CHAPTER III

LINEAR PROGRAMMING APPROACH FOR COST AND RISK
MINIMIZATION OF CSP-1.
CHAPTER III

LINEAR PROGRAMMING APPROACH FOR COST AND RISK
MINIMIZATION OF CSP - I

3.1 INTRODUCTION:

In chapter II, we considered the problem of optimal design of inspection plans for continuously manufactured items taking L consecutively produced items as a sample and tried to find out the optimal screening policy by skipping over certain number of successive samples when there was good indication that the process was running satisfactorily. The costs associated with screening policy consisted of cost of testing, cost of passing defective items and cost of taking wrong decisions. The plan considered above was an extension of the procedure A of the original plan of Dodge (1943) described below:

(i) At the outset, inspect 100 % of the units consecutively as produced and continue such inspection until i units in succession are found clear of defects.

(ii) When i units in succession are found clear of defects, discontinue 100 % inspection, and inspect only a fraction f of the units, selecting individual sample units one at a time from the flow of product, in such a manner as to assure an unbiased sample.
(iii) If a sample unit is found defective, revert immediately to a 100% inspection of succeeding units and continue until again i units in succession are found clear of defects.

(iv) Correct or replace with good units, all defective units found.

For the above plan among the things, Dodge provided the charts for determining values of f and i for a given AOQL.

Anscombe (1958) developed cost model for the continuously sampling plan (CSP) procedure of Dodge and tried to obtain f and i which determine the plan uniquely by minimizing the average total cost per item for operating the plan, assuming that the process fraction defective p is constant. For simplicity he took $f = 1/n$, i.e. assumed that only the nth unit is inspected out of every n consecutively produced items. He also considered the case of varying p by taking its few numerical values and calculating the pair $(n, i)$, using certain empirical relationships between excess average cost and the plan parameters. He recommended the use of uniform prior for p for further investigation.

Agrawal (1980) extended the work of Anscombe (1958) by considering beta prior for p. He prepared extensive tables giving excess average cost for different combinations
of \((f, l)\). His conclusion regarding the optimal choice of the plan was based on the empirical results.

Also, Agrawal (1980) following Marigutti (1955) calculated numerically the risks associated with the inspection plan as the pay-off entries and tried to predict about the optimal choice of the sampling plan. He assumed that the consecutively manufactured items are submitted for inspection in consecutive lots of varying sizes. In the background of this plan was the procedure B of Dodge (1943) for continuous inspection.

In the present chapter, we have first considered the cost-function proposed by Anscombe (1958) under uniform prior and applied Two-Stage Linear Programming approach to minimize the aggregate of excess average costs of inspection yielding optimal choice of \(n\) and \(l\). The case when \(p\) is fixed can be obtained as a special case. Secondly, game theoretic approach using linear programming has been applied to find out the optimal plan for continuous inspection taking the risk for different \(p\) as pay-off entry. Illustrative examples have been included to explain the application of the proposed procedure.

3.2 THE COST MODEL:

For the CSP-1 procedure A, mentioned in the previous section, cost model proposed by Anscombe (1958),
in brief, is described below in order to avoid excessive references. Let

\[ k = \text{cost of inspecting per article} \]
\[ p = \text{process fraction defective} \]
\[ F = \text{the average fraction of output inspected} \]
\[ n = \text{lot-size or sub-lot size of consecutively produced items taken for inspection} \]
\[ i = \text{run-length of conforming units in a batch of n consecutively produced items} \]

Thus for the CSP-1, the average total cost \( (C) \) of operating the plan, per article produced would be

\[ C = p + (k-p)F \quad (3.2.1) \]

when the process fraction defective \( p \) is constant.

The quantity \( F \) called the average fraction of the output can be obtained from the relation

\[ F = \frac{1}{1 + (n-1)q^i} \quad , \quad q = 1 - p \quad (3.2.2) \]

If values of \( p \) below and above \( k \) occur in equal proportions, we would expect \( F = 1/2 \) at \( p = k \), giving

\[ (n-1)(1-k)^i = 1 \quad (3.2.3) \]
or approximately (if $n$ and $1/k$ are not too small)

$$k_i = \log_{e} n$$  \hspace{1cm} (3.2.4)

Thus, $F$ can be approximated by the "logistic" function of $p$, given below:

$$F = \frac{1}{1 + e^{-1(p-k)}}$$  \hspace{1cm} (3.2.5)

If we assume the case $p \neq k$, the costs of inspection excess to (3.2.1) are

$$C = (k-p)F$$, when $p < k$  \hspace{1cm} (3.2.6)

$$C = (p-k)(1-F)$$, when $p > k$  \hspace{1cm} (3.2.7)

The above excess costs may be regarded as the cost of our not knowing the value of $p$, which are also termed as "regret".

