CHAPTER VI

INFORMATION THEORETIC APPROACH FOR OPTIMUM REDUNDANCY
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6.1 INTRODUCTION:

Black and Proschan (1959) considered the following two seemingly different problems, one arising in inventory control, the other in reliability design.

1. A complex system is to be placed in the field for a period of experimentation. How many spares for each of the essential components should accompany the system? Maximum assurance of continued operation of the system is desired for a fixed expenditure of spares. Component failure distributions are known.

2. A complex system (a missile, say) is to perform a mission. How should redundancy be designed into the system to give maximum reliability within the weight limitation? Component failure distributions are known.

The authors pointed out that both the problems have the same mathematical structure. Under the assumption of exponential failure rate they obtained
the solution for the composition of the spare parts kit which maximizes the assurance of continued operation subject to the fixed budget for spares. They remarked that the formulae are also applicable in determining the optimal allocation of redundancy in designing system reliability under weight or cost restraint. Bellman and Dreyfus (1973) has solved the same problem under simultaneous constraints of weight and cost. Ackoff and Sasieni (1978), in their study on replacement-maintenance and reliability problems of complex system included the case of rectangular failure distribution among other things. The general procedure followed to solve the above problem was to employ linear programming approach for the case of linear objective function and the dynamic programming approach or Kuhn-Tucker method, when the objective function is non-linear.

In this chapter, information theoretic approach has been developed to solve the problem posed by Black and Proschan (1959) under weight limitation. The method can be applicable for cost limitations too, with some minor modifications. The advantage of the proposed method lies in its simplicity for obtaining exact numerical results, apart from being new one.
6.2 **FORMULATION OF THE PROBLEM**

We shall make use of the notations of Black and Proschan (1959) for convenience in formulating the problem.

Let a system having \( k \) components be placed in the field for operation for a fixed period \( t_0 \) and that during the period, only the redundant standby units provided are used to replace the components that have failed, the occurrence of failures being independent. Let

- \( d_{ij} \) = number of components of type \( i \) consist in the system, scheduled for \( t_j \) hours of use ; \( j = 1, 2, \ldots, m ; i = 1, 2, \ldots, k \),
- \( w_i \) = weight of a single unit \( i \),
- \( \theta_i \) = failure rate per hour,
- \( n \) = total number of inventory of spare parts
  i.e. total number of redundant standby units to be provided in the system,
- \( n_i \) = inventory of spare parts to be provided for the \( i^{th} \) components ; \( i = 1, 2, \ldots, k \),
- \( N_{ij} \) = number of failures during the \( t_j \) hours of operation,
- \( N_i \) = total number of failures of \( i^{th} \) component during the \( t_0 \) hours,
\[ P_1(n_1) = \text{probability that } n_1 \text{ spares of type } 1 \]
\[ \text{will be adequate to replace the } i^{th} \]
\[ \text{component on failure}, \]

\[ P(n) = \text{probability that a spare parts kit consisting of } n_1 \text{ spares of type } 1 (i=1,2,\ldots,k) \]
\[ \text{will be adequate for successful operation of the system.} \]

If we assume that a single unit of type \( i \)
\[ \text{has an Exponential life density } \theta_i \exp(-\theta_i t), \]
\[ \text{with a failure rate per hour, } i = 1, 2, \ldots, k; \text{then } N_{ij} \]
\[ \text{will follow a Poisson distribution with parameter } \theta_i d_k t_j \]
\[ \text{and} \]
\[ N_i = \sum_{j=1}^{m} N_{ij}, \]
\[ \text{a Poisson distribution with parameter} \]
\[ \lambda_i = \sum_j \theta_i d_k t_j \]
\[ \text{(6.2.1)} \]

Hence, probability that there will be no more failures than \( n_1 \) for the \( i^{th} \) component will be given by

\[ P_1(n_1) = \Pr [ N_i \leq n_1 ] = \sum_{x=0}^{n_1} \frac{e^{-\lambda_i} \lambda_i^x}{x!} \]
\[ \text{(6.2.2)} \]

Since, we have assumed that the failure of different types of components are independent, probability that no more than \( n \) failures occur in the
whole system during $t_0$ hours of operation would be given by

$$P(n) = \prod_{i=1}^{k} P_1(n_i)$$

(6.2.3)

Our problem is to find the optimum composition of redundant standby units say $(n_1, n_2, \ldots, n_k)$ of a given total inventory of $n$ spares yielding maximum reliability of the system for a fixed weight $w_0$.

