S_{u^*}, \quad (2.68d)
\]
\[
D_j \hat{n}_j = \tilde{c}, \quad \text{on} \quad S_D, \quad (2.68e)
\]
\[
\Phi = \tilde{\Phi}, \quad \text{on} \quad S_\Phi, \quad (2.68f)
\]
\[
D_j^* \hat{n}_j = \hat{c}^*, \quad \text{on} \quad S_{D^*}, \quad (2.68g)
\]
\[
\Phi^* = \tilde{\Phi}^*, \quad \text{on} \quad S_{\Phi^*}; \quad (2.68h)
\]

where \( S_u, S_u^*, S_\sigma, S_\sigma^*, S_\Phi, S_{\Phi^*}, S_D, S_{D^*} \) are the parts of the surface \( S \) over which \( u, u^*, \sigma, \sigma^*, \Phi, \Phi^*, D \) and \( D^* \) are prescribed respectively and

\[
S = S_\sigma \cup S_{\sigma^*} \cup S_u \cup S_u^* \cup S_\Phi \cup S_{\Phi^*} \cup S_D \cup S_{D^*},
\]
and

\[
S_u \cap S_\sigma = \emptyset, \quad S_u^* \cap S_{\sigma^*} = \emptyset, \quad S_\Phi \cap S_D = \emptyset, \quad S_{\Phi^*} \cap S_{D^*} = \emptyset. \quad (2.69)
\]

Initial conditions:

\[
u_i(x, 0) = \tilde{u}_i(x), \quad u_i^*(x, 0) = \tilde{u}_i^*(x), \quad \tilde{u}_i(x, 0) = \hat{u}_i(x), \quad \hat{u}_i^*(x, 0) = \tilde{u}_i^*(x), \quad x \in \tilde{V}, \quad (2.70)
\]

where \( \tilde{u}_i, \tilde{u}_i^*, \tilde{\Phi}, \hat{\Phi}^*, \tilde{t}_i, \hat{t}_i^*, \hat{c}, \hat{c}^*, \tilde{u}_i, \tilde{u}_i^*, \hat{u}_i, \) and \( \hat{u}_i^* \) are the given functions.

The components of traction vectors and normal components of electric displacements can be denoted as

\[
\sigma_{ij} \tilde{n}_i = \tilde{t}_j, \quad \sigma^* \hat{n}_j = \hat{t}_j^*, \quad D_j \hat{n}_j = \hat{c}, \quad D_j^* \hat{n}_j = \hat{c}^*. \quad (2.71)
\]

Let us consider two external data systems

\[
L^{(\alpha)} = \{u_i^{(\alpha)}, u_i^{\ast (\alpha)}, \tilde{t}_i^{(\alpha)}, \hat{t}_i^{(\alpha)}, \tilde{\Phi}^{(\alpha)}, \hat{\Phi}^{\ast (\alpha)} \}, \quad (\alpha = 1, 2).
\]

Let \( A^{(\alpha)} = \{u_i^{(\alpha)}, u_i^{\ast (\alpha)}, \sigma_{ij}^{(\alpha)}, \sigma_{ij}^{\ast (\alpha)}, D_i^{(\alpha)}, D_i^{\ast (\alpha)}, \Phi^{(\alpha)}, \Phi^{\ast (\alpha)}, E_i^{(\alpha)}, E_i^{\ast (\alpha)} \} \) be a solution of an initial-boundary value problem corresponding to the given data system \( L^{(\alpha)} \).
The components of traction vectors and normal components of electric displacements corresponding to \( A^{(\alpha)} \) can be written as

\[
\sigma_{ij}^{(\alpha)} \hat{n}_j = \tilde{t}_j^{(\alpha)}, \quad \sigma^n_{ij} \hat{n}_j = \tilde{c}_j^{(\alpha)}, \quad D_{ij}^{(\alpha)} \hat{n}_j = \tilde{c}^{(\alpha)}, \quad D^n_{ij} \hat{n}_j = \tilde{c}^n_{(\alpha)}.
\]  

(2.72)

The Riemann convolution of two scalar functions \( u(x,t) \) and \( v(x,t) \) is defined as

\[
u \ast v = [u \ast v](x,t) = \int_0^t u(x,t-s) \, v(x,s) \, ds.
\]  

(2.73)

**Theorem 2.1 Dynamic Reciprocal Theorem**

The dynamic reciprocal theorem, sometimes also known as the Betti-Rayleigh Reciprocal theorem, presents a relation between two elastodynamic states of the same body.