From equations (3.2.6) and (3.2.7), total excess cost due to variation in $p$ can be written as

$$\Delta C = \Delta_1 C + \Delta_2 C$$  \hspace{1cm} (3.2.8)

and the total excess average cost as

$$E(\Delta C) = E(\Delta_1 C) + E(\Delta_2 C)$$  \hspace{1cm} (3.2.9)
We now introduce the use of prior in computing the costs. Let us denote by \( f(p) \) the density of \( p \), then excess average cost given by (3.2.9) can be written as

\[
E(\Delta C) = \int_0^k (k-p) F dp + \int_k^{2k} (p-k) (1-F) f(p) dp
\]  
\[\text{(3.2.10)}\]

In particular, for a uniform prior of \( p \) within the limit \((0, 2k)\) i.e.

\[
f(p) = \begin{cases} 
1/2k & , \quad 0 \leq p \leq 2k \\
0 & , \quad \text{otherwise}
\end{cases}
\]  
\[\text{(3.2.11)}\]

the total excess average cost would be

\[
E(\Delta C) = \frac{1}{k} \int_0^k (k-p) F dp + \frac{1}{k} \int_k^{2k} (p-k) (1-F) dp
\]  
\[\text{(3.2.12)}\]

### 3.3 Two-Stage Linear Programming for Minimizing \( E(\Delta C) \)

Our objective is to find the optimal plan i.e. a plan \((n, i)\) giving the minimum of aggregate of excess average cost. Since the \( p \) has random or uncertain behaviour during the inspection and it can vary within \((0, 2k)\), that may be either less than \(k\)
(p < k) or greater than k (p > k) or equal to k (p = k), it should be treated as random variable. Further, since for p = k no excess cost is incurred, we have only two possible cases i.e. p < k and p > k to be considered for optimization. The above problem can be formulated as a two-stage linear programming problem in the manner described below.

The two-stage linear programming technique is one which converts a stochastic linear programming into an equivalent deterministic problem. Here, the total excess average cost of operating the plan given by (3.2.12) is treated as the objective function. In particular, when p follows uniform prior the objective function would be (3.2.12). The constraints against optimization are

\[ \sum_{i} p_i = 1 \quad (3.3.1) \]

and

\[ 0 \leq p \leq 1 \quad (3.3.2) \]

We formulate our problem of minimizing \( \Delta C \) over varying p subject to the constraints given by (3.3.1) and (3.3.2) as follows so that two-stage linear programming approach can be directly applied.
Minimize \( E(\Delta C) = \begin{cases} 
\int_0^k F(k-p) f(p) \, dp 
\quad , \quad p < k \\
\int_0^k (k-p) F f(p) \, dp + \int_k^{2k} (p-k)(1-F) f(p) \, dp 
\quad , \quad k \leq p \leq 2k \\
\int_k^{2k} (p-k)(1-F) f(p) \, dp 
\quad , \quad p > k
\end{cases} \) 

subject to the constraints (3.3.1) and (3.3.2).

Since the constraints for \( p \) is probabilistic we require permanent feasibility condition to be satisfied. In fact, permanent feasibility condition along with convexity condition of the cost function as the objective function has already been established by Dantzig (1955) for \( m \)-stage linear programming problem.

It may be noted that if \( p \) is deterministic, the above problem reduces to that of general linear programming problem as the first and the third terms in \( E(\Delta C) \) disappear and so does prior.
The equation (3.3.3) for rectangular prior
given by (3.2.11) reduces to

\[
\begin{align*}
&\min_p \left[ E(\Delta C) \right] = \\
&\quad \begin{cases} \\
&\frac{1}{2k} \int_0^k \frac{x \, dx}{1 + e^{ix}}, & p < k \\
&\frac{1}{2k} \int_0^k \frac{x \, dx}{1 + e^{ix}} - \frac{1}{2k} \int_0^k \frac{x \, dx}{1 + e^{ix}} + \frac{k}{4}, & 0 \leq p \leq 2k \\
&\frac{-1}{2k} \int_0^k \frac{x \, dx}{1 + e^{ix}} + \frac{k}{4}, & p > k
\end{cases}
\end{align*}
\]

(3.3.4)

Explicit solutions of the integrals in
(3.3.4), if obtained, would be quadratic in \( k \) and
can be solved by following Dantzig (1955). However,
due to its complicated nature (in the series form)
and the approximations made in curtailing the series,
we prefer numerical analysis.