In other words we wish to

Maximize $P(n) = \prod_{i=1}^{k} P_1(n_i) = \prod_{i=1}^{k} \left[ \sum_{x=0}^{n_i} \frac{e^{-\lambda_i} \lambda_i^x}{x!} \right]$ 

subject to the constraints

$$w(n) = \sum_{i=1}^{k} w_i n_i \leq w_0,$$

$$\sum_{i=1}^{k} n_i = n,$$

and

$$n_i \geq 0 \quad i = 1, 2, \ldots, k$$

(6.2.5)

To solve the above non-linear programming problem with linear constraints, Black and Proschan (1959) used the Kuhn-Tucker method and arrived at approximate solution. Bellman and Dreyfus (1973) solved the same problem under cost, weight and space limitations by dynamic programming approach.
In this chapter, we have made an information-theoretic approach described in the following section to obtain the optimum inventory of spare parts which maximizes the reliability of operating system for a given total weight \( w \).

6.3 INFORMATION THEORETIC APPROACH:

To formulate the above problem using information theoretic approach, we recall that \( N_i = \sum_j N_{ij} \) is a Poisson variate with parameter

\[
\lambda_i = \theta_i \sum_{i=1}^{m} d_{ij} f_j; \quad i = 1, 2, \ldots, k.
\]

It is well known that if \( N_1, N_2, \ldots, N_k \) are \( k \) independent Poisson variates with parameters \( \lambda_1, \lambda_2, \ldots, \lambda_k \) respectively, then the conditional distribution of \( P( N_1 \cap N_2 \cap \ldots \cap N_k / N = n) \), where \( N = N_1 + N_2 + \ldots + N_k \) is fixed, is a multinomial distribution given by

\[
P( N_1 \cap N_2 \cap \ldots \cap N_k / N = n) = \frac{n!}{n_1! n_2! \ldots n_k!} p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}
\]

\[
= \frac{n!}{\prod_{i=1}^{k} n_i!} \prod_{i=1}^{k} p_i^{n_i}, \quad (6.3.1)
\]
where
\[ \sum_{i=1}^{k} n_i = n = N = \sum_{i=1}^{k} N_i \]  
(6.3.2)

and
\[ p_i = \frac{\lambda_i}{\lambda} ; \sum_{i=1}^{k} p_i = \frac{\sum \lambda_i}{\lambda} = 1 \]  
(6.3.3)

Let \( \pi(\theta) = \pi = (\pi_1, \pi_2, \ldots, \pi_k) \) be the probability distribution of spare parts which maximizes the reliability of the system under the weight limitation of \( w_0 \). If we have a set of observed probabilities \( p_1, p_2, \ldots, p_k \) of failures for \( k \) types of components then our approach to find the optimum value of \( \pi_1 \), hence of \( n_1 = n \pi_1 \) lies in minimizing with respect to \( \pi_1 \), the following discriminating information statistic for the multinomial distribution (6.3.1):

\[ I(\pi, p) = \sum_{i} \pi_i(\theta) \log_e \frac{\pi_i(\theta)}{p_i} \]  
(6.3.4)

subject to the constraints

\[ \sum_{i} w_i \pi_i \leq w_0 \]

and

\[ \pi_i \geq 0 ; i = 1, 2, \ldots, k \]  
(6.3.5)