If \( A^{(\alpha)} \) is a solution of a initial-boundary value problem corresponding to the external data system \( L^{(\alpha)} \) \( (\alpha = 1, 2) \), then

\[
\int [\gamma * (\tilde{t}_i^{(1)} * u_i^{(2)} + \tilde{c}^{(2)} * \Phi^{(2)}) + \rho_{11} f_i^{(1)} * u_i^{(2)} + \rho_{12} (f_i^{(1)} * u_i^{(2)} + F_i^{(1)} * u_i^{(2)}) + \rho_{22} F_i^{(1)} * u_i^{(2)}] \, dV +
\int [\gamma * (\tilde{t}_i^{(2)} * u_i^{(1)} + \tilde{c}^{(1)} * \Phi^{(1)}) + \rho_{11} f_i^{(2)} * u_i^{(1)} + \rho_{12} (f_i^{(2)} * u_i^{(1)} + F_i^{(2)} * u_i^{(1)}) + \rho_{22} F_i^{(2)} * u_i^{(1)}] \, dV,
\]

where

\[
f_i^{(\alpha)} = \tilde{u}_i^{(\alpha)} + \dot{u}_i^{(\alpha)}, \quad F_i^{(\alpha)} = \tilde{c}_i^{(\alpha)} + t \dot{u}_i^{(\alpha)}, \quad \gamma(t) = t, \, t \in T.
\]  

(2.74)

**Proof**

First we prove that

\[
E_{\alpha\beta}(r,s) = E_{\beta\alpha}(s,r), \quad r, s \in T,
\]  

(2.75)
\[ E_{\alpha\beta}(r, s) = \int \left[ t_i^{(\alpha)}(x, r) \ u_i^{(\beta)}(x, s) + \tilde{t}_i^{(\alpha)}(x, r) \ u_i^{(\beta)}(x, s) + \right. \]
\[ \left. \tilde{c}^{(\alpha)}(x, r) \ \Phi^{(\beta)}(x, s) + \tilde{c}^{(\alpha)}(x, r) \ \Phi^{(\beta)}(x, s) \right] \, dS - \]
\[ \int \left[ \rho_{11} \tilde{u}_i^{(\alpha)}(x, r) \ u_i^{(\beta)}(x, s) + \rho_{22} \tilde{u}_i^{(\alpha)}(x, r) \ u_i^{(\beta)}(x, s) + \right. \]
\[ \left. \rho_{12} (\tilde{u}_i^{(\alpha)}(x, r) \ u_i^{(\beta)}(x, s) + u_i^{(\beta)}(x, s) \ \tilde{u}_i^{(\alpha)}(x, r)) \right] \, dV. \]  

(2.77)

Consider the expression
\[ I_{\alpha\beta}(x, r, s) = \sigma_{ij}^{(\alpha)}(x, r) \ u_{ij}^{(\beta)}(x, s) + \sigma_{ij}^{(\alpha)}(x, r) \ u_{ij}^{(\beta)}(x, s) - \]
\[ D_i^{(\alpha)}(x, r) \ E_i^{(\beta)}(x, s) - D_i^{(\alpha)}(x, r) \ E_i^{(\beta)}(x, s). \]  

(2.78)