Anscombe (1958), suggested that for the
rectangular prior with \( 0 \leq p \leq k \), the excess average
cost would be obtained from the empirical formula:

\[
E(\Delta C) = 0.3 \frac{k}{\sqrt{n}}
\]

(3.3.5)
He further remarked that if \( 0 \leq p \leq 2k \), the answer can not be much different from the above. However, as will be seen later, these remarks do not hold true even for rectangular prior, not to speak of other priors such as beta discussed by Agrawal (1980).

3.4 NUMERICAL ILLUSTRATION:

In order to illustrate the implication of the different formulae obtained in section 3.3 we, for convenience of comparison, calculate the costs of operating the plans \( (n, i) \) considered by Anscombe (1958) given in Table 3.1. The minimum of the total excess average cost of operating the plan when we lack specific knowledge whether \( p < k \) or \( p > k \) and know only that \( p \) lies in the interval \((0, 2k)\) with rectangular prior, is obtained from the middle term of the equation (3.3.4).

The values of \( \min_p \left[ E(\Delta C) \right] \) have been calculated for different combinations of \( (n, i) \) when \( k = 0.10 \) ensuring that process fraction defective does not exceed 20 percent. The results are shown in column (iii) of Table 3.1. In columns (iv) and (v) we have quoted the results of Anscombe (1958) obtained from (3.3.5) and those of Agrawal (1980) for beta prior, respectively.
In case, we have specific knowledge that \( p < k \), \((= 0.10 \text{ say})\), the excess average cost is obtained from the first term of the equation (3.3.4). Similarly, for \( p > k \) we use the third term only. In above two cases, one must, however, be careful that range for prior would be 0 to 0.10 and 0.10 to 0.20 respectively. The excess average costs for one-sided situations are shown in column (vi) and (vii) of Table 3.1.

Table 3.1: Minimum Excess Average Costs per 1000 units with \( k = 0.10 \).

<table>
<thead>
<tr>
<th>n</th>
<th>i</th>
<th>Uniform prior ( 0 \leq p \leq 2k )</th>
<th>Beta-prior</th>
<th>( \mathbb{E}(\Delta \mathbb{C}) )</th>
<th>( \mathbb{E}(\Delta \mathbb{C}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(i)</td>
<td>(ii)</td>
<td>(iii)</td>
<td>(iv)</td>
<td>(v)</td>
</tr>
<tr>
<td>5</td>
<td>27</td>
<td>4.580</td>
<td>5.780</td>
<td>7.003</td>
<td>2.680</td>
</tr>
<tr>
<td>10</td>
<td>43</td>
<td>2.310</td>
<td>4.580</td>
<td>4.756</td>
<td>2.060</td>
</tr>
<tr>
<td>20</td>
<td>57</td>
<td>1.430</td>
<td>3.980</td>
<td>3.461</td>
<td>2.020</td>
</tr>
<tr>
<td>50</td>
<td>76</td>
<td>1.070</td>
<td>3.450</td>
<td>2.251</td>
<td>1.410</td>
</tr>
<tr>
<td>100</td>
<td>90</td>
<td>0.820</td>
<td>3.000</td>
<td>1.692</td>
<td>1.300</td>
</tr>
</tbody>
</table>

From the entries of Table 3.1, we can easily conclude that the minimized excess average cost by two-stage linear programming approach is quite low in
comparison to those obtained by Anacembe's formula and given by Agrawal for beta-prior. However, in all the three cases, a similar trend of change in costs is observed. Moreover, in all the three situations, the optimum plan amongst the considered ones is the same, that is, \( n = 100, i = 90 \).

3.5 MINIMIZATION OF RISK FOR OPTIMAL INSPECTION

So far, we have discussed how could the optimal choice of the plan for continuous inspection be based on the consideration of minimum excess average cost. In this section and onward we shall discuss how could the criterion of minimum risk be used for choosing an optimal inspection plan for continuously manufactured items. For the Dodge-type continuous sampling plan, where rejected lots undergo 100% inspection, the risk function \( R \) suggested by Moriguti (1955) is defined as follows:

\[
R = I \cdot n + D \cdot N \cdot P, \quad P_a(n, c) + N \cdot H \cdot \{1 - P_a(n, c)\} \tag{3.5.1}
\]

where

- \( p \) = process fraction defective
- \( I \) = inspection cost
- \( D \) = loss due to accepting an defective item
- \( H \) = loss due to rejecting a non-defective item
- \( P_a \) = probability of accepting a lot
\[ N = \text{lot-size} \]
\[ n = \text{sample-size} \]
\[ c = \text{acceptance number} \]

The quantity \( P_a(n, c) \), probability of acceptance of the lot of quality \( p \), is usually computed by using the Poisson - approximation given by

\[
P_a(n, c) = \sum_{x=0}^{c} \frac{e^{-np}(np)^x}{x!}
\quad \text{(3.5.2)}
\]

It may be noted that the risk function (or the penalty cost) is a known deterministic function, and the expected value of risk-function \( R \) is a convex function.