To minimize the non-linear objective function (6.3.4) under linear constraints (6.3.5), we have used the dynamic programming approach based on
calculus of variations. The recurrence relationship for this approach can be written as follows:

\[ f_x(\pi_k) = \min_{0 \leq \omega_k \pi_k \leq \omega_0} \left[ \sum_{i=1}^{k} \left( I_k(\pi_i, b) + f_{i+1}(\omega_0 - \pi_k \omega_i) \right) \right] \]

\[ f_k(\pi_k) = \min_{0 \leq \omega_k \pi_k \leq \omega_0} \left[ I_k(\pi_k, b) \right] \]

\[ = I_k(\pi_k, b) \quad ; \quad i = k \]  \hspace{1cm} (6.3.7)

where

\[ \pi_k \geq 0. \]

Once optimum value of \( \pi_k \) say \( \hat{\pi}_k \) and hence \( \hat{n}_1 = n \hat{\pi}_1 \) has been obtained, the maximum value of reliability of the operating system set for a fixed period \( t_0 \) can be obtained by equations (6.2.2) and (6.2.3).

6.4 Numerical Illustration

In order to illustrate the application of foregoing results we consider the data given in Table 1. of Black and Proschan (1959), but treat the cost \( c_i \) in dollars as weight in Kilograms.
Table 6.1:

<table>
<thead>
<tr>
<th></th>
<th>Component type</th>
<th>Failure rate/hour</th>
<th>Weight</th>
<th>Number in VHF, scheduled for 332 hours use each, $d_{ij}$</th>
<th>Number in VHF, scheduled for 2160 hrs of use each, $d_{ii}$</th>
<th>Expected number of failures</th>
<th>Observed probabilities of failures</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>1/2500</td>
<td>240</td>
<td>4</td>
<td>4</td>
<td>4.0</td>
<td>0.4592</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
<td>1/4000</td>
<td>1025</td>
<td>2</td>
<td>5</td>
<td>2.9</td>
<td>0.3329</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
<td>1/800</td>
<td>1158</td>
<td>4</td>
<td>0</td>
<td>1.7</td>
<td>0.1951</td>
</tr>
<tr>
<td>4</td>
<td>D</td>
<td>1/6000</td>
<td>750</td>
<td>2</td>
<td>0</td>
<td>0.11</td>
<td>0.0126</td>
</tr>
</tbody>
</table>

By using the ratios given in the last column of the table 6.1, a fixed number $n = 21$ of redundant standby units can be decomposed into a set of $(10, 7, 4, 0)$ units to meet the failures of components of type A to D respectively. The corresponding total weight obtained from the column 4 is then found to be 14,207 Kg. The reliability of the system worked out with the help of equation (6.2.2) and (6.2.3) comes out to be $0.866464$. These calculations provide the guide line for the exact solution and give an idea about $w_0$.

In view of the above calculations we take $w_0 = 14,000$ when $n = 21$. The values of $x_i$, $i = 1, 2, 3, 4$ are then found by using the recurrence
relationships (6.3.6) and (6.3.7) as

\[ \pi_1 = 0.494711, \pi_2 = 0.288670 \]
\[ \pi_3 = 0.191385, \pi_4 = 0.025234 \]

giving \( n_1 = 10, n_2 = 6, n_3 = 4 \) and \( n_4 = 1 \), and corresponding minimum weight = 13,932 Kgs. = \( w(n) \).

The reliability for this optimum composition of spare parts kit of redundant standby units comes out to be 93.342 %. These results, though, obtained from an entirely different approach are in complete agreement with those of Black and Proschan (1959).

It has been checked by numerical calculations that a too high value of \( w_0 \) for the same \( n = 21 \) makes the system less reliable whereas a too low value makes the solution infeasible.

Hence for an increase in the reliability of the system, we must have simultaneous increase in both i.e. in the total inventory level \( n \) and maximum allowable weight \( w_0 \). For instance, system can attain reliability of order say 99.9 % if we increase the total inventory level to \( n = 31 \) and allow a weight of 22,000 Kg. with a composition of (13,9,7,2) units.