Using the equations (2.38) and (2.65) in the equation (2.78), we obtain
\[ I_{\alpha\beta}(x, r, s) = \left[ c_{ijkl} \ u_{ij}^{(\alpha)}(x, r) + m_{ij} \ u_{ij}^{(\alpha)}(x, r) - e_{kl}^{(\alpha)}(x, r) - \zeta_{kl}^{(\alpha)}(x, r) \right] u_{ij}^{(\beta)}(x, s) + \]
\[ \left[ m_{ij} \ u_{ij}^{(\alpha)}(x, r) + R \ u_{ij}^{(\alpha)}(x, r) - \zeta_{ij}^{(\alpha)}(x, r) + e_{ij}^{(\alpha)}(x, r) \right] u_{ij}^{(\beta)}(x, s) - \]
\[ \left[ e_{ijk} \ u_{ij}^{(\alpha)}(x, r) + \zeta_{ij}^{(\alpha)}(x, r) + \zeta_{ij}^{(\alpha)}(x, r) + A_{ij} \ E_i^{(\alpha)}(x, r) + \right] \ E_i^{(\beta)}(x, s) - \]
\[ \left[ \zeta_{ij} \ u_{ij}^{(\alpha)}(x, r) + e_{ij}^{(\alpha)}(x, r) + A_{ij} \ E_i^{(\alpha)}(x, r) + \zeta_{ij}^{(\alpha)}(x, r) \right] \ E_i^{(\beta)}(x, s). \]

\[ = \sigma_{ij}^{(\beta)}(x, s) \ u_{ij}^{(\alpha)}(x, r) + \sigma_{ij}^{(\alpha)}(x, s) \ u_{ij}^{(\beta)}(x, r) - \]
\[ D_i^{(\beta)}(x, s) \ E_i^{(\alpha)}(x, r) - D_i^{(\alpha)}(x, s) \ E_i^{(\beta)}(x, r). \]

\[ \Rightarrow I_{\alpha\beta}(x, r, s) = I_{\beta\alpha}(x, s, r). \]  

(2.79)

Using the equations (2.64), the equation (2.78) can be written as
\[ I_{\alpha\beta}(x, r, s) = \left[ \sigma_{ij}^{(\alpha)}(x, r) \ u_{ij}^{(\beta)}(x, s) + \sigma_{ij}^{(\alpha)}(x, r) \ u_{ij}^{(\beta)}(x, s) + \right. \]
\[ \left. D_i^{(\alpha)}(x, r) \ \Phi^{(\beta)}(x, s) + D_i^{(\alpha)}(x, r) \ \Phi^{(\beta)}(x, s) \right] \right|_j - \]
\[ \left[ \rho_{11} \tilde{u}_i^{(\alpha)}(x, r) \ u_i^{(\beta)}(x, s) + \rho_{12} \tilde{u}_i^{(\alpha)}(x, r) \ u_i^{(\beta)}(x, s) + \right. \]
\[ \left. \rho_{12} \ u_i^{(\alpha)}(x, s) \ \tilde{u}_i^{(\alpha)}(x, r) + \rho_{22} \ u_i^{(\alpha)}(x, s) \ \tilde{u}_i^{(\alpha)}(x, r) \ u_i^{(\beta)}(x, s) \right]. \]  

(2.80)

Integrating both sides of the above equation over the volume \( V \) and applying the Gauss divergence theorem and using the equations (2.72) and (2.77), we obtain
\[ \int I_{\alpha\beta}(x, r, s) \, dV = E_{\alpha\beta}(r, s). \]  

(2.81)

The equations (2.79) and (2.81) imply that
\[ E_{\alpha\beta}(r, s) = E_{\beta\alpha}(s, r). \]  

(2.82)
Using the equations (2.75) and (2.77) in the equation (2.83), we obtain

\[
\int \left[ \mathcal{I}_i^{(1)} * u_i^{(2)} + \mathcal{I}_i^{(1)} * u_i^{(2)} + \mathcal{I}_i^{(1)} * \Phi^{(1)} + \mathcal{I}_i^{(1)} * \Phi^{(2)} \right] dS -
\]

\[
\int [\rho_{11} \mathcal{I}_i^{(1)} * u_i^{(2)} + \rho_{12} (\mathcal{I}_i^{(1)} * u_i^{(2)} + \mathcal{I}_i^{(1)} * u_i^{(2)}) + \rho_{22} \mathcal{I}_i^{(1)} * u_i^{(2)} ] dV
\]