As a modified form of procedure B of Dodge (1943) for continuous inspection, the material is assumed to be offered as a flow of consecutive lots in order of their production and single sampling plan \((N, n, c)\) is operated for judging the acceptability of a lot. All the \( n \) items in a sample are required to be inspected whenever in the first \((n-1)\) items we observe \((c-1)\) or fewer defectives; otherwise inspection of fewer than \( n \) items would lead to the decision of rejecting the lot and terminating the inspection thereby. Thus, the number of items to be
inspected in a lot to arrive at a decision is a random
variable with its expected value given by

\[ E (1) = R = \frac{n}{1 = c + 1} \frac{1 \cdot P_i}{\sum_1 P_i} \quad (3.5.3) \]

where

\[ P_i = \frac{n}{1 = c + 1} \left( \frac{i - 1}{c} \right) \cdot p^{c + 1} \cdot q^{i - 1 - c} \quad (3.5.4) \]

By taking \( \bar{R} \) in place of \( n \) in the equation
(3.5.1), the risk function for the continuous sampling
plan can be defined as

\[ R = I \cdot R + H \cdot N \cdot P \cdot P_a(n, c) + D \cdot N \left[ 1 - P_a(n, c) \right] \quad (3.5.5) \]

Agrawal (1980), treated the risk \( R \) as pay-off entries of a rectangular game by taking the plan
parameters as strategies and the values of \( p \) (process
continuous defectives) as states of nature. He, however,
instead of solving the game, tried to evaluate numerically
average risks over \( p \) for different plans, and took the
one with minimum average risk as the optimal choice. Since
our game is rectangular one, it can be transformed into a
linear programming problem and solved easily. In the
present investigation, we have solved the game theoretically
by minimizing the maximum risks and found the minimax value
of the game. In other words, we have found the best strategy
(plan) for the statistician against nature (p) and the corresponding minimum risk.

3.6 **LINEAR PROGRAMMING APPROACH FOR OPTIMUM RISK**

The problem of continuous inspection, like any other problem in statistics, could be viewed as a game against nature where the quality of the incoming lots say \( p_1, p_2, \ldots, p_n \) are taken as the strategies of nature (player A) with the courses of action \( (x_1, x_2, \ldots, x_m) \) and the choice of different sampling plans \( S_j, j = 1, 2, \ldots, n \) as the strategies of statistician or the inspector (player B) with courses of action \( (y_1, y_2, \ldots, y_n) \). A sampling plan \( S_j \) is characterized by a particular combination of \( (N, n, c) \), for example, the two plans \( (300, 30, 0) \) and \( (300, 65, 1) \) are taken as two different strategies. Let us denote by \( R_{ij} \) the risk in choosing the plan \( S_j \) when the lot fraction defective is \( p_i \) and treat it as pay-off to A (i.e., nature).

We shall now deal with the problem of maximizing pay-off to A irrespective of the strategies of B which is equivalent to minimizing the risk of the statistician. This, in fact, offers a general solution to the choice of optimal sampling plan with expected minimum risk.
Let us denote by \( \gamma \), the expected pay-off to A. Then we shall try to maximize \( \gamma \) by using linear programming approach as formulated below:

Maximize \( \gamma \)

subject to the constraints

\[
R_{11} x_1 + R_{21} x_2 + \ldots + R_{m1} x_m \geq \gamma
\]
\[
R_{12} x_1 + R_{22} x_2 + \ldots + R_{m2} x_m \geq \gamma
\]
\[
\vdots
\]
\[
R_{1n} x_1 + R_{2n} x_2 + \ldots + R_{mn} x_m \geq \gamma
\]

\[
x_1 + x_2 + \ldots + x_m = 1
\]

and

\[
x_1, x_2, \ldots, x_m \geq 0 . \quad (3.6.1)
\]

Here we have assumed that \( \gamma \) is nonnegative.