\[
= \int \left[ \mathcal{I}_i^{(2)} * u_i^{(1)} + \mathcal{I}_i^{(2)} * u_i^{(1)} + \mathcal{I}_i^{(2)} * \Phi^{(1)} + \mathcal{I}_i^{(2)} * \Phi^{(2)} \right] dS -
\]

\[
\int [\rho_{11} \mathcal{I}_i^{(2)} * u_i^{(1)} + \rho_{12} (\mathcal{I}_i^{(2)} * u_i^{(1)} + \mathcal{I}_i^{(2)} * u_i^{(1)}) + \rho_{22} \mathcal{I}_i^{(2)} * u_i^{(1)} ] dV.
\]

Taking convolution on both sides of the equation (2.84) with the function \( \gamma \), we get

\[
\int \gamma * [\mathcal{I}_i^{(1)} * u_i^{(2)} + \mathcal{I}_i^{(1)} * u_i^{(2)} + \mathcal{I}_i^{(1)} * \Phi^{(1)} + \mathcal{I}_i^{(1)} * \Phi^{(2)} ] dS -
\]

\[
\int [\rho_{11} (\gamma * \mathcal{I}_i^{(1)} * u_i^{(2)} + \rho_{12} (\gamma * \mathcal{I}_i^{(1)} * u_i^{(2)} + \gamma * \mathcal{I}_i^{(1)} * u_i^{(2)}) + \rho_{22} (\gamma * \mathcal{I}_i^{(1)} * u_i^{(2)} ) ] dV
\]

\[
= \int \gamma * [\mathcal{I}_i^{(2)} * u_i^{(1)} + \mathcal{I}_i^{(2)} * u_i^{(1)} + \mathcal{I}_i^{(2)} * \Phi^{(1)} + \mathcal{I}_i^{(2)} * \Phi^{(2)} ] dS -
\]

\[
\int [\rho_{11} (\gamma * \mathcal{I}_i^{(2)} * u_i^{(1)} + \rho_{12} (\gamma * \mathcal{I}_i^{(2)} * u_i^{(1)} + \gamma * \mathcal{I}_i^{(2)} * u_i^{(1)}) + \rho_{22} (\gamma * \mathcal{I}_i^{(2)} * u_i^{(1)} ) ] dV.
\]

(2.85)

Using definition of Riemann convolution, we get

\[
\gamma * \mathcal{I}_i^{(1)} * u_i^{(2)} = (\gamma * \mathcal{I}_i^{(1)} ) * u_i^{(2)} = [u_i^{(1)}( \mathbf{x}, t) - u_i^{(1)}( \mathbf{x}, 0) - t \mathcal{I}_i^{(1)}( \mathbf{x}, 0) ] * u_i^{(2)}( \mathbf{x}, t).
\]

Using the equations (2.70) and (2.74), the above equation can be written as

\[
\gamma * \mathcal{I}_i^{(1)} * u_i^{(2)} = u_i^{(1)} * u_i^{(2)} - \mathcal{I}_i^{(1)} * u_i^{(2)} - t \mathcal{I}_i^{(1)} * u_i^{(2)} = u_i^{(1)} * u_i^{(2)} - f_i^{(1)} * u_i^{(2)}.
\]

(2.86)

Similarly, we can write

\[
\gamma * \mathcal{I}_i^{(1)} * u_i^{(2)} = u_i^{(1)} * u_i^{(2)} - \mathcal{I}_i^{(1)} * u_i^{(2)} - t \mathcal{I}_i^{(1)} * u_i^{(2)} = u_i^{(1)} * u_i^{(2)} - f_i^{(1)} * u_i^{(2)}.
\]

(2.87)

\[
\gamma * \mathcal{I}_i^{(1)} * u_i^{(2)} + \gamma * \mathcal{I}_i^{(1)} * u_i^{(2)}
\]

\[
[u_i^{(1)} * u_i^{(2)} + u_i^{(1)} * u_i^{(2)}] - f_i^{(1)} * u_i^{(2)} - f_i^{(1)} * u_i^{(2)}.
\]

(2.88)
Making use of the equations (2.86)-(2.88) in the equation (2.85), the stated result is established. This result is the generalized version of the theorem of reciprocity in the linear theory of electro-elastodynamics for porous piezoelectric materials.

**Theorem 2.2**

If

\[
G(r, s) = \int [\tilde{t}_i(x, r) u_i(x, s) + \tilde{t}_i^*(x, r) u^*_i(x, s) + \tilde{c} (x, r) \Phi(x, s) + \tilde{c}^* (x, r) \Phi^*(x, s)] \, dS,
\]

\[r, s \in \mathcal{T},\]

then show that

\[
\frac{d}{dt} \left[ \rho_{11} (u_j)^2 + \rho_{12} u_j u^*_j + \rho_{22} (u^*_j)^2 \right] \, dV
\]

\[
= \int (G(t+s,t-s) - G(t-s,t+s)) \, ds + \int \rho_{11} [u_i(x,0) \dot{u}_i(x, 2t) + u_i(x, 2t) \dot{u}_i(x, 0)] \, dV +
\]

\[
\int \rho_{12} [u^*_i(x,0) \dot{u}_i(x, 2t) + u^*_i(x, 2t) \dot{u}_i(x, 0) + u_i(x, 0) \dot{u}^*_i(x, 2t) + u_i(x, 2t) \dot{u}^*_i(x, 0)] \, dV +
\]

\[
\int \rho_{22} [u^*_i(x,0) \ddot{u}_i(x, 2t) + u^*_i(x, 2t) \ddot{u}_i(x, 0)] \, dV,
\]

(2.89)

**Proof**

Using the equation (2.82), we can write

\[
\int_0^t E_{11}(t+s,t-s) \, ds = \int_0^t E_{11}(t-s,t+s) \, ds.
\]

(2.91)

Using the equation (2.77), we can write

\[
\int_0^t E_{11}(t+s,t-s) \, ds =
\]

\[
\int_0^t \int [\tilde{t}_i^{(1)}(x, t+\tau) u_i^{(1)}(x, t-s) + \tilde{t}_i^{*(1)}(x, t-s) u_i^{*(1)}(x, t+\tau) +
\]

\[
\tilde{c}^{(1)}(x, t+\tau) \Phi^{(1)}(x, t-s) + \tilde{c}^{*(1)}(x, t+s) \Phi^{*(1)}(x, t-s)] \, dS \, d\tau -
\]

\[
\int_{V} \left[ \rho_{11} \ddot{u}_i^{(1)}(x, t+s) u_i^{(1)}(x, t-s) + \rho_{22} \ddot{u}_i^{*(1)}(x, t+s) u_i^{*(1)}(x, t-s) +
\]

\[
\rho_{12} \ddot{u}_i^{*(1)}(x, t+s) u_i^{*(1)}(x, t-s) + u_i^{(1)}(x, t-s) \dot{u}_i^{*(1)}(x, t+s)] \, dV \, ds.
\]

(2.92)
Replacing \( u_i^{(1)} \), \( u_i^{(n)} \), \( \tilde{v}_i^{(1)} \), \( \tilde{v}_i^{(n)} \), \( \tilde{c}^{(1)} \), \( \tilde{c}^{(n)} \) by \( u_i, u_i^s, \tilde{v}_i, \tilde{v}_i^s, \tilde{c}, \tilde{c}^s \) in the above equation and using the equation (2.89), we obtain

\[
\int_0^t E_{11}(t+s,t-s) \, ds = \int_0^t G(t+s,t-s) \, ds - \\
\int_0^t \int_0^V \left[ \rho_{11} \tilde{u}_i(x,t+s) u_i(x,t-s) + \rho_{22} \tilde{u}_i^s(x,t+s) u_i^s(x,t-s) + \rho_{12} (\tilde{u}_i(x,t+s) u_i^s(x,t-s) + u_i(x,t-s) \tilde{u}_i^s(x,t+s)) \right] \, dV \, ds.
\]