Now, let us make the transformations

\[
\frac{x_1}{\gamma} = x_1, \quad \frac{x_2}{\gamma} = x_2, \quad \ldots, \quad \frac{x_m}{\gamma} = x_m . \quad (3.6.2)
\]

Since

\[
\max \gamma = \min \left( \frac{1}{\gamma} \right)
\]

\[
= \min \left\{ \frac{x_1 + x_2 + \ldots + x_m}{\gamma} \right\}
\]

\[
= \min \left\{ x_1 + x_2 + \ldots + x_m \right\} \quad (3.6.3)
\]
The maximization problem (3.6.1) can be written in the form of minimization problem, as given below:

\[
\text{Min} \left( \frac{1}{\gamma} \right) = \text{Min} \left( x_1 + x_2 + \ldots + x_m \right)
\]

subject to the constraints

\[
\begin{align*}
R_{11} x_1 + R_{21} x_2 + \ldots + R_{m1} x_m & \geq 1 \\
R_{12} x_1 + R_{22} x_2 + \ldots + R_{m2} x_m & \geq 1 \\
\vdots & \quad \vdots \\
R_{1n} x_1 + R_{2n} x_2 + \ldots + R_{mn} x_m & \geq 1 \\
\end{align*}
\]

\[x_1 + x_2 + \ldots + x_m = 1 \quad \text{and} \quad x_1, x_2, \ldots, x_m \geq 0 \quad (3.6.4)\]

The above problem can be restated according to Karlin (1962) as follows:

\[
\text{Minimize} \left( \frac{1}{\gamma} \right)
\]

subject to the constraints

\[
\begin{align*}
\sum_{j=1}^{n} R_{ij} x_j & \leq 1, \quad \text{for all } j \\
x_1 + x_2 + \ldots + x_m & = 1 \\
\end{align*}
\]

\[x_1, x_2, \ldots, x_m \geq 0, \quad 1 = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n. \quad (3.6.5)\]
The above equation can be solved by applying the simplex method of solving the general linear programming problem. A computer program has also been developed for solving the above problem numerically and is useful when the number of alternatives are quite large. The results are discussed in the next section by way of an illustration. It is found that in a number of situations game can be solved analytically by applying principle of dominance.

3.7 NUMERICAL ILLUSTRATION:

In order to explain the computational procedure of the risk ($R_{ij}$) and the method of obtaining the optimal plan we consider an example with $I = 0.10, D = 5.00$ and $H = 2.00$. In defining the above costs, it is assumed that the cost of inspecting an item is always much less than the loss due to the acceptance of a defective item or the rejection of a non-defective item, whereas, the cost of rejecting a non-defective item is less than the cost of accepting a defective one. Reasons for such an assumption have been explained by Grubbs (1954). Let us consider a plan $(N, n, c) = (300, 33, 0)$ having $AQL = 1.0 \%$. Then, if we assume the incoming lot quality to be $p = 0.001$, then its probability of acceptance $P_a(n, c)$ computed by using (3.3.2) is found to be $0.970$. The average number of items inspected per lot in a continuous
sampling scheme is worked out from the equation (3.5.3) giving a value of 17. The risk, \( R_{ij} = 21.150 \) is then obtained from equation (3.5.5).

Assuming that the incoming lots do not contain more than 20% defective items, we choose \( p_1 = 0.001 \), \( p_2 = 0.005 \), \( p_3 = 0.010 \), \( p_4 = 0.015 \) and \( p_5 = 0.020 \) as the states of nature and try to find out the optimal plan amongst \( S_1 = (300,33,0), S_2 = (300,65,1), S_3 = (400,34,0) \) and \( S_4 = (400,70,1) \), all having the same AOQL i.e. 1.0%. The pay-off matrix is shown in Table 3.2.

Table 3.2: Pay-off Matrix of Risks \( (R_{ij}) \) for Different Plans \( S_j \) and Incoming Lot Quality \( p_i \), with AOQL = 1.0%.

<table>
<thead>
<tr>
<th>( S_j ) ( (N,n,c) )</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_3 )</th>
<th>( P_4 )</th>
<th>( P_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(300,33,0)</td>
<td>21.150</td>
<td>99.070</td>
<td>181.120</td>
<td>249.390</td>
<td>306.810</td>
</tr>
<tr>
<td>(300,65,1)</td>
<td>8.390</td>
<td>35.090</td>
<td>99.610</td>
<td>172.980</td>
<td>249.710</td>
</tr>
<tr>
<td>(400,34,0)</td>
<td>27.740</td>
<td>138.100</td>
<td>247.900</td>
<td>321.250</td>
<td>393.490</td>
</tr>
<tr>
<td>(400,70,1)</td>
<td>7.690</td>
<td>52.310</td>
<td>145.280</td>
<td>247.560</td>
<td>353.580</td>
</tr>
</tbody>
</table>
The minimax solution of the above game has been obtained by feeding the data of Table 3.2 to a computer through FORTRAN IV language. This yielded the optimal plan as \((400, 34, 0)\) with a minimum risk of 25.00 (cost unit).