Similarly, we obtain

\[
\int_0^t E_{11}(t-s,t+s) \, ds = \int_0^t G(t-s,t+s) \, ds - \\
\int_0^t \int_0^V \left[ \rho_{11} \tilde{u}_i(x,t-s) u_i(x,t+s) + \rho_{22} \tilde{u}_i^s(x,t-s) u_i^s(x,t+s) + \rho_{12} (\tilde{u}_i(x,t-s) u_i^s(x,t+s) + u_i(x,t+s) \tilde{u}_i^s(x,t-s)) \right] \, dV \, ds.
\]

The equations (2.91), (2.93) and (2.94) imply that

\[
\int_0^t [G(t+s,t-s) - G(t-s,t+s)] \, ds \\
= \int_0^t \int_0^V \left[ \rho_{11} \left( \tilde{u}_i(x,t+s) u_i(x,t-s) - \tilde{u}_i(x,t-s) u_i(x,t+s) \right) \right] \, dV + \\
\int_0^t \int_0^V \left[ \rho_{12} \left( \tilde{u}_i^s(x,t+s) u_i^s(x,t-s) + u_i(x,t-s) \tilde{u}_i^s(x,t+s) - \tilde{u}_i(x,t-s) u_i^s(x,t+s) \right) \right] \, dV + \\
\int_0^t \int_0^V \left( \rho_{22} \left( \tilde{u}_i^s(x,t+s) u_i^s(x,t-s) - \tilde{u}_i^s(x,t-s) u_i^s(x,t+s) \right) \right) \, dV.
\]

Some of the integrals in the equation (2.95) are solved as follows.

\[
\int_0^t \left[ \tilde{u}_i(x,t+s) u_i(x,t-s) - \tilde{u}_i(x,t-s) u_i(x,t+s) \right] \, ds \\
= u_i(x,0) \dot{u}_i(x,2t) + u_i(x,2t) \dot{u}_i(x,0) - 2 u_i(x,t) \ddot{u}_i(x,t),
\]
Making use of the equations (2.96)-(2.98) in the equation (2.95), the required result (2.90) is obtained.

In this sub-section, the uniqueness of the solution of the initial-boundary value problem in the linear theory of electro-elastodynamics for PPM is proved without assumption of positive definiteness of $\frac{1}{2}c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$. This fact finds its importance in the case of small disturbances superimposed upon an initially stressed state.

**Theorem 2.3 (Uniqueness Theorem)**

Let $\{u^{(\alpha)}_i, \sigma^{(\alpha)}_{ij}, \Phi^{(\alpha)}, \Phi^{*(\alpha)}, E^{(\alpha)}_i, E^{*(\alpha)}_i, D^{(\alpha)}_i, D^{*(\alpha)}_i\}; (\alpha = 1, 2)$ be any two sets satisfying the equations (2.64)-(2.65) with the same boundary and initial conditions and

$$
\begin{align*}
    u^{(0)}_i &= u^{(1)}_i - u^{(2)}_i; \quad u^{*(0)}_i = u^{*(1)}_i - u^{*(2)}_i; \quad \Phi^{(0)} = \Phi^{(1)} - \Phi^{(2)}; \quad \Phi^{*(0)} = \Phi^{*(1)} - \Phi^{*(2)}; \\
    \sigma^{(0)}_{ij} &= \sigma^{(1)}_{ij} - \sigma^{(2)}_{ij}; \quad \sigma^{*(0)} = \sigma^{*(1)} - \sigma^{*(2)}; \quad D^{(0)}_i = D^{(1)}_i - D^{(2)}_i; \quad D^{*(0)}_i = D^{*(1)}_i - D^{*(2)}_i; \\
    E^{(0)}_i &= E^{(1)}_i - E^{(2)}_i; \quad E^{*(0)}_i = E^{*(1)}_i - E^{*(2)}_i,
\end{align*}
$$

then

$$
\begin{align*}
    u^{(0)}_i &= 0; \quad u^{*(0)}_i = 0; \quad \Phi^{(0)} = \text{constant}; \quad \Phi^{*(0)} = \text{constant on } \bar{V} \times T.
\end{align*}
$$
Proof

\[ \{ u_i^{(1)}, u_i^{s(1)}, \sigma_{ij}^{(1)}, \sigma^{s(1)}, \Phi^{(1)}, \Phi^{s(1)}, E_i^{(1)}, E_i^{s(1)}, D_j^{(1)}, D_j^{s(1)} \} \]

and

\[ \{ u_i^{(2)}, u_i^{s(2)}, \sigma_{ij}^{(2)}, \sigma^{s(2)}, \Phi^{(2)}, \Phi^{s(2)}, E_i^{(2)}, E_i^{s(2)}, D_j^{(2)}, D_j^{s(2)} \} \]

are the solutions of the equations (2.64) satisfying boundary conditions either (2.66) or (2.67) or (2.68) and initial conditions (2.70).

\[ \{ u_i^{(0)}, u_i^{s(0)}, \sigma_{ij}^{(0)}, \sigma^{s(0)}, \Phi^{(0)}, \Phi^{s(0)}, E_i^{(0)}, E_i^{s(0)}, D_j^{(0)}, D_j^{s(0)} \} \] is also a solution of (2.64) satisfying the boundary conditions:

(a) \( \sigma_{ij}^{(0)} \hat{n}_i = 0, \sigma^{s(0)} \hat{n}_j = 0, D_j^{(0)} \hat{n}_j = 0, D_j^{s(0)} \hat{n}_j = 0 \); \hspace{1cm} (2.100)

or

(b) \( u_j^{(0)} = 0, u_j^{s(0)} = 0, \Phi^{(0)} = 0, \Phi^{s(0)} = 0 \); \hspace{1cm} (2.101)

or

(c) \[ \sigma_{ij}^{(0)} \hat{n}_i = 0, \text{ on } S_{\sigma}, \] \hspace{1cm} (2.102a)
\[ u_j^{(0)} = 0, \text{ on } S_u, \] \hspace{1cm} (2.102b)
\[ \sigma^{s(0)} \hat{n}_j = 0, \text{ on } S_{\sigma'}, \] \hspace{1cm} (2.102c)
\[ u_j^{s(0)} = 0, \text{ on } S_{u'}, \] \hspace{1cm} (2.102d)
\[ D_j^{(0)} \hat{n}_j = 0, \text{ on } S_D, \] \hspace{1cm} (2.102e)
\[ \Phi^{(0)} = 0, \text{ on } S_\Phi, \] \hspace{1cm} (2.102f)
\[ D_j^{s(0)} \hat{n}_j = 0, \text{ on } S_{D'}, \] \hspace{1cm} (2.102g)
\[ \Phi^{s(0)} = 0, \text{ on } S_{\Phi'}, \] \hspace{1cm} (2.102h)

and initial conditions:

\[ u_i^{(0)}(x, 0) = 0, u_i^{s(0)}(x, 0) = 0, \dot{u}_i^{(0)}(x, 0) = 0, \dot{u}_i^{s(0)}(x, 0) = 0, \] \hspace{1cm} (2.103)

\[ x \in \bar{V}. \]
Using the equation (2.90), we can write

\[
\frac{d}{dt} \int [\rho_{11} u_i^{(0)} + \rho_{12} u_i^{(0)} u_i^{(0)} + \rho_{22} (u_i^{(0)})^2] \, dV = 
\int [G(t+s,t-s) - G(t-s,t+s)] \, ds + 
\int \rho_1 [u_i^{(0)}(x,0) \dot{u}_i^{(0)}(x,t) + u_i^{(0)}(x,2t) \dot{u}_i^{(0)}(x,0)] \, dV + 
\int \rho_2 [u_i^{(0)}(x,0) \dot{u}_i^{(0)}(x,2t) + u_i^{(0)}(x,2t) \dot{u}_i^{(0)}(x,0)] \, dV + 
\int \rho_2 [u_i^{(0)}(x,0) \dot{u}_i^{(0)}(x,2t) + u_i^{(0)}(x,2t) \dot{u}_i^{(0)}(x,0)] \, dV.
\] (2.104)

Making use of the equation (2.89) and boundary and initial conditions in the above equation, we obtain

\[
\frac{d}{dt} \int [\rho_{11} u_i^{(0)} + \rho_{12} u_i^{(0)} u_i^{(0)} + \rho_{22} (u_i^{(0)})^2] \, dV = 0.
\] (2.105)

\[
\Rightarrow \int [\rho_{11} u_i^{(0)} + \rho_{12} u_i^{(0)} u_i^{(0)} + \rho_{22} (u_i^{(0)})^2] \, dV = \text{constant}.
\]

\[
\therefore \quad u_i^{(0)} \text{ and } u_i^{(0)} \text{ are zero initially}
\]

\[
\therefore \quad \text{constant} = 0.
\] (2.106)

\[
\Rightarrow u_i^{(0)} = u_i^{(0)} = 0.
\]

\[
\Rightarrow u_i^{(1)} = u_i^{(2)} \text{ and } u_i^{(1)} = u_i^{(2)}.
\]

Hence \( u_i \) and \( u_i^* \) are unique.

Making use of the equation (2.106) in the equations (2.65), we obtain

\[
D_i^{(0)} = \xi_{ij} E_j^{(0)} + A_{ij} E_j^{(0)},
\] (2.107)

\[
D_i^{(1)} = A_{ij} E_j^{(0)} + \xi_{ij} E_j^{(0)}.
\] (2.108)
\[ \int \left[ E_i^{(0)} D_j^{(0)} + E_i^{*(0)} D_j^{*(0)} \right] dV = \]
\[ \int \left[ \xi_{ij} E_i^{(0)} E_j^{(0)} + \xi_{ij}^{*} E_i^{*(0)} E_j^{*(0)} + A_{ji} E_i^{(0)} E_j^{*(0)} + A_{ji} E_i^{*(0)} E_j^{(0)} \right] dV. \] (2.109)

Also, we have
\[ \int \left[ E_i^{(0)} D_i^{(0)} + E_i^{*(0)} D_i^{*(0)} \right] dV = \]
\[ \int \left[ D_{i,i}^{(0)} \Phi^{(0)} + D_{i,i}^{*(0)} \Phi^{*(0)} \right] dV - \int \left[ D_{i,i}^{(0)} \Phi^{(0)} + D_{i,i}^{*(0)} \Phi^{*(0)} \right] \hat{n}_i \, dS. \] (2.110)

Making use of the equations (2.64) and boundary conditions in the above equation, we obtain
\[ \int \left[ E_i^{(0)} D_i^{(0)} + E_i^{*(0)} D_i^{*(0)} \right] dV = 0. \] (2.111)

The equations (2.109) and (2.111) imply that
\[ \int \left[ \xi_{ij} E_i^{(0)} E_j^{(0)} + \xi_{ij}^{*} E_i^{*(0)} E_j^{*(0)} + A_{ji} E_i^{(0)} E_j^{*(0)} + A_{ji} E_i^{*(0)} E_j^{(0)} \right] dV = 0. \]
\[ \Rightarrow E_i^{(0)} = E_i^{*(0)} = 0. \]
\[ \Rightarrow \Phi^{(0)} = \Phi^{*(0)} = \text{constant}. \]

Hence the uniqueness of solution of an initial boundary value problem in the linear theory of electro-elastodynamics for PPM is proved.

2.6 Conclusion

The linear theory of elasticity for porous piezoelectric materials is established. The constitutive equations, equations of motion and boundary conditions are derived for anisotropic porous piezoelectric materials. Some standard theorems of classical elasticity are generalized for the linear theory of elasticity for porous piezoelectric materials. The uniqueness theorem and the reciprocity theorem are also proved for the linear theory of elasticity for a porous piezoelectric body. The uniqueness of solution of initial-boundary value problem in the linear theory of electro-elastodynamics for porous piezoelectric materials is proved.